H.P. Sculp.

Aristippus Philosophus Socraticus, naufragio cum ejectus ad Rhodiensium litus animadvertisset Geometrica schemata descripta, exclamavisse ad comites ita dicitur,

Bene speremus, Hominum enim vestigia video.

Vitruv. Architect. lib. 6. Praef.

FIRST VOLUME

OF THE

INSTRUCTIONS

GIVEN IN THE

DRAWING SCHOOL

ESTABLISHED BY THE

DUBLIN-SOCIETY,

Purſuant to their RESOLUTION of the Fourth of FEBRUARY, 1768;

To enable Youth to become PROFICIENTS in the different Branches of that Art, and to purſue with Succeſs, GEOGRAPHICAL, NAUTICAL, MECHANICAL, COMMERCIAL, and MILITARY STUDIES.

Under the Direction of *JOSEPH FENN,* heretofore Profeſſor of PHILOSOPHY in the Univerſity of NANTS.

Quid munus Reipublicæ majus aut melius afferre poſſumus, quam ſi Juventutem bene Erudiamus? CICERO.

DUBLIN:

Printed by ALEX. M'CULLOH, in Henry-ſtreet, M,DCC,LXIX.

PLAN *of the* **INSTRUCTIONS** *given in the* **DRAWING-SCHOOL** *established by the* **DUBLIN SOCIETY,** *to enable* **YOUTH** *to become Proficients in the different Branches of that Art, and to pursue with Success geographical, nautical, mechanical, commercial or military Inquiries.*

L'Oisiveté & l' Ignorance sont les deux Sources empoisonnées de tous les Desordres, & les plus grands Fléaux de la Societé.

Wise Regulations relative to the Education of Youth, in England, Scotland, and other Parts of Europe.

THE Education of Youth is considered in all Countries as the Object which interests most immediately the Happiness of Families, as well as that of the State. To this End, the ablest Hands are employed in forming Plans of Instruction, the best calculated for the various Professions of Life, and Societies are formed, composed of Men distinguished, as well by their Birth and Rank, as by their Experience and Knowledge, under whose Inspection, and by whose Care they are carried into Execution, by Persons of acknowledged Abilities in their different Departments: And thus the Education of Youth is conducted, from their earliest Years, in a Manner the best suited to engage their Minds in the Love of useful Knowledge, to improve their Understandings, to form their Taste and ripen their Judgments, to fix in them an Habit of Thinking with Steadiness and Attention, to promote their Address and Penetration, and to raise their Ambition to excel in their respective Provinces.

Fatal Consequences resulting from the Neglect of this Object

However necessary such Regulations may appear to every reasonable Person, however wished for by every Parent who feels the Loss of a proper Education in his own Practice; nevertheless they had not been even thought of in this Country, where that Extent of Knowledge, requisite to prepare Youth to appear with Dignity in the various Employments of Life, or to enable them to bring to Perfection the different Arts for which they are designed, being not attended to; Education was regarded as a puerile Object, and of Course abandoned to illiterate Persons, who from their illiberal and mechanic Methods of teaching gave Youth little or no Information.

How far the Drawing-School established under the Inspection of the Dublin-Society put on a proper Footing, has supplied this Defect.

To remove so general and well grounded a Complaint, it was proposed that the Youth of this Kingdom should receive in the Drawing-School established by the DUBLIN SOCIETY, the Instructions necessary to enable them to become Proficients in the different Branches of that Art, and, to pursue with Success, geographical, nautical, mechanical, commercial or military Enquiries: in this View, an Abstract of the following Plans were delivered to their Secretaries and Treasurer in the Month of October, 1764, to be laid before the Society; and to prevent an Undertaking of National Utility, to be defeated through the Suggestions of Design or Ignorance, the Plans were printed; which being received by the Public with general Approbation, the DUBLIN-SOCIETY, pursuant to the Report of their Committee appointed to examine into the Merit of the Plans, and the Character of the Proposer, resolved, the 4th of February, 1768, that they should be carried into Execution by the Author, under their immediate Inspection.

The PLANS *are as follow.*

I.

PLAN of a Course of pure Mathematicks, absolutely necessary for the right understanding any Branches of practical Mathematicks in their Application to geographical, nautical, mechanical, commercial, and military Enquiries.

II.

PLAN of the physical and moral System of the World, including the Instructions relative to young Noblemen and Gentlemen of Fortune.

III.

PLAN of the military Art, including the Instructions relative to Engineers, Gentlemen of the Artillery, and, in general, to all Land-Officers.

IV.

PLAN of the merchantile Arts, or the Instructions relative to those who are intended for Trade.

PLAN

V.

PLAN of the naval Art, including the Instructions relative to Ship-Buiders, Sea-Officers, and to all those concerned in the Business of the Sea.

VI.

PLAN of a School of Mechanic Arts, where all Artists, such as Architects, Painters, Sculptors, Engravers, Clock-makers, &c. receive the Instructions in Geometry, Perspective, Staticks, Dynamicks, Physicks, &c. which suit their respective Professions, and may contribute to improve their Taste and their Talents.

The Youth of this Kingdom destitute of the most important Means of Instruction.

THOSE PLANS have convinced the Noblemen and Gentlemen of Fortune of this Kingdom, that their Children, and in general, the Youth of this Country, were destitute of the most important Means of Instruction, and would ever be destitute of them, until they had resolved that Men of Genius and Education should be encouraged to appear as Teachers.

PLAN *of a Course of pure Mathematicks, absolutely necessary for the right understanding any Branches of practical Mathematicks in their Application to geographical, nautical, mechanical, commercial, and military Inquiries.*

Vix quicquam in universa Mathesi ita difficile aut arduum occurrere posse, quo non inoffenso Pede per hanc Methodum penetrare liceat.

I.

Utility of the Mathematicks

PURE Mathematicks comprehend Arithmetick, and Geometry. Practical Mathematicks, their Application to particular Objects, as the Laws of Equilibrium, and Motion of solid and fluid Bodies, the Motion of the heavenly Bodies, &c. they extend to all Branches of human Knowledge, and strengthening our intellectual Powers, by forming in the Mind an Habit of Thinking closely, and Reasoning accurately, serve to bring to Perfection, with an entire Certitude, all Arts which Man can acquire by his Reason alone. It is therefore of the highest Importance, that the Youth * of this Country should be methodically brought acquainted with a Course of pure Mathematicks, to serve as an Introduction to such Branches of Knowledge as are requisite to qualify them for their future Stations in Life. The Noblemen and Gentlemen of Fortune, therefore, have unanimously resolved, that such a Course should be given on the most approved Plan, in the DRAWING SCHOOL established under their Inspection, by a Person, who, on account of the Readiness and Knowledge he has acquired in these Matters, during the many Years that he has made them his principal Occupation, is qualified for making the Entry to those abstruse Sciences, accessable to the meanest Capacity.

* The proper Age to commence this Course is 14.

II.

Method of teaching Mathematicks.

As to the Method of teaching Mathematicks, the synthetic Method being necessary to discover the principal Properties of geometrical Figures, which cannot be rightly deduced but from their Formation, and suiting Beginners, who, little accustomed to what demands a serious Attention, stand in Need of having their Imagination helped by sensible Objects, such as Figures, and by a certain Detail in the Demonstrations, is followed in the Elements (a). But as this Method, when applied to any other Research, attains its Point, but after many Windings and perplexing Circuits, etc. by multiplying Figures, by describing a vast many Lines and Arches, whose Position and Angles are carefully to be observed, and by drawing from these Operations a great Number of incidental Propositions which are so many Accessaries to the Subject; and very few having Courage enough, or even are capable of so earnest an Application as is necessary to follow the Thread of such complicated Demonstrations, afterwards a Method more easy and less fatiguing to the [illegible] is pursued. This Method is the analitic Art, the ingenious [illegible] of reducing Problems to the most simple and easiest [illegible] the Question proposed can admit of; it is the universal Key of Mathematicks, and has opened the Door to a great Number of Persons, to whom it would be ever shut, without its Help; by [illegible] Genius, and Genius, aided by Art so useful, [illegible] would never have obtained by its own Force [illegible] the Theory of curve Lines have been unfolded [illegible] in different Orders, Classes, Genders, [illegible] as in an Arsenal, where Arms are properly arranged [illegible] ready [illegible] which serve in the Re[illegible] either in Mathematicks, Astronomy, [illegible] has [illegible] the great Sir *Isaac Newton* [illegible] he has made, and enabled the Men of [illegible] them. The Method of [illegible] Exercises of it, the first be-

[illegible] Schools established in England, [illegible] the Elements of Geometry; we shall [illegible] from the first of [illegible] each of which the Master [illegible] the first six Books of [illegible] the Elements of Fortification and [illegible] the [illegible] and [illegible] Books of [illegible] of Astronomy. The [illegible] Sir Isaac Newton's Principia [illegible] explained to them: the [illegible] Those in the fourth Class [illegible] Newton's Principia, with [illegible] Mathematicians of Europe.

ing the Art of finding Magnitudes infinitely ſmall, which are the Elements of finite Magnitudes; the ſecond the Art of finding again, by the Means of Magnitudes infinitely ſmall, the finite Quantities to which they belong; the firſt as it were reſolves a Quantity, the laſt reſtores it to its firſt State; but what one reſolves, the other does not always reinſtate, and it is only by analitic Artifices that it has been brought to any Degree of Perfection, and perhaps, in Time, will be rendered univerſal, and at the ſame Time more ſimple. What cannot we expect, in this Reſpect, from the united and conſtant Application of the firſt Mathematicians in *Europe*, who, not content to make uſe of this ſublime Art, in all their Diſcoveries, have perfected the Art itſelf, and continue ſo to do.

This Method has alſo the Advantage of Clearneſs and Evidence, and the Brevity that accompanies it every where does not require too ſtrong an Attention. A few Years moderate Study ſuffices to raiſe a Perſon, of ſome Talents, above theſe Geniuſes who were the Admiration of Antiquity; and we have ſeen a young Man of Sixteen, publiſh a Work, (*Traite' des Courbes à double Courbure par Clairaut*) that *Archimedes* would have wiſhed to have compoſed at the End of his Days. The Teacher of Mathematicks, therefore, ſhould be acquainted with the different Pieces upon the analitic Art, diſperſed in the Works of the moſt eminent Mathematicians, make a judicious Choice of the moſt general and eſſential Methods, and lead his Pupils, as it were, by the Hand, in the intricate Roads of the Labyrinth of Calculation; that by this Means Beginners, exempted from that cloſe Attention of Mind, which would give them a Diſtaſte for a Science they are deſirous to attain, and methodically brought acquainted with all its preliminary Principles, might be enabled in a ſhort Time, not only to underſtand the Writings of the moſt eminent Mathematicians, but, reflecting on their Method of Proceeding, to make Diſcoveries honourable to themſelves and uſeful to the Public.

Has the Advantage of Clearneſs, Evidence, and Brevity.

III.

Arithmetick comprehends the Art of Numbering and Algebra, conſequently is diſtinguiſhed into particular and univerſal Arithmetick, becauſe the Demonſtrations which are made by Algebra are general, and nothing can be proved by Numbers but by Induction. The Nature and Formation of Numbers are clearly ſtated, from whence the Manner of performing the principal Operations, as Addition, Subtraction, Multiplication and Diviſion are deduced. The Explication of the Signs and Symbols uſed in Algebra follow, and the Method of reducing, adding, ſubtracting, multiplying, dividing, algebraic Quantities ſimple and compound. This prepares the Way for the Theory of vulgar, algebraical, and decimal Fractions, where the Nature, Value, Man-

How Arithmetick numeral and ſpecious is treated.

Manner of comparing them, and their Operations, are carefully unfolded. The Composition and Resolution of Quantities comes after, including the Method of raising Quantities to any Power, extracting of Roots, the Manner of performing upon the Roots of imperfect Powers, radical or incommensurable Quantities, the various Operations of which they are susceptible. The Composition and Resolution of Quantities being finished, the Doctrine of Equations presents itself next, where

The Art of solving Equations.

their Genesis, the Nature and Number of their Roots, the different Reductions and Transformations that are in Use, the Manner of solving them, and the Rules imagined for this Purpose, such as Transposition, Multiplication, Division, Substitution, and the Extraction of their Roots, are accurately treated. After having considered Quantities in themselves, it remains to examine their Relations; this Doctrine comprehends arithmetical and geometrical Ratios, Proportions and Progressions: The Theory of Series follow, where their Formation, Methods for discovering their Convergency, or Divergency, the Operations of which they are susceptible, their Reversion, Summation, their Use in the Investigation of the Roots of Equations, Construction of Logarithms, *&c.* are

The Nature and Laws of Chance.

taught. In fine, the Art of Combinations, and its Application for determining the Degrees of Probability in civil, moral and political Enquiries are disclosed. *Ars cujus Usus et Necessitas ita universale est, ut sine illa, nec Sapientia Philosophi, nec Historici Exactitudo, nec Medici Dexteritas, aut Politici Prudentia, consistere queat. Omnis enim horum Labor in conjectando, et omnis Conjectura in Trutinandis Causarum Complexionibus aut Combinationibus versatur.*

IV.

Division of Geometry into Elementary, Transcendentel and Sublime.

GEOMETRY is divided into ELEMENTARY, TRANSCENDENTAL and SUBLIME.

To open to Youth an accurate and easy Method for acquiring a Knowledge of the Elements of Geometry, all the Propositions in *Euclid* (a) in the Order they are found in the best Editions, are retained with

(a) "Perspicuity in the Method and Form of Reasoning, is the peculiar Characteristic of "Euclid's Elements, not as interpolated by Campanus and Clavius, anatomised by Herigone and "Barrow, or depraved by Tacquet and Deschales, but of the Original, handed down to us by "Antiquity. His Demonstrations being conducted with the most express Design of reducing "the Principles assumed to the fewest Number, and most evident that might be, and in a Me-"thod the most natural, as it is the most conducive towards a just and complete Comprehension "of the Subject, by beginning with such Particulars as are most easily conceived, and flow most "readily from the Principles laid down; thence by gradually proceeding to such as are more ob-"scure, and require a longer Chain of Argument, and have therefore been regarded in all Ages, "as the most perfect in their Kind." Such is the Judgment of the ROYAL SOCIETY, who have express'd at the same Time their Dislike to the new modelled Elements that at present every where abound; and to the illiberal and mechanic Methods of teaching those most perfect Arts; which is to be hoped, will never be countenanced in the Public Schools in England and Scotland, &c.

all poſſible Attention, as alſo the Form, Connection and Accuracy of his Demonſtrations. The eſſential Parts of his Propoſitions being ſet forth with all the Clearneſs imaginable, the Senſe of his Reaſoning are explained and placed in ſo advantageous a Light, that the Eye the leaſt attentive may perceive them. To render theſe Elements ſtill more eaſy, the different Operations and Arguments eſſential to a good Demonſtration, are diſtinguiſhed in ſeveral ſeparate Articles; and as Beginners, in order to make a Progreſs in the Study of Mathematicks, ſhould apply themſelves chiefly to diſcover the Connection and Relation of the different Propoſitions, to form a juſt Idea of the Number and Qualities of the Arguments, which ſerve to eſtabliſh a new Truth; in fine, to diſcover all the intrinſical Parts of a Demonſtration, which it being impoſſible for them to do without knowing what enters into the Compoſition of a Theorem and Problem, Firſt, The Preparation and Demonſtration are diſtinguiſhed from each other. Secondly, The Propoſition being ſet down, what is ſuppoſed in this Propoſition is made known under the Title of Hypotheſis, and what is affirmed, under that of Theſis. Thirdly, All the Operations neceſſary to make known Truths, ſerve as a Proof to an unknown one, are ranged in ſeparate Articles. Fourthly, The Foundation of each Propoſition relative to the Figure, which forms the Minor of the Argument, are made known by Citations, and a marginal Citation recalls the Truths already demonſtrated, which is the Major: In one Word, nothing is omitted which may fix the Attention of Beginners, make them perceive the Chain, and teach them to follow the Thread of geometrical Reaſoning.

Methodical Order in which the Elements of Euclid are digeſted.

V.

Tranſcendental Geometry preſuppoſes the algebraic Calulation; it commences by the Solution of the Problems of the ſecond Degree by Means of the Right-line and Circle: This Theory produces important and curious Remarks upon the poſitive and negative Roots, upon the Poſition of the Lines which expreſs them, upon the different Solutions that a Problem is ſuſceptible of; from thence they paſs to the general Principles of the Application of Algebra to curve Lines, which conſiſt, Firſt, In explaining how the Relation between the Ordinates and Abciſſes of a Curve is repreſented by an Equation. Secondly, How by ſolving this Equation we diſcover the Courſe of the Curve, its different Branches, and its Aſymptots. Thirdly, The Manner of finding by the direct Method of Fluxions, the Tangents, the Points of Maxima, and Minima. Fourthly, How the Areas of Curves are found by the inverſe Method of Fluxions.

Tranſcendental Geometry.

In what it conſiſts.

The Conic Sections follow; the beſt Method of treating them is to conſider them as Lines of the ſecond Order, to divide them into their Species. When the moſt ſimple Equations of the Parabola,

Beſt Method of treating Conic Sections.

Ellipse, and Hyperbola are found, then it is easily shewn that these Curves are generated in the Cone. The Conic Sections are terminated by the Solution of the Problems of the third and fourth Degree, by the Means of these Curves.

The different Orders of Curves.

The Conic Sections being finished, they pass to Curves of a superior Order, beginning by the Theory of multiple Points, of Points of Inflection, Points of contrary Inflection, of Serpentment, &c. These Theories are founded partly upon the simple algebraic Calculation, and partly on the direct Method of Fluxions. Then they are brought acquainted with the Theory of the Evolute and Caustiques by Reflection and Refraction. They afterwards enter into a Detail of the Curves of different Orders, assigning their Classes, Species, and principal Properties, treating more amply of the best known, as the Folium, the Conchoid, the Cissoid, &c.

The mechanic Curves follow the geometrical ones, beginning by the exponential Curves, which are a mean Species between the geometrical Curves and the mechanical ones; afterwards having laid down the general Principles of the Construction of mechanic Curves, by the Means of their fluxional Equations, and the Quadrature of Curves, they enter into the Detail of the best known, as the Spiral, the Quadratrice, the Cycloid, the Trochoid, &c.

VI.

Sublime Geometry.

Sublime Geometry comprehends the inverse Method of Fluxions, and its Application to the Quadrature, and Rectification of Curves, the cubing of Solids, &c.

Its Division.

Fluxional Quantities, involve one or more variable Quantities; the natural Division therefore of the inverse Method of Fluxions is into the Method of finding the Fluents of fluxionary Quantities, containing one variable Quantity, or involving two or more variable Quantities; the Rule for finding the Fluents of fluxional Quantities of the most simple Form, is laid down, then applied to different Cases, which are more composed, and the Difficulties which some Times occur, and which embarrass Beginners, are solved.

What the first Part comprehends.

These Researches prepare the Way for finding the Fluents of fluxional Binomials, and Trinomials, rational Fractions, and such fluxional Quantities as can be reduced to the Form of rational Fractions; from thence they pass to the Method of finding the Fluents of such fluxional Quantities which suppose the Rectification of the Ellipse and Hyperbola, as well as the fluxional Quantities, whose Fluents depend on the Quadrature of the Curves of the third Order; in fine, the Researches which Mr. *Newton* has given in his Quadrature of Curves, relative to the Quadrature of Curves whose Equations are composed of three or four Terms;

and this first Part is terminated by the Methods of finding the Fluents of fluxional, logarithmetical, and exponential Quantities, and those which are affected with many Signs of Integration, and the various Methods of Approximation, for the Solution of Problems, which can be reduced to the Quadrature of Curves.

The second Part of the inverse Method of Fluxions, which treats of fluxional Quantities, including two or more variable Quantities, commences by shewing how to find the Fluents of such fluxional Quantities as require no previous Preparation; the Methods for knowing and distinguishing these Quantities or Equations; afterwards they pass to the Methods of finding the Fluents of fluxional Quantities, which have need of being prepared by some particular Operation, and as this Operation consists most commonly in separating the indeterminate Quantities, after being taught how to construct differential Equations, in which the indeterminate Quantities are separated, they enter into the Detail of the different Methods for separating the variable Quantities in a proposed Equation, either by Multiplication, Division, or Transformations, being shewed their Application, first to homogeneous Equations, and after being taught how to construct these Equations in all Cases, the Manner of reducing Equations to their Form is then explained. How the Method of indeterminate Co-efficients can be employed for finding the Fluents of fluxional Equations, including a certain Number of variable Quantities, and how by this Method, the Fluent can be determined by certain Conditions given of a fluxional Equation. Fluxional Quantities of different Orders follow; it is shewn, first, that fluxional Equations of the third Order, have three Fluents of the second Order, but the last Fluent of a fluxionary Equation of any Order is simple; then the various Methods imagined by the most eminent Mathematicians for finding these Fluents, supposing the Fluxion of any one variable Quantity constant, are explained, and the Whole, in fine, terminated by the Application of this Doctrine to the Quadrature and Rectification of Curves, Cubing of Solids, &c.

What the second Part comprehends.

VII.

Such is the Plan of a Course of pure Mathematicks traced by *Newton*, improved by *Cotes*, *Bernoully*, *Euler*, *Clairaut*, *D'Alembert*, *M'Laurin*, *Simpson*, *Fontain*, * &c. which serves as a Basis to the Instructions requisite to qualify Youth to appear with Dignity in the different Employments of Life, or to enable them in Time, to bring to Perfection the various Arts for which they are intended.

Conclusion.

* Quadratura curvarum, harmonia mensurarum, &c.

PLAN *of the System of the Physical and Moral World, including the Instructions relative to young Noblemen and Gentlemen of Fortune.*

PLAN *of the System of the Physical World.*

——— Nubem pellente mathesi,
Claustra patent cœli, rerumque immobilis ordo:
Jam superum penetrare domos, atque ardua cœli
Scandere, sublimis genii concessit acumen.

I.

Utility of the Study of the System of the World.

Is a Preservative against the Passions.

Leads to Virtue.

STUDY in general is necessary to Mankind, and essentially contributes to the Happiness of those who have experienced that active Curiosity which induceth them to penetrate the Wonders of Nature. It is, besides, a Preservative against the Disorders of the Passions; a kind of Study therefore which elevates the Mind, which applies it closely, consequently, which furnishes the most assured, arms against the Dangers we speak of, merits particular Distinction. "It is not "sufficient, says *Seneca*, to know what we owe to our Country, to our "Family, to our Friends, and to ourselves, if we have not Strength of "Mind to perform those Duties, it is not sufficient to establish Precepts, "we must remove Impediments, *ut ad præcepta quæ damus possit animus* "*ire, solvendus est*. (Epist. 95.) Nothing answers better this Purpose than the Application to the Study of the System of the World; the Wonders which are discovered captivate the Mind, and occupy it in a noble Manner; they elevate the Imagination, improve the Understanding, and satiate the Heart: The greatest Philosophers of Antiquity have been of this Opinion. *Pythagoras* was accustomed to say, that Men should have but two Studies, that of Nature, to enlighten their Understandings, and of Virtue to regulate their Hearts; in effect to become virtuous, not through Weakness but by Principle, we must be able to reflect and think closely; we must by Dint of Study be delivered from Prejudices which makes us err in our Judgments, and which are so many Impediments to the Progress of our Reason, and the Improvement of our Mind. *Plato* held the Study of Nature in the highest Esteem; he even goes so far as to say, that Eyes were given to Man to contemplate the Heavens: To which alludes the following Passage of *Ovid*.

Finxit in effigiem moderantum cuncta deorum,
Pronaque cum spectant animalia cetera terram,
Os homini sublime dedit, cœlumque tueri
Jussit, et erectos ad sidera tollere vultus.

II.

The Poets who have illustrated *Greece* and *Italy*, and whose Works are now sure of Immortality, were perfectly acquainted with the Heavens, and this Knowledge has been the Source of many Beauties in their Works: *Homer*, *Hesiod*, *Aratus*, among the *Greeks*: *Horace*, *Virgil*, *Ovid*, *Lucretius*, *Manilius*, *Lucan*, *Claudian*, among the *Latins*; make use of it in several Places, and have expressed a singular Admiration for this Science. Is celebrated by the Poets.

Ovid after having anounced in his Fasti, that he proposes celebrating the Principles on which the Division of the Roman Year is founded, enters on his Subject by the following pompous Elogium of the first Discoverers of the System of the World.

Felices animos, quibus haec cognoscere primis,
Inque domos superas scandere cura fuit,
Credibile est illos pariter vitiisque locisque,
Altius humanis exeruisse caput.
Non venus aut vinum sublimia pectora fregit,
Officiumve fori militiaeque labor,
Nec levis ambitio perfusaque gloria fuco,
Magnarumve fames sollicitavit opum.
Admovere oculis distantia sydera nostris,
Ætheraque ingenio supposuere sua.
Sic petitur caelum.

Claudian in the following Verses, celebrates *Archimedes* on his Invention of a Sphere admirably contrived to represent the celestial Motions.

Jupiter in parvo cum cerneret aethera vitro,
Risit, et ad superos talia dicta dedit:
Huccine mortalis progressa potentia curae!
Jam meus in fragili luditur orbe labor.
Jura poli, rerumque fidem legesque deorum
Ecce Syracusius transtulit Arte senex;
Inclusus Variis famulatur spiritus astris,
Et vivum certis motibus urget opus;
Percurrit proprium mentitus signifer, annum,
Et simulata novo Cynthia mense redit:
Jamque suum volvens audax industria mundum
Gaudet, et humana sidera mente regit.

Virgil ſeems deſirous of renouncing all other Study, to contemplate the Wonders of Nature.

Me vero primum dulces ante omnia muſæ,
Quarum ſacra fero ingenti percuſſus amore,
Accipiant, cœlique vias et ſydera monſtrent
Defectus ſolis varios, lunæque labores,
Unde tremor terris, qua vi maria alta tumeſcant
Objicibus ruptis, rurſuſque in ſeipſa reſidant,
Quid tantum oceano properent ſe tingere ſoles
Hyberni, vel quæ tardis mora noctibus obſtet . . . . .
Felix qui potuit rerum congnoſcere cauſas.
GEOR. II. 475.

La Fontaine imitates the Regrets of *Virgil* in a maſterly Manner, where he ſays,

Quand pourront les neuf ſæurs loin des cours et des villes,
M'occuper tout entier, et m' apprendre des cieux
Les divers mouvements inconnus à nos yeux,
Les noms et les vertues de ces clartes errantes.
Songe dun habitant du Mogol.

Voltaire, the firſt Poet of our Age, has teſtified in many Parts of his Works, his Taſte for Aſtronomy, and his Eſteem for Aſtronomers, whom he has celebrated in the fineſt Poetry. What he ſays of *Newton* is worthy of Attention.

Confidens du Tres Haut, Subſtances eternelles,
Qui parez de vos feux, qui couvrez des vos ailes,
Le trone ou votre maitre eſt aſſis parmis vous :
Parles ! du grand NEWTON *n'etiez vous point jaloux.*

To which we can only oppoſe what *Pope* has ſaid on the ſame Subject:

Nature and Nature's Laws lay hid in Night;
God ſaid, Let *Newton* be, and all was Light.

The great Geniuſes of every Species have been ſurprized at the Indifference which Men ſhew for the Spectacle of Nature. *Taſſo* puts Reflections in the Mouth of *Rinaldo*, which merit to be recited for the Inſtruction of thoſe to whom the ſame Reproach may be applied; it is at the Time when marching before Day towards *Mount Olivet*, he contemplates the Beauty of the Firmament.

Con gli occhi alzati contemplando intorno,
Quinci notturne e quindi matutine
Bellezze, incorruptibili e divine;
Fra se stesso pensava, o quanto belle
Luci, il tempio celeste in se raguna!
Ha il suo gran carro il de, l'aurata stelle
Spiega la notte, e l'argentata Luna;
Ma non è chi vagheggi o questa, o quelle;
E miriam noi torbida luce e bruna,
Ch'un girar d'occhi, un balenar di riso
Scopre in breve confin di fragil viso.
JERUS. Cant. xviii. St. 12, 13.

III.

The Knowledge of the System of the World has delivered us from the Apprehensions which Ignorance occasions; can we recal without Compassion, the Stupidity of those People, who believed that by making a great Noise when the Moon was eclipsed, this Goddess received Relief from her Sufferances, or that Eclipses were produced by Inchantments (a)?

Effects which the Ignorance of the System of the World has produced.

Cum frustra resonant Æra auxiliaria Lunæ. Met. iv. 333.
Cantus et e Curru Lunam deducere tentant,
Et faceret si non Æra repulsa sonent. Tib. El. 8.

The Knowledge of the System of the World has dissipated the Errors of Astrology, by whose foolish Predictions Mankind had been so long abused. The Adventure of 1186, should have covered with Shame the Astrologers of *Europe*; they were all, Christians, Jews and Arabians, united to anounce, seven Years before, by Letters published throughout *Europe*, a Conjunction of all the Planets, which would be attended with such terrible Ravages, that a general Dissolution of Nature was much to be dreaded, so that nothing less than the End of the World was expected: this Year notwithstanding passed as others. But a hundred Lies, each as well attested, would not be sufficient to wain ignorant and credulous Men from the Prejudices of their Infancy. It was necessary that a Spirit of Philosophy, and Research, should spread itself among Mankind, open their Understandings, unveil the Limits of Nature, and accustom them not to be terrified without Examination, and without Proof.

The Knowledge of the System of the World has dissipated the Errors of Astrology.

IV.

The Comets, as it is well known, were one of the great Objects of Terror which the Knowledge of the System of the World has, in fine,

(a) Seneca, Hipolit. 787. Tacit. Ann. Plutarch in Pericle, et de defectu Oraculorum.

removed. It is not without Concern we find such strange Prejudices in the finest Poem of the last Age, whereby they are transmitted to the latest Posterity.

Qual colle chiome sanguinose horende,
Splender cometa suol per l'aria adusta,
Che i regni muta, ei fieri morti adduce,
Ai purperei tiranni infausta luce. Jerus. Lib. 7. St. 52.

The Charms of Poetry are actually employed in a Manner more philosophical and useful, witness the following fine Passage.

Cometes que l'on craint à legal du tonnerre,
Cessez d'epouvanter les peuples de la terre;
Dans une Ellipse immense achevez votre cours,
Remontez, descendez pres de l'astre des jours;
Lancez vos feux, volez, et revenant sans cesse,
Des mondes epuisez ranimez la vieillesse.

Thus the profound Study of the System of the World has dissipated absurd Prejudices, and re-established human Reason in its inalienable Rights.

V.

The Knowledge of the System of the World useful in Geography and Navigation, and consequently of the greatest Importance to these Kingdoms.

To the Knowledge of the System of the World, are owing the Improvements in Cosmography, Geography, and Navigation; the Observation of the Height of the Pole, taught Men that the Earth was round, the Eclipses of the Moon taught how to determine the Longitudes of the different Countries of the World, or their mutual Distances from East to West. The Discovery of the Satellites of *Jupiter*, has contributed more effectually to improve geographical or marine Charts, than ten thousand Years Navigation; and when their Theory will be better known, the Method of Longitudes will be still more exact and more easy. The Extent of the *Mediterranean* was almost unknown in 1600, and To-Day, is as exactly determined as that of *England* or *Ireland*. By it the new World was discovered. *Christopher Columbus* had a more intimate Knowledge of the Sphere, than any Man of his Time, since it gave him that Certainty, and inspired him with that Confidence with which he directed his Course towards the West, certain to rejoin by the East the Continent of *Asia*, or to find a new one. And nothing seems to be wished for, to render Navigation more perfect and secure, but a Method for finding with Ease, the Longitude at Sea, which is now obtained by the Means of the Moon: And if the Navigators of this Kingdom were initiated in Astronomy, by able Teachers, as is practised in other Parts of *Europe*, their Estimation would approach within twenty Miles of the Truth, whilst in ordinary Voyages, the Uncertainty amounts to more than three hundred Leagues, by which the Lives and Fortunes of Thousands are endangered. The Utility therefore of the Marine to those Kingdoms, where Empire, Power, Commerce, even Peace and War, are decided at Sea, proves that of the Knowledge of the System of the World.

VI.

The actual State of the Laws, and of the ecclesiastical Administration, is essentially connected with the System of the World; St. *Augustine* recommended the Study of it particularly for this Reason; St. *Hyppolite* applied himself to it, as also many Fathers of the Church, notwithstanding our Kalendar was in such a State of Imperfection, that the *Jews* and *Turks* were astonished at our Ignorance. *Nicholas* V, *Leon* X, *&c.* had formed a Design of re-establishing Order in the Kalendar, but there were at that Time no Philosophers, whose Reputation merited sufficient Confidence. *Gregory* the XIIIth, governed at a Time when the Sciences began to be cultivated, and he alone had the Honour of this Reformation.

The Reformation of the Kalendar depended on it.

VII.

Agriculture borrowed formerly from the Motions of the celestial Bodies, its Rules and its Indications; *Job, Hesiod, Varro, Eudoxus, Aratus, Ovid, Pliny, Columella, Manilius,* furnish a thousand Proofs of it. The *Pleyades, Arcturus, Orion, Syrius,* gave to *Greece* and *Egypt* the Signal of the different Works; the rising of *Syrius* anounced to the *Greeks* the Harvest; to the *Egyptians* the overflowing of the *Nile*. The Kalendar answers this Purpose actualy.

Is useful in Agriculture

VIII.

Ancient Chronology deduces from the Knowledge and Calculation of Eclipses, the most fixed Points which can be found, and in remote Times we find but Obscurity. The *Chinese* Chronology is entirely founded upon Eclipses, and we would have no Uncertainty in the ancient History of Nations as to the Dates, if there were always Philosophers. (See the Art of verifying Dates.)

Is the Foundation of Chronology

IX.

It is from the System of the World we borrow the Division of Time, and the Art of regulating Clocks and Watches; and it may be said, that the Order and Multitude of our Affairs, our Duties, our Amusements, our Taste, for Exactness and Precision, our Habitudes have rendered this Measure of Time almost indispensable, and has placed it in the Number of the Necessaries of Life; if instead of Clocks and Watches, Meridians and solar Dials are traced, it is an Advantage that the Knowledge of the System of the World has procured us, Dial-

Furnishes the Means of measuring Time.

ling being the Application of spherical Trigonometry; a Projection of the Sphere upon a Plane, or a Section of a Cone, according to the Forms given to a Dial.

X.

Is useful in Physick.

The Knowledge of the Changes of the Air, Winds, Rain, dry Weather, Motions of the Thermometer, Barometer, have certainly an essential and immediate Relation with the Health of the human Body; the Knowledge of the System of the World will be of sensible Utility, when, by repeated Observations, the physical Influences of the Sun and Moon upon the Atmosphere, and the Revolutions which result will be discovered. *Galen* advises the Sick not to call to their Assistance Physicians, who are not acquainted with the Motions of the celestial Bodies, because Remedies given at unseasonable Times are useless or hurtful, and the ablest Physicians of our Days are convinced, that the Attractions which elevate the Waters of the Ocean twice a Day, influence the State of the Atmosphere, and that the Crisis and Paroxisms of Disorders correspond with the Situation of the Moon in respect of the Equator, Sysigies, and Apsides. See *Mead*, *Hofman*, &c.

XI.

Cultivated in all Ages by all the civilized Nations of the World.

Those Advantages which result from the Knowledge of the System of the World, has caused it to be cultivated and held in singular Esteem by all the civilized People of the Earth. The ancient Kings of *Persia*, and the Priests of *Egypt*, were always chosen amongst the most expert in this Science. The Kings of *Lacedemon* had always Philosophers in their Council. *Alexander* was always accompanied by them in his military Expeditions, and *Aristotle* gave him strict Charge to do nothing without their Advice. It is well known how much *Ptolemeus* the second King of *Egypt*, encouraged this Science; in his Time flourished *Hyparchus*, *Calimachus*, *Apollonius*, *Aratus*, *Bion*, *Theocrites*, *Conon*. *Julius Cæsar* was very curious in making Experiments and Observations, as it appears by the Discourse which *Lucan* makes him hold with *Achore* Priest of *Egypt*, at the Feast of *Cleopatra*.

——— *Media inter prelia semper*
Stellarum cœlique plagis superisque vacavi,
Nec meus Eudoxi vincetur fastibus annus. PHAR.

Has been the favorite Study of great Princes.

The Emperor *Tiberius* applied himself to the Study of the System of the World, as *Suetonius* relates; the Emperor *Claudius* foresaw there would be an Eclipse the Day of his Anniversary, and fearing it might occasion Commotions at *Rome*, he ordered an Advertisement to be published, in which he explains the Circumstances, and the Causes of this Phenomenon. It was cultivated particularly by the Emperors *Adrian* and *Severus*, by *Charlemagne*, by *Leon* V, Emperor of *Constantinople*, by *Alphonso* X, King of *Castile*, by *Frederick* II, Emperor of the * *West*, by *Calife Almamon*, the Prince *Ulubeigh*, and many other Monarchs of *Asia*.

Among the Heroes who alſo cultivated it, are reckoned *Mahomet* II, Conqueror of the *Greek* Empire; the Emperor *Charles* V, and *Lewis* XIV. In fine, the Eſtabliſhments of different Philoſophical Societies in *England*, *Scotland*, *France*, *Italy*, *Germany*, *Poland*, *Sweden*, *Ruſſia*, &c. have given the Monarchs, Nobility, and Gentry of thoſe Countries, a Taſte for the more refined Pleaſures attending the Study of the Sciences, and particularly of the Syſtem of the World, an Example worthy to be imitated by thoſe of this Kingdom.

XII.

Publick Schools eſtabliſhed in the different Parts of Europe for inſtructing young Noblemen and Gentlemen of Fortune in what regards the Syſtem of the World.

Beſides thoſe renowned Societies which have all contributed to the Progreſs of every Branch of human Knowledge, and particularly of the Syſtem of the World, there has been eſtabliſhed in the different Parts of *Europe* public Schools, conducted by Men of ſuperior Talents and Abilities, who make it their Buſineſs to guide and inſtruct the young Nobility and Gentry in this noble Science, and furniſh thoſe who diſcover ſingular Diſpoſitions with every Means of Improvement.

Foundation of Henry Saville.

An illuſtrious *Engliſhman*, *Henry Saville*, founded in the Univerſity of *Oxford* two Schools, which have been of vaſt Utility to *England*; the Maſters have been Men all eminent in this Science, *John Bainbridge* in 1619, *John Greaves* in 1643, *Seth Ward*, *Chriſtopher Wren*, *Edward Bernard* in 1673, *David Gregory* in 1691, *Briggs*, *Wallis*, and *J. Caſwell* in 1708, *Keill* in 1712, *Hornſby*, &c.

Foundations of Lownds and Lucas.

The Schools eſtabliſhed at *Cambridge*, among whoſe Maſters were *Barrow*, *Newton*, *Cotes*, *Wiſton*, *Smyth*, and *Long*, all celebrated Aſtronomers.

College of Greſham.

The School of *Greſham* at *Biſhops-Gate* in *London*, which has eſſentially contributed to the Progreſs of Aſtronomy; among the Maſters of this School were Doctor *Hook*, and other eminent Men.

Mathematical School of Chriſt's Hoſpital.

The Royal mathematical School at *Chriſt's-Hoſpital*, where *Hodgſon*, *Robertſon*, &c. have bred up a great Number of expert Navigators and Aſtronomers.

Mathematical Schools in Scotland.

The Schools of *Edinburgh*, *Glaſgow*, and *Aberdeen*, are known all over *Europe*; the Nobility, and Gentlemen of Fortune of *Scotland*, ſuperintending them, and taking every Method of encouraging both Maſters and Students to Aſſiduity and Attention, to go through their reſpective Taſks with Alacrity and Spirit; the Names of *Gregory*, *Mc Laurin*, *Stuart*, *Simpſon*, &c. the famous Maſters, will never be forgotten.

* He ordered the Works of Ptolemey to be tranſlated into Latin, and publickly to be taught at Naples.

The Royal College.

The Royal School of *France*, founded by *Francis* I, has essentially contributed to the Progress of the Knowledge of the System of the World. *Orance*, *Finé*, *Stadius*, *Morin*, *Gassendi*, *de la Hire*, *de Lisle*, who were successively Masters of it, have been celebrated Astronomers, &c.

XIII.

Observatories and Schools of Experimental Philosophy.

Experiments and Observations are the Foundation of all real Knowledge, those which serve as a Basis to the Discoveries relative to the System of the World, are made and learned in Experimental Schools and Observatories: The first Observatory of any Celebrity, was built by *William* V, Landgrave of *Hesse*, where he collected all the Instruments, Machines, Models, &c. which were known in his Time, and put it under the Direction of *Rothman* and *Byrgius*, the first an Astronomer, the second an expert Instrument-Maker: The Duke of *Broglio*, General of the *French* Army, having rendered himself Master of *Cassel* in 1760, took a Copy of the Observations and Experiments made in this Observatory, and deposited it in the Library of the Academy.

Of Cassel.

Of Uranibourg.

Frederick I. King of *Denmark*, being informed of the singular Merit of *Ticho Brahe*, granted him the Island of *Venusia*, opposite *Copenhagen*, and built for him the Castle of *Uranibourgh*, furnished it with the largest, and the most perfect Instruments, and gave Pensions to a Number of Observers, Calculators, and Experiment-Makers, to assist him, which enabled him in the Space of 16 Years, to lay the Foundation of the System of the World, in a Manner more stable, than was ever before effected. The most eminent Men took Pleasure in visiting this incomparable Philosopher: The King of *Scotland* going to espouse the Princess *Anne*, Sister of the King of *Denmark*, passed into the Island of *Venusia* with all his Court, and was so charmed at the Operations and Success of *Tycho*, that he composed his Elogium in *Latin* Poetry: So much Merit raised him Enemies, and the Death of King *Frederick* II, furnished them the Means of succeeding in their Machinations. A Minister called *Walchendorp*, (whose Name should be devoted to the Execration of the Learned of all Ages) deprived him of his Island of *Venusia*, and forbad him to continue at *Copenhagen* his Experiments and Observations.

XIV.

Of Dantzick.

The first Observatory of the last Age, was that of *Hevelius*, established at *Dantzick*; it is described in his great Work, intitled, *Machina Celestis*.

Of Copenhagen.

The Astronomical Tower of *Copenhagen* was finished in 1656, built by *Christian* IV, at the Solicitation of *Longomontanus*.

Of Pekin.

There has been an Experimental School and Observatory at *Pekin* these 400 Years, built on the Walls of the City: Father *Verbiest* being made President of the Tribunal of Mathematicks in 1669, obtained of the Emperor *Cam-by*, that all the *European* Instruments, Machines, Models, &c. should be added to those with which it was already furnished. (See the Description of *China* by *Dubald.)* There has been made there a vast Collection of useful Experiments and Observations, a Copy of which is deposited in the *French* Academy.

XV.

The Royal Observatory of *England* was built by *Charles* II. under the Direction of Sir *J. Moore*, four Miles from *London*, to the Eastward upon a high Hill: It will be for ever famous by the immortal Labours of *Flamstead, Halley*, and *Bradley*; *Flamstead* was put in Possession of this Observatory in 1676, where, during the Space of 33 Years, he made a prodigious Number of Observations contained in his History of the Heavens: *Halley* succeeded him, and was, without Doubt, the greatest Astronomer *England* produced; at the Age of Twenty he went to the Island of *St. Helen*, to form a Catalogue of the Southern Stars, which he published in 1679; then he went to *Dantzick* to confer with *Hevelius*, he travelled also through *Italy* and *France* for his Improvement; in 1683 he published his Theory of the Variation of the Magnetic Needle; in 1686 he superintended the Impression of the *Principia Mathematica Philosophiæ Naturalis*, which its immortal Author could not resolve with himself to publish. The same Year he published his History of the Trade Winds; in 1698 he received the Command of a Vessel to traverse the Atalantic Ocean, and visit the *English* Settlements, in order to discover whether the Variation of the Magnetic Needle, found by Experiment, agreed with his Theory, and to attempt new Discoveries; he advanced as far as 52 Degrees South Latitude, where the Ice impeded his further Progress; he visited the Coast of *Brasil*, the *Canaries*, the Islands of *Cape Verde, Barbadoes*, &c. and found every where the Variation of the Compass comformable to his Theory; in 1701 he was commissioned to traverse the *English* Channel, to observe the Tides, and to take a Survey of the Coasts; in 1708 he visited the Ports of *Trieste* and *Boccari* in the Gulph of *Venice*, and repaired the first, accompanied by the chief Ingineer of the Emperor; he published in 1705 the Return of the Comets of which he was the first Discoverer; and we have seen in 1759 the Accomplishment of his Prediction; in 1713 he was made Secretary of the Royal Society; he examined the different Methods for finding the Longitude at Sea, and proved that those which depend on the Observations of the Moon were the only practicable ones, and as those Methods required accurate Tables of this Planet, which did not differ from Observation more than two Minutes, he set about rectifying them, having discovered that to obtain this Point it was sufficient to determine, every Day during 18 Years, the Place of the Moon by Observation, and to know how much the Tables differed from it, the Errors every Period afterwards being the same, and returning in the same Order: It was

The Royal Observatory and Experimental School at Greenwich rendered famous by the Labours of Flamsted Halley and Bradley.

in 1722 that this courageous Aſtronomer, in the 65th Year of his Age, undertook this immenſe Work, and after having completed it, and publiſhed the Succeſs of his Labours for foretelling accurately the Moon's Place, and deducing the Longitude at Sea; we loſt this great Man the 25th of *January* 1742. *Bradley* ſucceeded him, who inriched Aſtronomy with his Diſcoveries and accurate Obſervations. He departed this Life the 13th of *July* 1762, in the 70th Year of his Age. *M. Maſkeline*, his Succeſſor, continues his Obſervations with the moſt active Zeal and happy Diſpoſitions.

Other Obſervatories and Experimental Schools in England.

The Royal Obſervatory not being ſufficient for all thoſe who purſue the Study of natural Philoſophy, there has been formed ſeveral Obſervatories in *London* and the different Parts of *England*, for Example, the Obſervatory of *Sherburn* near *Oxford*, where the Lord *Maclesfield*, late Preſident of the Royal Society, *M. Hornſby*, &c. have made Experiments and Obſervations for many Years.

Thoſe of Edinburgh, &c.

The Experimental School and Obſervatory of *Edinburgh*, built by the Subſcription of the Nobility and Gentry of that Kingdom, has been rendered famous by *M'Laurin*. The Royal Academy of Sciences deputed in 1747 the King's Aſtronomer, *Le Monier*, to obſerve there an annulary Eclipſe of the Sun.

XVI.

The Royal Obſervatory of Paris.

The Royal Obſervatory of *Paris*, the moſt ſumptuous Monument that ever was conſecrated to Aſtronomy, was built under the Direction of the great *Colbert*, immortal Protector of the Arts and Sciences. It is near 200 Feet in Front, 140 from North to South, and 100 in Height, the Vaults are near eighty Feet deep; there are alſo ſeveral others in *Paris*, and in other Parts of *France*, as that of *M. Lemonier* at the *Capuchines* of *St. Honore*, that of *M. Deliſle* at the Hotel *de Cluny*, that of *M. La Caille* at the College of *Maſarin*, that of the Palace of *Luxemburgh*, that of *M. de Fouchy* in *Rue des Poſtes*, and that of *M. Pingre* at *St. Genevieve*; the Obſervatory of *Marſeilles* which *F. Pezenas* has rendered famous, that of *Lyons* where *F. Beraud* made Experiments and Obſervations for a long Time, that of *Rowen* and *Toulouſe* from which *M. Bowin* and *M. Dulange*, *M. d' Auguier* ſend annually to the Academy a great Number of uſeful and curious Experiments and Obſervations; that of *Straſburgh* where *M. Brakenaſſer* has made ſome.

Other Obſervatories and Experimental Schools in France.

XVII.

Of Nuremberg in 1678.

The Senate of the Republic of *Nuremberg*, erected an Obſervatory in 1678, and put it under the Direction of *Geo. Chriſtopher Eimmart. Phil. Wurzelban* built another in 1692, deſcribed in his Book *Uranies Noricæ Baſis*. The Adminiſtrators of the Univerſity of *Leyden*, eſtabliſhed in 1690, an Experimental School and Obſervatory. *Frederick* I. King of *Pruſſia*, having founded in 1700, an Academy of Sciences at

Of Leiden in 1690.

Berlin, built an Experimental School, with an Obſervatory. The preſent King of *Pruſſia*, added a ſuperb Edifice, where the Academy actually holds its Aſſemblies. The Inſtitution of *Bologn*, a famous Academy, eſtabliſhed in 1709, by the Count of *Marſigli*, with the Permiſſion of *Clement* XI. has a fine Experimental School and Obſervatory, which *Manfredi* and *Zanotti* have rendered famous. There are four Experimental Schools, with Obſervatories, at *Rome*; that of *Blancbini*, that of the Convent of *Ara Cæli*, that of the Convent of *Minerva*, and that of *Trinite du Mont*. There is alſo one at *Genoa*, founded by the Marquis of *Salvagi*; one at *Florence*, which *Ximenes* has rendered famous; one at *Milan*, erected in the College of *Brera*, in 1713. The Superiors of the Univerſity of *Altorf*, in the Territory of *Nuremberg*, erected an Experimental School, and an Obſervatory, and furniſhed it with all the neceſſary Implements. In 1714, the Landgrave of *Heſſe*, *Charles* I. Heir of the States and Talents of the celebrated Landgrave we have already ſpoke of, built a new Experimental School and Obſervatory, and put it under the Direction of *Zumback*. In 1722, the King of *Portugal*, *John* V. erected an Experimental School and Obſervatory, in his Palace at *Liſbon*; there is alſo one in the College of St. *Antony*. The Experimental School and Obſervatory at *Peterſbourg*, is one of the moſt magnificent in *Europe*, it is ſituated in the Middle of the ſuperb Edifice of the Imperial Academy of *Peterſbourg*, it is compoſed of three Flights of Halls, adapted for making Experiments and Obſervations, and is 150 Feet high. In 1726, the Magiſtrates of the Republic of *Utrecht*, built an Experimental School, and an Obſervatory, in which the famous *Muſchembroek* made his Experiments and Obſervations. In 1739, the King of *Sweden* erected one at *Upſal*, and put it under the Direction of *Wargentin*. In 1740, the Prince of *Heſſe Darmſtad*, erected another at *Gieſſen*, near *Marborough*. There are two Experimental Schools and Obſervatories, at *Vienna*, where *F. Hell*, and *F. Liganig*, diſtinguiſh themſelves actually. There is one at *Tyrnaw* in *Hungary*; one in *Poland*, at *Wilna*, &c. &c.

Of Berlin in 1700. Of Italy in 1709 and 1713. Of Altorf in 1714. Of Liſbon in 1722. Of Peterſbourg in 1725. Of Utrecht in 1726. Of Upſal in 1739. Of Vienna. Of Wilna.

Such are the renowned Eſtabliſhments to which we are indebted for our Knowledge of the Syſtem of the World, and the Improvements it receives every Day; but there are a great many Branches, which require ſuch long Operations, and ſo great a Space of Time, that Poſterity will always have new Obſervations and Diſcoveries to make. *Multum egerunt qui ante nos fuerunt, ſed non peregerunt, multum adbuc reſtat. Operis multumque reſtabit; nec ulli nato poſt mille Sæcula præcludetur Occaſio aliquid adbuc adjiciendi.* (SENEC. Epiſ. 64.)

XVIII.

Thoſe great Examples of all the civilized Nations of the World, have at length brought the Noblemen and Gentlemen of this Country, to a true Senſe of the Importance of procuring to their Children, thoſe Means of Inſtruction, which may prevent their regretting in a more advanced Age, the miſ-ſpent Time of their Youth; which is the only Period of Life in which they can apply themſelves with Succeſs, to the Study of Nature: In this happy Age, when the Mind begins to think, and the Heart has no Paſſions voilent enough to trouble it. Shortly, the Paſſions and Pleaſures of their Age will engroſs their Time, and when the Fire of Youth is abated, and they have paid to the Tumult of the World the Tribute of their Age and Rank, Ambition will gain the Aſcendant. And though in a more advanced Age, which will not however be more ripe, they ſhould apply themſelves to the Study of the Sciences, their Minds having loſt that Flexibility which they had in their youthful Days, it is only by the Dint of Study, they can attain what they might acquire before with the greateſt Eaſe.

Publick School eſtabliſh'd in the City of Dublin for inſtructing Youth in evry Branch of pure and mixt Mathematicks purſuant to the Reſolution of the Noblemen and Gentlemen of Fortune of the Kingdom of Ireland the 4th of February 1768.

To improve therefore the Dawn of their Reaſon, to ſecure them from Ignorance, ſo common among People of Condition, which expoſes them daily to be ſcandalouſly impoſed upon, to accuſtom them early to the Habit of thinking and acting on rational Principles, a SCHOOL has been eſtabliſhed on the moſt approved PLAN, where, after having ſpent ſome Time in learning ELEMENTARY MATHEMATICKS, they are initiated in the Miſteries of SUBLIME GEOMETRY, and of the INFNITESIMAL CALCULATION; from thoſe abſtract Truths, they are led to the Diſcovery of the Phenomena of Nature, they are taught how to diſcern their Cauſes, and meaſure their Effects; from thence they are conducted as far as the Heavens, thoſe immenſe Globes which roll over our Heads with ſo much Majeſty, Variety and Harmony, letting themſelves be approached; they are taught how to obſerve their Motions, and inveſtigate the Laws according to which this material World, and all Things in it, are ſo wiſely framed, maintained and preſerved.

To relax their Minds after thoſe Speculations, they are brought back to Earth, where, free from all Spirit of Syſtem and Reſearch of Cauſes, they are taught how to contemplate the Wonders of Nature in detail. But as it preſents an immenſe Field, whoſe whole Extent the greateſt Genius cannot compaſs, and the Inquiries the moſt valuable, and the only worthy of a true Citizen are thoſe by which the Good of Society is promoted, they are confined particularly to the Study of what may contribute to the Perfection of uſeful Arts, ſuch as AGRICULTURE and COMMERCE, that thus initiated in the true Principles of the different Branches of Knowledge ſuitable to their Rank, having completed their Studies in this School, far from being obliged to forget what they have learned, as hitherto has been the Caſe, thev may be enabled to purſue with Succeſs, ſuch Inquiries as are beſt adapted to their Genius.

Progress of the Discoveries relative to the System of the World.

I.

THE first Views which Philosophers had of the System of the World, were no better than those of the Vulgar, being the immediate Suggestions of Sense; but they corrected them; thus the first System supposed the Earth to be an extended Plane, and the Center round which the Heavenly Bodies revolved.

First views of Philosophers of the System of the World

II.

The *Babylonians* from examining the Appearances of Sence were the first who discovered the Earth to be round, and the Sun to be the Center of the Universe *(a)* in these Points they were followed by *Pythagoras* and his School.

Discoveries of the Babylonians, and of Pythagoras.

III.

The true System of the World being discovered, it may appear surprizing that the Notion of the Earth's being the Center of the Celestial Motions should generally prevail: for tho' on a superficial Survey it seems to be recommended by its Simplicity, and to square exactly with the Appearances of Sence, yet on Examination it is found entirely insufficient to explain the Phenomena, and to account for the Heavenly Motions: This constrained *Ptolemy* and his followers to incumber and embarrass the Heavens with a Number of Circles and Epicycles equally arduous to be conceived and employed, for nothing so difficult as to substitute Error in the room of Truth.

Efforts that have been, made to maintain the Earth to be at rest. System of Ptolomy.

Probably the Influence of *Aristotle*'s Authority, whose Writings in *Ptolomy*'s Time were held in the highest Esteem, and considered as the Standard of Truth, lead this Philosopher into Error: But why did not *Aristotle* declare in favour of the true System, which he knew, since he endeavoured to overthrow it: this Reflection is sufficiently mortifying to the Pride of the Human Understanding, whatever was the Cause, thus much is certain, that the Ptolomaic System generally prevailed to the Time of *Copernicus*.

IV.

This great Man revived the ancient System of the *Babylonians*, and of *Pythagoras* which he confirmed by so many Arguments and Discoveries that Error could no longer maintain its Ground against the Evidence of Demonstration; thus the Sun was reinstated by *Copernicus* in the Center of the World, or to speak more exactly, in the Center of our Planetary System.

Copernicus revives the ancient System of Pythagoras.

(a) NEWTON in his Book DE SYSTEMATE MUNDI attributes this Opinion to Numa Pompilius, and says, (Page 1.) it was to represent the Sun in the Center of the Celestial Orbits that Numa caused a round Temple to be built in honour of Vesta, the Goddess of Fire in the Middle of which a perpetual Fire was preserved.

V.

System of Ticho Brahe

The Copernican System easily accounts for all the Celestial Phenomena, and tho' Observation and Argument are equally favourable to it, yet *Ticho-Brahe* an eminent Philosopher of that Age refused his assent to the Evidence of these Discoveries, whether deluded by an ill-formed Experiment, (b) or carried away by the Vanity of making a new System, he composed one which steers a middle Course between those of *Ptolomy* and *Copernicus*; he supposed the Earth to be at rest and the other Planets which move round the Sun, to revolve with him round the Earth, in the Space of 24 Hours; thus retaining the most exceptionable Part of Ptolomy's System, viz. the inconceivable Rapidity with which the *primum Mobile* is supposed to revolve, from whence we may learn into what dangerous Errors the misapplication of Genius may lead us.

The Discoveries relative to the System of the World, improved by Tycho.

Tho' *Tycho* erred in the Manner he made the Celestial Bodies move, yet he contributed very much to the Progress of the Discoveries relative to the System of the World, by the Accuracy and long Series of his Observations. He determined the Position of a vast Number of Stars to a Degree of exactness unknown before; he discovered the Refraction of the Atmosphere, by which the Celestial Phenomena are so much influenced; he was the first who proved from the Parallax of the Comets, that they ascend above the Moon; he was the first who observed what is called the *Moon's variation*; and in fine, it is from his Observations on the Motions of the Planets, that Kepler who resided with him, near Prague, during the last Years of his Life, deduced his admirable Theory of the Motions of the Heavenly Bodies.

VI.

How much remained to be discovered after Copernicus.

Copernicus undoubtedly rendered important Services to Human Reason by re-establishing the true System of the World: It was already a great point gained that Human Vanity condescended to place the Earth in the Number of the simple Planets; but much still remained to be discovered: neither the Forms of the Planetary Orbits, nor the Laws by which their Motions are regulated, were known; for these important Discoveries we are indebted to Kepler.

(b) It was objected to Copernicus, that the Motion of the Earth would produce Effects which did not take Place; that, for Example, if the Earth moved, a Stone dropp'd from the Top of a Tower, ought not to fall at the Foot of it, because the Earth moved during the Time of the Stone's descent, that notwithstanding it falls at the Foot of the Tower. COPERNICUS replied, that the Situation of the Earth with respect to Bodies that fall on its Surface was the same as that of a Ship in Motion, with respect to Bodies that are made to fall in it; he asserted, that a Stone let fall from the Top of the Mast of a Vessel in Motion, would fall at the Foot of it. This Experiment which is now incontestible was then ill-made, and was the Cause or the Pretext which made Ticho refuse his assent to the Discoveries of Copernicus.

This eminent Philosopher found out, that the Notion which generally prevailed before his time, that the Planets revolved in circular Orbits, was erroneous; and he discovered, by the means of Ticho's Observations, that the Planets move in Ellipses, the Sun residing in one of the Foci: and that they move over the different Parts of their Orbit, with different Velocities, so that the Area described by a Planet, that is, the Space included between the straight lines drawn from the Sun to any two Places of the Planet, is always proportional to the time which the Planet employs to pass from one to the other.

Discoveries of Kepler the elipticity of the orbits. the proportionality of the areas and the times.

Some years afterwards, comparing the Times of the Revolutions of the different Planets about the Sun, with their different Distances from him, he found that the Planets which are placed the farthest from the Sun to move slowest, and examining whether this Proportion was that of their Distances, he discovered after many Trials, in the Year 1618, that the Times of their Revolutions were as the Square Roots of the Cubes of their mean Distances from the Sun.

Relation which subsists between the periodic times and the distances.

VII.

Kepler not only discovered these two Laws, which retain his Name, and which regulate the Motions of all the Planets, and the Curve they describe, but had also some Notion of the Force which makes them describe this Curve; in the Preface to his Commentaries on the Planet Mars, we discover the first Hints of the attractive Power; he even goes so far as to say, that the Flux and Reflux of the Sea, arises from the gravity of the Waters towards the Moon: but he did not deduce from this Principle what might be expected from his Genius and indefatigable Industry. For in his Epitome of Astronomy(c) he proposes a physical Account of the planetary Motions from quite different Principles; and in this same Book of the Planet Mars, he supposes in the Planets a friendly and a hostile Hemisphere, that the Sun attracts the one and repels the other, the friendly Hemisphere being turned to the Sun in the Planets descent to its Perhihelium, and the Hostile in its Recess.

VIII.

The Attraction of the Celestial Bodies was suggested much more clearly by M. Hook, in his Treatise on the Motion of the Earth, printed in the Year 1674, twelve Years before the Principia appeared. *These are his Words*, Page 27, "I shall explain hereafter a System of the World, different in ma-
" ny Particulars from any yet known, answering in all Things to the com-
" mon Rules of Mechanical Motions. This depends on the three following
" Suppositions."

(c) See Gregory, Book 1, Page 69.

Singular anecdote concerning attraction.

" 1st That all celestial Bodies, whatever, have an Attraction, or gravitating " Power towards their own Centers, whereby they attract, not only their " own Parts and keep them from flying from them, as we may observe the " Earth to do, but that they do also attract all the other celestial Bodies that " are within the Sphere of their Activity; and consequently not only the " Sun and the Moon have an Influence upon the Body and Motion of the " Earth, and the Earth on the Sun and Moon, but also, that Mercury, Ve- " nus, Mars, Jupiter aud Saturn, by their attractive Powers, have a consi- " derable Influence upon the Motion of the Earth, as in the same Manner " the corresponding attractive Power of the Earth hath a considerable influ- " ence upon the Motion of the Planets."

" 2d That all Bodies whatever that are put into a direct and simple Motion, " will so continue to move forward in a streight Line, till they are by some " other effectual Power deflected and turned into a Motion, describing a Cir- " cle, an Ellipse, or some other more compounded Curve Line."

" 3d That these attractive Powers are so much the more powerful in ope- " rating, by how much the nearer the Body wrought upon is to their own " Center."

" These several Degrees I have not yet experimentally verified, but it is " a Notion which if fully prosecuted as it ought to be, will mightily assist the " Astronomer to reduce all the celestial Motions to a certain Rule, which I " doubt will never be done true without it. He that understands the Na- " ture of the circular Pendulum and circular Motion, will easily understand " the whole Ground of this Principle, and know where to find Directions " in Nature for the true stating thereof. This I only hint at present to such " as have a Capacity and Opportunity of prosecuting this Enquiry, &c."

IX.

We are not to imagine, that this Hint thrown out casually by *Hook*, detracts from the Glory of *Newton*, who even took Care to make Mention of it in his Book *de Systemate mundi* (d). the Example of *Hook* and *Kepler* makes us perceive the wide Difference between having a Notion of the Truth, and being able to establish it by irrefragable Demonstration; it also shews us how little the greatest Sagacity can penetrate into the Laws and Constitution of Nature, without the Aid and Direction of Geometry.

X.

Strange notions of Kepler.

Kepler, who made such important Discoveries, whilst he followed this unerring Guide, affords us a convincing Proof of the Errors into which the brightest Genius may be seduced, by indulging the pleasing Vanity of inventing Systems; who could believe, for Instance, that such a Man could

(d) page. 3 Edition of 1731.

adopt the wild Fancies and whimsical Reveries of the Pythagoreans, concerning Numbers: yet he thought that the Number and Interval of the primary Planets bore some Relation to the five regular Solids of Elementary Geometry (e), imagining that a Cube inscribed in the Sphere of Saturn would touch the Orb of Jupiter with its six Planes, and that the other four regular Solids, in like Manner, fitted the Intervals that are betwixt the Spheres of the other Planets: afterwards on discovering that this Hypothesis did not square with the Distances of the Planets, he fancied that the celestial Motions are performed in Proportions corresponding with those, according to which a Cord is divided in order to produce the Tones which compose the Octave in Music (f);

Kepler having sent to Ticho a Copy of the Work, in which he attempted to establish those Reveries. Ticho recommended to him, in his Answer(g), to relinquish all Speculations deduced from first Principles, all reasoning a Priori, and rather study to establish his Researches on the sure and firm Ground of Observation. Wise counsel of Ticho to Kepler.

The great *Hughens* himself (h) believed that the fourth Satellite of Saturn, which retains his Name, making up with our Moon and the four Satellites of Jupiter six secundary Planets, the Number of the Planets was complete, and it was labour lost to attempt to discover any more, because the principle Planets are also six in Number, and the Number Six is a *perfect* Number, as being equal to the Sum of its aliquot Parts, 1, 2 and 3. Whimsical notion of Hughens.

XI.

It was by never deviating from the most profound Geometry, that *Newton* discovered the Proportion in which Gravity acts, and that in his Hands the Principle of which *Kepler* and *Hook* had only some faint Notion, became the Source of the most admirable and unhoped for Discoveries.

One of the Causes which prevented Kepler from applying the Principles of Attraction to explain the Phœnomena of Nature with Success, was his Ignorance of the true Laws of Motion. Newton had the Advantage over Kepler of profiting of the Laws of Motion, established by Hughens, which he has carried to so great a Height in his Mathematical Principles of Natural Philosophy. Advantages of Newton over Kepler in his time, the theory of motion was better understood.

XII.

The Mathematical Principles of Natural Philosophy consist of three Books, besides the Definitions, the Laws of Motion and their Corollaries; the first Book is composed of fourteen Sections, the second contains nine, Analysis of the principia.

(e) Mysterium Cosmographicum.

(f) Mysterium Cosmographicum.

(g) Uti suspensis speculationibus a priori descendentibus animum potius ad observationes quas simul offerebat considerandas adjicerem (it is Kepler who speaks) notæ in secundam editionem mysterii cosmographici and the third, the Application of the two first to the Explication of the Phœnomena of the System of the World.

XIII.

Definitions.

The Principia commence with eight Definitions; *Newton* shews in the two first how the *Quantity of Matter and the Quantity of Motion* should be measured; he defines in the third, the *Vis intertiæ*, or resisting Force, which all Matter is endued with; he explains in the fourth what is to be understood by *active Force*; he defines in the fifth *the centripetal Force*, and lays down in the sixth, seventh and eighth the Manner of measuring its *absolute Quantity*, *its motrix Quantity*, and *its accelarative Quantity*; afterwards he establishes the three following Laws of Motion.

XIV.

Laws of motion.

1st. That a Body always perseveres of itself, in its State of Rest, or of uniform Motion in a straight Line.

2d. That the change of Motion, is proportional to the Force impressed, and is produced in the straight Line in which that Force acts.

3d. That Action and Reaction are always equal with opposite Directions.

XV.

First book, the 1st section contains the principles of infinitessimal geometry

Newton having explained those Laws, and deduced from them several Corollaries, commences his first Book with eleven Lemmas, which compose the first Section, he unfolds in those eleven Lemmas his Method of *Prime and ultimate Ratios*; this Method is the Foundation of infinitessimal Geometry, and by its Assistance, this Geometry is rendered as certain as that of the Ancients.

the other 13 general propositions on the motion of bodies.

The thirteen other Sections of the first Book of the Principia, are employed in demonstrating general Propositions on the Motion of Bodies, Abstracting from the Species of these Bodies and of the Medium in which they move.

It is in this first Book that *Newton* unfolds all his Theory of the gravitation of the celestial Bodies, but does not confine himself to examine the Questions relative to it; he has rendered his Solutions general, and has given a great Number of Applications of those Solutions.

XVI.

Second book it treats of the motion of bodies in resisting mediums. intended to overthrow the vortices of Descartes

In the second Book, *Newton* treats of the Motion of Bodies in resisting Mediums.

This second Book which contains a very profound Theory of Fluids, and of the Motion of Bodies which are immersed in them, seems to have been destined to over throw the System of Vortices, though it is only in the Scholium of the last Proposition, that *Newton* openly attacks *Descartes*, and proves that the celestial Motions are not produced by Vortices.

XVII.

In fine, the third Book of the Principia treats of the Syſtem of the World; in this Book, *Newton* applies the Propoſitions of the two firſt: in this Application we ſhall endeavour to follow *Newton*, and point out the Connection of his Principles, and ſhew how naturally they unravel the Mechaniſm of the Univerſe.

Third book. it treats of the ſyſtem of the world.

XVIII.

The Term, Attraction, I employ in the Senſe in which *Newton* has defined it, underſtanding by it nothing more than that Force, by which Bodies tend towards a Center, without pretending to aſſign the Cauſe of this Tendency.

What is meant by the word attraction.

Principal Phenomena of the Syſtem of the World.

I.

THE Knowledge of the Diſpoſition and Motions of the Celeſtial Bodies muſt precede a juſt Enquiry into their Cauſes. It will not therefore appear unneceſſary to prepare our Readers by a ſuccinct deſcription of our planetary Syſtem for our Account of the manner *Newton* demonſtrates the powers which govern the Celeſtial Motions and produce their mutual Influences. This Deſcription muſt neceſſarily comprize ſome Truths, diſcovered by that illuſtrious Philoſopher, the Manner he attained them will be deſcribed in the Sequel.

II.

The celeſtial Bodies that compoſe our planetary Syſtem, are divided into *Primary Planets*, that is, thoſe which revolve round the Sun, as their *Center* and *Secondary Planets*, otherwiſe, called *Satellites*, which revolve round their reſpective Primaries as Centers: There are ſix Primary Planets whoſe Names and Characters are as follows,

Firſt diviſion of the celeſtial bodies of our planetary ſyſtem into principal and ſecondary planets.

☿ *Mercury*,
♀ *Venus*,
♁ *The Earth*,
♂ *Mars*,
♃ *Jupiter*,
♄ *Saturn*.

Names and characters of the principal planets

In enumerating the Primary Planets, we follow the Order of their Diſtances from the Sun, commencing with thoſe which are neareſt to him.

Which are the planets that have ſatellites.

The Earth, Jupiter, and Saturn, are the only Planets which have been diſcovered to be attended by Secondaries: The Earth has only one Satellite, namely, the Moon; Jupiter has four, and Saturn five, excluſive of his Ring, ſo that our Planetary Syſtem is compoſed of eighteen celeſtial Bodies, including the Sun and the Ring of Saturn.

enumeration of the celeſtial bodies of our planetary ſyſtem.

III.

The Primary Planets are divided into *ſuperior* and *inferior Planets*, the *inferior Planets* are thoſe which are nearer the Sun than the Earth is; theſe

Second diviſion of the planets into ſuperior and inferior, which are the inferior planets and what is their arraingement.

are *Mercury* and *Venus*; the Orbit (a) of Venus includes that of Mercury and also the Sun, and the Orbit of the Earth is exterior to those of Mercury and of Venus, and incloses them and the Sun also.

how this order has been discovered.

This order is discovered, by Venus and Mercury sometimes appearing to be interposed between the Sun and us, which could never happen unless these Planets revolved nearer the Sun than the Earth, and it is very perceivable that Venus recedes farther from the Sun than Mercury does, and consequently its Orbit includes that of Mercury.

which are the superior planets and what is their arraingement.

The superior Planets are those which are more distant from the Sun than the Earth is, these are three in Number, *Mars*, *Jupiter* and *Saturn*; we know that the Orbits of these Planets inclose the Orbit of the Earth, because the Earth is sometimes interposed between them and the Sun.

The Orbit of Mars incloses that of the Earth, the Orbit of Jupiter that of Mars, and the Orbit of Saturn that of Jupiter; so that of the three superior Planets Saturn is the remotest from the Earth, and Mars is the nearest.

how it has been discovered.

This Arraingement is discovered by those Planets which are nearer the Earth (b) sometimes coming between the Eye and the Remoter, and intercepting them from our View.

IV.

The planets are opaque Bodies.

All the Planets are opaque Bodies; this appears of Venus and Mercury, because when they pass between us and the Sun, they resemble black Spots traversing his Body, aud assume all those various Appearances which are called Phases, that is, the Quantity of their Illumination depends on their Position in respect to the Sun and us.

The planets are spherical

For the same Reason, since Mars has *Phases* we infer his Opacity, and the same Conclusion is extended to Jupiter and Saturn, because their Satelites do not appear illuminated while their Primaries are between them, and the Sun which proves that that Hemisphere of those Planets which is turned from the Sun is opaque: Lastly, we know that the Planets are spherical Bodies, because, whatever be their Position, in respect of us, their Surface always appears to be terminated by a Curve.

We conclude that the Earth is spherical, because in Eclipses her Shadow, always appears to be bounded by a Curve, and when a Ship sails out of sight, it gradually disappears, first the Hulk, next the Sails, and lastly the Mast, sinking to the Eye and vanishing, and moreover, it the Earth was an extended Plane, Navigation would have discovered its Limits and Boundaries the contrary of which is proved by many Voyagers, such as Drake, Forbish, and Lord Anson, who have sailed round the World.

(a) Orbit is the Curve which a Planet describes in revolving round the Body which serves it as a Center.

(b) Wolf's Elements of Astronomy.

V.

All that we know therefore concerning the primary Planets, proves that they are opaque, solid and spherical Bodies.

The Sun appears to be a Body of a Nature entirely different from the Planets; we know not whether the Parts of which it is composed be solid or fluid; all that we can discover is, that those Parts emit light & heat, and burn when condensed and assembled in sufficient Quantity; hence we may probably conclude, that the Sun is a Globe of Fire resembling terrestrial Fire, since the Effects produced by this and the solar Rays, are exactly the same.

The planets appear to be all of the same nature,

It is probable that the Sun is a globe of fire.

VI.

All the celestial Bodies compleat their Revolutions round the Sun in Ellipses (c), more or less excentic, the Sun residing in the common Focus of all their Orbits; hence the Planets in their Revolutions sometimes approach nearer, and sometimes recede farther from the Sun; a right Line passing through the Sun and terminating in the two Points of the Orbit of a Planet, which are nearest and remotest from the Sun, is called the *Line of the Apsides*, the Point of the Orbit which is nearest the Sun is called the *Perihelium*; and the Point of the Orbit which is remotest from the Sun is called the *Aphelium*.

In what curve the celestial bodies revolve about the sun.

what is the line of the apsides the aphelium and perihelium.

The primary Planets in their Revolutions round the Sun, carry also their Satellites, which at the same Time revolve round them as their Centers.

All these Revolutions are performed in a direction from West to East (d).

In what direction the planets revolve.

There appear from Time to Time Stars that move in all Directions, and with astonishing Rapidity, when they are sufficiently near to be visible, these are called Comets.

Of the comets.

We have not yet collected Observations sufficient to determine their Number, all that we know concerning them, and 'tis but lately that the Discovery has been made; is that they are Planets revolving round the Sun like the other Bodies of our System, and that they describe Ellipses so very excentric as to be visible only while they are moving over a very small Part of their Orbit.

The comets are planets.

VII.

All the Planets in their Revolutions round the Sun, observe the two Laws of *Kepler*.

Observations evince, that the Comets observe the first of these Laws, namely, that which makes the celestial Bodies (e) describe equal Areas in e-

The planets and comets observe the laws of Kepler

(c) A Species of Curve, which is the same with what is commonly called an OVAL, the foci are the points in which Gardeners fix their pegs in order to trace this curve of which they make a frequent use.

(d) The Spectator is supposed to be placed on the Earth.

(e) By the Word Area, in general is understood a Surface, here it signifies the Space included between two Lines drawn from the Center to two Points where the Planet is found;

qual Times; and in the sequel it will be shewn, that all the Observations that have hitherto been made, concerning their Motions, render it highly probable that they are regulated by the second Law, that is, that their periodic (f) Times are in the sesquiplicate ratio of their mean Distances.

VIII.

Proofs of the motion of the earth

Admitting these two Laws of *Kepler*, confirmed by all astronomical Observations, from them we may derive several convincing Proofs of the Motion of the Earth, a Point which had been so long contested; for supposing the Earth to be the Center of the Celestial Motions, these two Laws are not observed; the Planets do not describe Areas proportional to the Times around the Earth, and the periodic Times of the Sun and the Moon, for instance, round this Planet, are not as the Square Roots of the Cubes of their mean Distances from the Earth; for the periodic Time of the Sun around the Earth, being nearly thirteen Times greater than that of the Moon, its Distance from the Earth would be, according to Kepler's Rule, between five and six Times greater than that of the Moon, but Observations demonstrate, that this Distance is about four-hundred Times greater, therefore, admitting the Laws of Kepler, the Earth is not the Center of the celestial Revolutions.

The centripetal Force(g) which Newton has demonstrated to be the Cause of the Revolutions of the Planets renders the Curve they describe around their Center concave (h) towards it, since this Force is exerted in drawing them off from the tangent (i); now the Orbits of Mercury and Venus, in some Parts, are convex to the Earth; of consequence, the inferior Planets do not revolve round the Earth.

The same may easily be proved of the superior Planets; for these are

those Areas are proportional to the Times, that is, they are greater or less, as the Times in which they are described are longer or shorter.

(f) Periodical Time is the Time that a Planet employs in compleating its Revolution in its Orbit. An Example, of Sesquiplicate Ratio will render it more intelligible than a Definition; Suppose then the mean Distance of Mercury from the Sun, to be 4, that of Venus 9, the periodical Time of Mercury 40 Days, and let the periodical Time of Venus be required, cubing the two first Numbers 4 and 9, there will result 64 and 729; afterwards extracting the Square-Roots of these two Numbers, there will be found 8 for that of the first, and 27 for that of the second, and by the Rule of three you will have 8 : 27 : : 40 : 135, That is the Square-Root of the Cube of the mean Distance of Mercury from the Sun, is to the Square Root of the Cube of the mean Distance of Venus from the Sun, as the periodic Time of Mercury round the Sun is to the periodic Time sought of Venus round the Sun, which is found to be 135, according to the Suppositions which have been made, and this is what is called Sesquiplicate Ratio.

(g) The Word CENTRIPETAL FORCE carries its Definition along with it, for it signifies no more than that Force which makes a Body tend to a Center.

(h) The two Sides of the Crystal of a Watch may serve to explain those Words CONCAVE and CONVEX; the Side exterior to the Watch is CONVEX, and that which is on the Side of the Dial-plate is CONCAVE.

(i) A Tangent is a right Line which touches a Curve, without cutting it.

sometimes observed to be *direct* (k), sometimes *stationary*, and afterwards *retrograde*; all those Irregularities are only apparent and would vanish if the Earth was the Center around which the heavenly Bodies revolved, for none of these Appearances would be observed by a Spectator placed in the Sun, since they result only from the Motion of the Earth in its Orbit combined with the Motion of those planets in their respective Orbits; from hence we may see the Reason why the Sun and the Moon are the only heavenly Bodies that appear always direct; for as the Sun describes no Orbit, its Motion cannot be combined with that of the Earth, and as the Earth is the Center of the Moon's Motion, to us she should always appear direct; as would all the Planets to a Spectator placed in the Sun.

Objection made to Copernicus taken from the planet venus his answer to this objection

When *Copernicus* first proposed his System, an Objection was raised against it, taken from the Planet Venus by some who alledged, that if that Planet revolved round the Sun she should appear to have Phases as the Moon, to which *Copernicus* answered, if your Eyes were sufficiently acute you would actually observe such Phases, and that perhaps in Time some Art may be discovered so to improve and enlarge the visual Powers, as to render those Phases perceivable: This Prediction of Copernicus was first verified by *Galileo*, and every Discovery that has been made since on the Motion of the heavenly Bodies has confirmed it.

IX.

Under what angle the Orbits intersect

The Planes (l) of the Orbits of all the Planets intersect in right Lines passing through the center of the Sun, so that a Spectator placed in the Center of the Sun would be in the Planes of all those Orbits.

What is understood by the nodes & the line of the nodes of an orbit

The Right Line, which is the common Section of the Plane of each Orbit, with the Plane of the Ecliptic, that is, the Plane in which the Earth moves, is called the *Line* of the nodes of that Orbit, and the extreme Points of this Section, are called the *Nodes* of that Orbit.

Inclination of the Orbits to the Ecliptic

The Quantities of the Inclination of the Planes of the different Orbits, with the Plane of the Ecliptic, are as follows, the Plane of the Orbit of Saturn is inclined to the Plane of the Ecliptic in an Angle of $2^{d}\frac{1}{2}$, that of Jupiter $1^{d}\frac{1}{3}$, that of Mars in an angle somewhat less than 2^{d}, that of Venus somewhat more than $3^{d}\frac{1}{3}$, and that of Mercury about 7^{d}.

X.

The Orbits of the primary Planets being Ellipses, having the Sun in one of their Foci, all these Orbits are consequently excentric, and are more or less so, according to the Distance between their Centers and the Point where the Sun is placed.

(k) A Planet is said to be DIRECT when it appears to move according to the Order of the Signs, that is, from Aries to Taurus, from Taurus to Gemini, &c. which is also said to move in consequentia, it is stationary when it appears to correspond for some Time to the same Points of the Heavens, and in fine it is RETROGRADE when it appears to move contrary to the Order of the Signs, which is also said to move in Antecedentia, that is, from Gemini to Taurus, from Taurus to Aries, &c.

(l) The plane of the Orbit of a Planet is the surface on which it is supposed to move.

excentricity of the planets in semi diameters of the earth

The excentricity of all those Orbits have been measured, and have been found as follows, in decimal Parts of the semidiameter of the Earth's orbit, supposed to be divided into 100,000 Parts,

That of Saturn,	54207 Parts.
That of Jupiter,	25058
That of Mars,	14115
That of the Earth,	4692
That of Venus,	500
And in fine, that of Mercury,	8149 Parts.

excentricity of the planets in semi diameters of their great Orbit

The excentricity of the Planets measured in decimal Parts of the semidiameter of their Orbits, supposed to be divided into 100,000 Parts, are as follows,

That of Saturn,	5683 Parts.
That of Jupiter,	4822
That of Mars,	9263
That of the Earth,	5700
That of Venus,	694
That of Mercury,	21000 Parts

Whence it appears that the Excentricity of Mercury is almost insensible.

XI.

Proportion of the diameters of the planets.

The Planets are of different Magnitudes; of the Earth alone we know the absolute Diameter, because this Planet is the only one whose Circumference admits of actual Mensuration, but the relative Magnitudes of the Diameters of the other Planets have been discovered, and the Diameter of the Sun being taken for a common Measure, and supposed to be divided into 1000 Parts:

That of Saturn is	137
That of Jupiter	181
That of Mars	6
That of the Earth	7
That of Venus	12
That of Mercury	4

Hence we see that Mercury is the least of all the Planets, for Spheres are as the Cubes of their Diameters.

XII.

Distances of the planets from the sun

The Planets are placed at different Distances from the Sun, taking the Distance of the Earth from the Sun for a common Measure, and supposing it divided into 100,000 Parts, the mean Distances of the Planets are as follows,

That of Mercury is	38710
That of Venus	7233
That of the Earth	10000
That of Mars	152369
That of Jupiter	520110
In fine, that of Saturn	953800

The mean Distances of the Sun and the Planets from the Earth, have also been computed in Semidiameters of the Earth; the mean Distances of the Sun, Mercury and Venus from the Earth are nearly equal, and amount to 22000 Semidiameters of the Earth, that of Mars is 33500, that of Jupiter 115000, and that of Saturn 210000.

Distances of the planets from the earth

XIII.

The Times of the Revolutions of the Planets round the Sun, are less in Proportion of their Proximity, thus Mercury the nearest revolves in 87 Days, Venus next in Order revolves in 224, the Earth in 365, Mars in 686, Jupiter in 4332, and Saturn the remotest from the Sun in 10759, the whole in round Numbers.

Periodic times of the planets round the sun

XIV.

The Planets, besides their Motion of Translation round the Sun, have another Motion of Rotation round their Axis, called their *Diurnal Revolution.*

Rotation of the planets.

We only know the diurnal Revolution of the Sun and of four Planets, namely of the Earth, Mars, Jupiter and Venus; this Revolution has been discovered by Means of the Spots observed on their Discs, (m) and which successively appear and vanish; Mars, Jupiter and Venus having Spots on their Surface, by the regular Return and successive Disappearance of the same Spots it has been found, that these Planets turn round their Axes, and in what Time they compleat their Rotation; thus it has been observed, that Mars makes his Rotation in 23h. 20m. and Jupiter in 9h. 56m.

Means employed to discover it

In what planets this rotation has been perceived

Astronomers are not agreed about the Time in which Venus revolves round its Axis; most suppose the Time of rotation to be about 23 h. But Sign. Bianchini who observed the Motions of this Planet with particular Attention, thinks she employs 24 Days in turning round; but as he was compelled to remove his Instruments during the Time he was observing, an House having intercepted Venus from his View; and as he lost an Hour in this Operation, 'tis probable that the Spot he was observing during this Interval changed its Appearance; however this be his authority in Astronomical Matters deserves we should suspend our Judgment till more accurate Observations have decided the Point.

Incertitude with regard to the time of the rotation of Venus

M. de la Hire observed with a Telescope 16 Feet long, Mountains in Venus higher than those of the Moon.

The extraordinary brightness of Mercury arising from his proximity to the Sun, prevents our discovering by Observation its Rotation; and Saturn is too remote to have his Spots observed.

In the Year 1715 *Cassini* observed with a Telescope 118 Feet long; three Belts in Saturn resembling those observed in Jupiter, but probably those Observations could not be pursued with accuracy sufficient to conclude the Rotation of Saturn about its Axis.

The rotation of Mercury and of Saturn cannot be discovered by observation and why

(m). By the Disk of a Planet is understood that Part of its surface which is visible to us.

but analogy authorises us to conclude that those planets revolve round their Axes.

As Mercury and Saturn are subject to the same Laws that direct the Courses of the other Planets, and as far as has been discovered appear to be Bodies of the same Nature, Analogy authorizes us to conclude that they also revolve, round their Axes; and perhaps future Astronomers may be able to observe this Motion, and to determine its Period.

XV.

There appear from Time to Time Spots upon the Sun, which have served to discover that it has a rotatory Motion about its Axis.

How the rotation of the Sun about its axis has been discovered

It was long after the Discovery of those Spots, before Astronomers could observe any, sufficiently durable and permanent, to enable them to determine the Time of his Revolution. Keill in the 5th Lecture of his Astronomy, relates, that some Spots have been observed to pass from the Western Limb of the Sun to the Eastern Margent in 13 Days and half, and after 13 Days and half to re-appear in the Western Verge of his Disk, from whence he infers that the Sun revolves round its Axis in the Space of about 27 Days from West to East, that is in the same Direction of the Planets; by means of those Spots it has been discovered, that the Axis round which the Sun revolves, is inclined to the plane of the Ecliptic in an Angle of 7d.

Jaquier, in his Commentary on *Newton*, has made some Reflections on these Spots that deserve to be remarked; as no Observations prove the Times of their Occultation to be equal, but on the contraty, all the Observations he could collect, prove them to be unequal; and, that the Time during which they are concealed, has been always longer than that, during which they have been visible, from hence he concluded (as also *Wolf* Art. 411. of his Astronomy) that those Spots are not inherent to the Sun, but removed from his Surface to some distance.

The Solar Spots were first discovered in *Germany*, in the Year 1611, by *John Fabricius*, (n) who from thence concluded, the diurnal Revolution of the Sun. They were afterwards observed by *Scheiner*, (o) who published the Result of his Observations. The same Discovery was made by *Galileo* in *Italy*.

Scheiner observed more than fifty Spots on the Surface of the Sun; this may serve to account for a Phenomenon, related by many Historians, that the Sun, sometimes for the Space of a whole Year, has appeared very Pale, as this Effect would naturally follow from a Number of Spots sufficiently large and permanent, to obscure a considerable Portion of his Surface.

(n) Wolf. Elementa Astronomiæ Cap. 1.

(o) Scheiner having informed his Superior that he had discovered Spots in the Sun, he gravely replied, "that is impossible, I have read Aristotle two or three times over, and have found not the least mention of it."

It is no longer doubted that the Earth turns round her Axis in 23h 56m which compose our astronomical Day; from this Rotation arise the changes of Day and Night, which all the Climates of the Earth enjoy.

XVI.

This Motion of the Celestial Bodies about their Centers alters their Figures, for it is known that Bodies revolving in Circles, acquire a Force which is so much the greater, the Time of their Revolution being the same as the Circle which they describe is greater. This Force is called *Centrifugal Force*; that is, the Force which *repels them from the Center*; wherefore, from their diurnal Rotation, the Parts of the Planets acquire a Centrifugal Force, so much greater as they are nearer the Equators of these Planets: (since the Equator is the greatest Circle of the Sphere,) and so much less as they are nearer the Poles (p); supposing therefore the Heavenly Bodies in their State of Rest, to have been perfect Spheres, their Rotation about their Axes must have elevated their equatorial and depressed their polar Regions, and of Consequence changed their spherical Figures into that of Oblate Spheroids, flat towards the Poles. the effects of the rotatory motion of the planets consist in raising their equators. what is the centripetal force.

The Theory thus leads us to conclude, that all the Planets, in Consequence of their Rotation, should be flat towards the Poles, but this is only sensible in Jupiter and the Earth. In the Sequel it will appear, that the Proportion of the Axes (q), in the Sun, is assignable from Theory, but is too inconsiderable to be observed. which are the planets in which the elevation of the equator is perceived.

The Measures of Degrees of the Meridian, taken at the Polar Circle in *France*, and at the Equator, fix the Proportion of the Axes of the Earth to be as 173 to 174. By the Help of Telescopes the oblate Figure of Jupiter has been perceived. And the Disproportion of his Diameters is much greater than that of the Earth, because this Planet is a great deal bigger, and revolves with greater Rapidity about its Axis than the Earth; the Proportion of the Axes of Jupiter is esteemed to be as 13 to 14.

XVII.

As the Spots of Venus, Mars and Jupiter are variable, and frequently change their Appearance, it is probable that these Planets, like our Earth, are surrounded by dense Atmospheres, the Alterations in which, produce these Phenomena in respect of the Sun, as his Spots are not inherent on his Disk, and as they frequently appear and disappear, it is manifest that he is surrounded by a gross Atmosphere, contiguous to his Body, in which these Spots are successively generated and dissolved. observation proves that the Earth, Mars, Jupiter, Venus and the Sun are surrounded by atmospheres.

(p) The Poles are the Points about which the Body revolves, and the Equator, the Circle equi distant from those Points dividing the Sphere into two equal Parts.

(q) Axis or Diameter, in general, is a Line which passes through the Center. and is terminated at the Circumference, In the present Case, the Axes are two Lines which pass through the Center, one of which is terminated at the Poles, and the other at the Equator.

XVIII.

What has hitherto been ſet forth was known before the Time of *Newton*, but no one thought before him, that it was poſſible to diſcover the Quantities of Matter in the Planets, their Denſities, and the different Weights of one and the ſame Body ſucceſſively transferred to the Surfaces of the different Planets. How *Newton* attained to thoſe aſtoniſhing Diſcoveries will be explained in the Sequel; at preſent it ſuffices to ſay, that he found out that the Maſſes of the Sun, Jupiter, Saturn, and the Earth, that is the Quantities of Matter thoſe Bodies contain, are to one another, as 1 $\frac{1}{1067}$, $\frac{1}{3021}$ & $\frac{1}{169282}$, ſuppoſing (r) the Parallax of the Sun to be 10'' 3'''; that their Denſities are as 100, 94, 67, and 400; & that the Weights of the ſame Body, placed ſucceſſively on the Surfaces of the Sun, Jupiter, Saturn, and the Earth, would be as 10000, 943, 529, and 435; in determining thoſe Proportions, *Newton* has ſuppoſed the Semidiameters of the Sun, Jupiter, Saturn, and the Earth, to be as 10000, 997, 791, and 109. it will be ſhewn hereafter why neither the Denſity, nor the Quantity of Matter of Mercury, Venus, and Mars, or the Weights of Bodies at their reſpective Surfaces, are known.

Maſſes of the Sun, Jupiter, Saturn, and the Earth.
Their denſities
Weights of the ſame body at their ſurface.
why thoſe proportions are not diſcovered in the other planets

XIX.

It follows from all thoſe Proportions that Saturn is nearly 500 Times leſs than the Sun, and contains 3000 Times leſs Matter, that Jupiter is 1000 Times leſs than the Sun, and contains 1033 Times leſs Matter. Compared with the Sun the Earth is only as a Point, being 100,0000 Times leſs; and in fine, that the Sun is 116 Times greater, than all the Planets togeher.

proportions of the bulks and maſſes of the planets and of the Sun.

XX.

Comparing the Planets with one another, we find that Mercury and Mars are the only Planets leſs than the Earth; that Jupiter is not only the biggeſt of all the Planets, but is bigger than all the Planets together, and that this Planet is two thouſand Times bigger than the Earth.

XXI.

The Earth beſides her annual and diurnal Motion, has alſo a third Motion, by which her Axis recedes from its Parallelism, (ſ) & after a certain Time is directed to different Points of the Heavens, from this Motion ariſes what is called *the Preceſſion of the Equinoxes* that is, the Regreſſion of the equinoctial Points, or thoſe Points in which the tereſtrial Equator cuts the Ecliptic. The equinoctial Points move contrary to the Order of the Signs, and their Motion is ſo very ſlow, that they do not compleat a Revolution in leſs than 25920 Years, they recede a Degree in 72 Years, and the annual Quantity is about, 50''.

The preceſſion of the equinoxes.
In what direction it is performed
and in what time it is accompliſhed.
its annual quantity

(r) The parallax of the Sun, is the Angle, under which the Semidiameter of the Earth is ſeen from the Sun, and in general the parallax of any celeſtial Body, with reſpect to the Earth, is the Angle under which the Semidiameter of the Earth would be ſeen from that Body.

(ſ) A line is ſaid to be parallel when it always preſerves the ſame poſition with reſpect to a Point ſuppoſed fixed.

Newton found, as will appear in the Sequel, the Cause of this Motion in the Attraction of the Sun and Moon on the Elevation of the equatorial Parts of the Earth.

The Preceſſion of the Equinoxes has cauſed a Diſtinction of the Year into the *tropical* and *ſydereal*. The tropical Year is the Interval of Time elapſed between two ſucceſſive vernal or autumnal Equinoxes, in two annual Revolutions of the Earth. This Year is ſomewhat ſhorter than the ſydereal Year, or the Time intervening the Earth's Departure from any Point of her Orbit, and her Return to the ſame. Tropical year. Sydereal year.

XXII.

It remains to deſcribe the ſecondary Planets, which excluſive of the Ring of Saturn, are 10 in Number; namely, the 5 Satellites of Saturn, the 4 of Jupiter, and the Moon, the only Satellite attending the Earth. The ſecondary planets.

Obſervation proves that theſe Satellites in revolving round their Primaries, obſerve the Laws of *Kepler*. They obſerve the laws of Kepler.

The Satellites of Jupiter have been but lately diſcovered: The Diſcovery before the Invention of Teleſcopes was impoſſible. *Gallileo* diſcovered the four Satellites of Jupiter, which in Honour of his Patron, he termed the *Medicean* Stars. These are of the greateſt Utility in Geography and Aſtronomy. Diſcovery of the ſatellites of Jupiter.

Hughens was the firſt who diſcovered one of Saturn's Satellites; it ſtill retains his Name, and is the fourth. Afterwards *Caſſini* diſcovered the four others. And of theſe of ſaturn.

XXIII.

Taking the Semidiameter of Jupiter as a common Meaſure, his 4 Satellites revolve at the following Diſtances; the firſt at the Diſtance of 5 Semidiameters, the ſecond of 9, the third of 14, and the fourth of 25, neglecting Fractions. Theſe Determinations have been deduced by *Caſſini* from his Obſervations of their Eclipſes. Diſtances of the moons of Jupiter from this planet.

Their periodic Times round Jupiter are ſo much the longer as they are remoter from this Planet. The firſt revolves in 42 Hours, the ſecond in 85, the third in 171, and the fourth in 400, neglecting the Minutes. Their periodic times about Jupiter

The diurnal Rotations, Diameters, Bulks, Maſſes, Denſities, and attractive Forces of theſe Satellites, have not as yet been diſcovered; and the beſt Teleſcopes repreſent them ſo vaſtly ſmall, that there is no Hopes of ever attaining Certainty in theſe points; the ſame is the Caſe with regard to the Satellites of Saturn: Theſe are placed ſtill further beyond the reach of our Reſearches.

XXIV.

Taking the Diameter of Saturn's Ring for a common Meaſure, the Diſtances of the Satellites of Saturn commencing with the innermoſt, are in the following Proportions. Diſtances of the moons of ſaturn from ſaturn.

& their periodic times round this planet. The first is expressed by 1, the second by 2, the third by 3, the fourth by 8, and the fifth by 24, neglecting Fractions; and their periodic Times, according to *Cassini*, are 45^h, 65^h, 109^h, 382^h, and 1903^h respectively.

The Moons of Saturn, all revolve in the Plane of the Equator of that Planet, except the fifth, which recedes from it about 15 or 16 Degrees.

Conjectures of Hughens concerning a sixth satellite of Saturn. Several Philosophers, and among them *Hughens*, have suspected, that if Telescopes were once brought to perfection, a sixth Satellite of Saturn between the fourth and fifth would be discovered, the Distance between those two Satellites being two great in Proportion to that which separates the others; but there would then occur this other Difficulty, that this Satellite, which would be the fifth, notwithstanding must be less than any of the four interior Moons, since with our most perfect Telescopes it cannot be perceived.

The Orbits of the Satellites of Jupiter, and of Saturn, are nearly concentric to those Planets.

Observation of Maraldi concerning the satellites of Jupiter. *Maraldi* has observed Spots on the Moons of Jupiter, but no Consequences could as yet be derived from this Observation, which if properly pursued and accurately repeated, might conduct us to the Knowledge of several interesting Particulars respecting the Motions of the Satellites.

XXV.

Of the ring of Saturn. It does not adhere to the body of this planet. Its distance from the body of the planet. Its diameter. Its breadth. Its thickness. It is an opaque body subject to phases. Saturn, exclusive of his five Moons, is also surrounded by a Ring, no where adhering to his Body; for through the Interval which separates his Body from the Ring, we can view the fixed Stars: The Diameter of this Ring is to the Diameter of Saturn as 9 to 4, according to *Hughens*, that is more than the Double of the Diameter of Saturn; the Distance of the Body of Saturn from his Ring, is nearly equal to his Semidiameter; so that the Breadth of the Ring is nearly equal to the Distance between its interior Limb and the Globe of Saturn. Its Thickness is very inconsiderable, for when it turns its Edge to the Eye, it is no longer visible, but only appears as a black Line extended across the Globe of Saturn. Thus this Ring undergoes Phases according to the Position of Saturn in his Orbit, which proves it to be an opaque Body; and which like the other Bodies that compose our planetary System, shines only by reflecting the Light it receives from the Sun.

We cannot discover whether the Ring of Saturn has any Motion of Rotation, as no Changes in its Aspect are observed to authorise us to conclude this Rotation.

The Plane of this Ring always forms with the Plane of the Ecliptic an Angle of $23° \frac{1}{2}$, hence its Axis remains always parallel to itself in its Revolution round the Sun.

Of the discovery of this ring. Opinion concerning it be- The Discovery of the Ring of Saturn, the only Phenomenon of the Kind observed in the Heavens is due to *Hughens*. Before his Time, Astronomers observed Phases in Saturn, for they confounded Saturn with his Ring; but those Phases were so different from those of the other Planets as to be utterly inexplicable. In *Hevelius* may be seen the Names he gives to those Appearances of Saturn, and how far (t) he was from assigning the true Cause. fore Hughens.

Hughens comparing the different Appearances of Saturn, found they were produced by a Ring surrounding his Body; and this Supposition is so conformable with all Telescopic Discoveries, as to be now generally received.

Gregory describing the Notion of *Halley*, that the terrestrial Globe is only an Assemblage of Shells concentric to an internal Nucleus, proposes a Conjecture concerning this Ring, that it is formed of several concentric Shells detached from the Body of that Planet, whose Diameter was formerly equal to the Sum of its actual Diameter, and the Breadth of the Ring. Notion of Gregory concerning this ring.

Another Conjecture has also been proposed, that the Ring of Saturn is only an Assemblage of Moons, which from the immense Distance appear to be contiguous; but those Conjectures are not grounded on any Observation.

By the Shadows of the Satellites of Jupiter and Saturn projected on their Primaries, it has been discovered, that they are spherical Bodies. The satellites of Jupiter and Saturn are spherical bodies.

XXVI.

The Earth has only one Satellite, namely the Moon; but her Proximity has enabled us to push our Enquiries concerning this Satellite much further than about the others. Of the moon

The Moon performs its Revolution round the Earth in an Ellipse, the Earth being placed in one of the Foci: The Form and Position of this Ellipse is continually changing; these Variations are caused by the Action of the Sun, as will appear in the Sequel. What curve it describes round the earth.

The Moon in her Revolution round the Earth observes the first of the two Laws of *Kepler*, and recedes from it only by the Action of the Sun upon her; she compleats her Revolution round the Earth from West to East in 27 d. 7 h. 43 m. which is called its *periodical Month*. Its periodic month.

The Disc of the Moon is sometimes totally, and at other times partially, illuminated by the Sun. The illuminated Part is greater, or less, according to its Position with respect to the Sun and the Earth; these are called her *Phases*. She assumes all those various Phases during the Time of her *synodic* Revolution, or the Interval between two successive Conjunctions with the Sun. This synodic Month of the Moon consists of 29 Days ½ nearly. Her phases. Her synodic month.

The Phases of the Moon prove that she is an opaque Body, shining only by reflecting the Light of the Sun. The moon is an opaque and spherical body.

We know that the Moon is a spherical Body, because she always appears to be bounded by a Curve.

The Earth enlightens the Moon during her Nights, as the Moon does the Earth during ours; and it is by the reflected Light of the Earth that we see the Moon, when she is not illustrated by the Sun. The earth enlightens the moon during her nights.

(t) Hevelius in opusculo de Saturni Nativa facie distinguishes the different Aspects of Saturn by the Names of Monosphericum, Trisphericum, Spherico-ansatum, ellipti-coansatum, sphericocuspidatum, and subdivides them again into other Phases.

Proportion of this illumination.

As the Surface of the Earth is about 14 times greater than that of the Moon, the Earth seen from the Moon would appear 14 times brighter, and reflect 14 times more rays to the Moon, than the Moon does to us, supposing both equally capable of reflecting Light.

Inclination of the orbit of the moon

The Plane of the lunar Orbit forms with the Plane of the Ecliptic, an Angle of about 5ᵈ·

The great Axis of the Ellipse which the Moon describes round the Earth, is called *the Line of the Apsides* (u) *of the Moon.*

The Moon accompanies the Earth in her annual Revolution round the Sun.

If the Orbit of the Moon had no other Motion but that by which it is carried round the Sun along with the Earth, the Axis of this Orbit would always remain parallel to itself; and Moon being in her *Apogee*, and in her *Perigee*, would be always at the same Distances from the Earth, and would always correspond to the same Points of the Heavens; but the Line of the Apsides of the Moon revolves with an angular Motion round the Earth, according to the Order of the Signs; and the Apogee and Perigee of the Moon do not return to the same Points in less than 9 Years, which is the Time of the Revolution of the Line of the Apsides of the Moon.

Time of the revolution of the line of the apsides.

Revolution of the nodes of the moon

The Orbit of the Moon intersects the Orbit of the Earth in two Points, which are called her *Nodes*; these Points are not always the same, but change perpetually by a retrogressive Motion that is contrary to the Order of the Signs, and this Motion is such, that in the space of 19 Years the Nodes perform a whole Revolution, after which they return to the same Points of the Orbit of the Earth, or of the Ecliptic.

Time of its revolution.

Excentricity of the moon.

The Excentricity of the Orbit of the Moon changes also continually; this Excentricity sometimes increases, sometimes diminishes, so that the Difference of the greatest and least Excentricity exceeds half the least.

It will be explained in the Sequel how *Newton* discovered the Cause of all those Inequalities of the Moon.

Its motion round its axis.

The only uniform Motion that the Moon has, is its Motion of Rotation about her Axis; this Motion is performed exactly in the same Time as its Revolution about the Earth, hence its Days consist of 27 of our Days, 7ʰ· 43ᵐ·

In what time it is performed.

This equality of the lunar Day and the periodic Month makes the Moon always present to us nearly the same Disc.

Libration of the moon.

The uniform Motion of the Moon about its Axis, combined with the Inequality of its Motion round the Earth, produces the apparent Oscillation of the Moon about her Axis, sometimes Eastward, and at other times Westward, and this is what is called *her Libration*; by this Motion she presents

(u) The Line of the Apsides of the Moon is the Line which passes through the Apogee and Perigee; apogee is the Point of the Orbit the Remotest from the Earth, and the Perigee is the Point of the Orbit the nearest to the Earth; and in general, the Apsides of any Orbit are the Points the Remotest from, and nearest to, the central Point.

to us sometimes Parts which were concealed, and conceals others that were visible.

This Libration of the Moon arises from her Motion in an Elliptic Orbit, for if she revolved in a circular Orbit, having the Earth for its Center, and turned about her Axis in the Time of her periodic Motion round the Earth, she would in all Positions turn the same Disç exactly towards the Earth. Its cause.

We are ignorant of the Form of the Surface of the Moon, which is on the other Side of her Disc with Respect to us. Some Philosophers have even attempted to explain its Libration, by assigning a conical Figure to that Part of its Surface, which is concealed from us, and who deny her Rotation round her Axis.

The Surface of the Moon is full of Eminences and Cavities, for which reason she reflects on every Side the Light of the Sun, for if her Surface was even and polished like a Mirror, she would only reflect to us the Image of the Sun.

The mean Distance of the Moon from the Earth is nearly 60 ½ Semidiameters of the Earth. Distance of the moon from the earth.

The Diameter of the Moon is to the Diameter of the Earth, as 100 to 365, its Mass is to the Mass of the Earth, as 1 to 39, 788 and its Density is to the Density of the Earth, as 11 to 9. Its diameter Its mass. Its density.

And lastly, a Body which would weigh 3 Pounds at the Surface of the Earth, transferred to the Surface of the Moon would weigh one Pound. What bodies weigh on its surface.

All these Proportions are known in the Moon and not in the other Satellites, because this Planet supplies a peculiar Element, namely her Action on the Sea, which *Newton* knew how to measure and to employ for determining her Mass, the Method he pursued in this Enquiry will be unfolded in the Sequel.

Theory of the Primary Planets.

I.

In accounting for the celestial Motions, the first Phenomenon that occurs to be explained is the perpetual Circulation of the Planets round the Center of their Revolutions.

By the first Law of Nature every Body in Motion perseveres in that rectilinear Course in which it commenced, therefore that a Planet may be deflected from the straight Line it tends to describe incessantly, it is Necessary that a Force different from that which makes it tend to describe this straight Line should incessantly Act on it in order to bend its Course into a Curve, in the same Manner as when a Stone is whirled round in a Sling. The Sling incessantly restrains the Stone from flying off in the Direction of the Tangent to the Circle it describes.

To explain this Phenomenon, the Ancients invented their solid Orbs and *Descartes* Vortices, but both one and the other of those Explications How the ancient philosophers and Descartes explain the circulation of the planets in their orbits.

were mere Hypotheses devoid of Proof, and though *Descartes* Explanation was more Philosophical, it was no less Fictitious and Imaginary.

II.

It is a centripetal force which hinders the planets from flying off by the tangent.

Newton begins with proving in the first Proposition (a), that the Areas described by a Body revolving round an immoveable Center to which it is continually urged, are proportional to the Times, and reciprocally in the Second, that if a Body revolving round a Center describes about it Areas proportional to the Times, that Body is actuated by a Force directed to that Center. Since therefore according to *Kepler*'s Discoveries, the Planets describe round the Sun Areas proportional to the Times, they are actuated by a centripetal Force, urging them towards the Sun, and retaining them in their Orbits.

Newton has also shewn (Cor. 1. Prop. 2.) that if the Force acting on a Body, urges it to different Points, it would accelerate or retard the Description of the Areas, which would consequently be no longer proportional to the Times: Therefore if the Areas be proportional to the Times, the revolving Body is not only actuated by a centripetal Force, directed to the central Body, but this Force makes it tend to one and the same Point.

III.

And the projectile force hinders them from falling to the center

As the Revolutions of the Planets in their Orbits prove the Existance of a centripetal Force drawing them from the Tangent, so by their not descending in a straight Line towards the Center of their Revolution, we may conclude that they are acted upon by another Force different from the Centripetal. *Newton* has examined (b) in what Time each Planet would descend from its present Distance to the Sun if they were actuated by no other Force but the Sun's Action, & he has found (P.36) that the different Planets would employ in their Descent, the Half of the periodic Time of the Revolution round the Sun of a Body placed at Half their present Distances, and consequently these Times would be to their periodic Times, as 1 to $4\sqrt{2}$. Thus, *Venus* for Example would take about 40 Days to descend to the Sun, for $40 : 224 :: 1 : 4\sqrt{2}$ nearly; *Jupiter* would employ two Years and a Month in his Descent, and the Earth and the Moon sixty-six Days and nineteen Hours, &c. since then the Planets do not descend to the Sun, some Force must necessarily counteract the Force which make them tend to the Sun, and this Force is called the *Projectile Force*.

IV.

Of the centrifugal force of the planets.

The Effort exerted by the Planets in Consequence of this Force to recede from the Center of their Motion, is what is called their *Centrifugal Force*, hence in the Planets, the centrifugal Force is that Part of the projectile Force, which removes them directly from the Center of their Revolution.

(a) When the Propositions are quoted without quoting the Book, they are the Propositions of the first Book.

(b) De systemate mundi, page 31. edition 1731.

V.

The projectile Force has the same Direction in all the Planets, for they all revolve round the Sun from West to East.

Supposing the Medium in which the Planets move to be void of all Resistance, the Conservation of the projectile Motion in the Planets, is accounted for from the Inertia of Matter, and the first Law of Motion, but its Physical Cause, and the Reason of its Direction are as yet unknown.

VI.

Newton discovers the force urging the planets to the Sun to be in the inverse ratio of the square of their distances from the ratio of their periodic times and distances. First in the supposition of their orbits being circular.

After having proved that the Planets are retained in their Orbits by a Force directed to the Sun, *Newton* demonstrates (Prop. 4.) that the centripetal Forces of Bodies revolving in Circles are to one another as the Squares of the Arcs of those Circles described in equal Times, divided by their Rays, from whence he deduces (cor. 6) that if the periodic Times of Bodies revolving in Circles be in the sesquiplicate Ratio of their Rays, the centripetal Force which urges them to the Center of those Circles, is in the inverse Ratio of the Squares of those same Rays, that is of the Distance of those Bodies from the Center: But by the second Law of *Kepler*, which all the Planets observe, their periodic Times are in the sesquiplicate Ratio of their Distances from their Center; consequently, the Force which urges the Planets towards the Sun, decreases as the Square of their Distance from the Sun increases, supposing them to revolve in Circles concentric to the Sun.

VII.

Before Kepler's time it was thought that the planets revolved about the Sun in excentric circles.

The first and most natural Notion that we form concerning the Orbits of the Planets, is that they perform their Revolutions in concentric Circles; but the Difference in their apparent Diameters, and more accuracy in the Observations, have long since made known that their Orbits cannot be concentric to the Sun; their Courses therefore, before *Kepler's* Time, were explained by excentric Circles, which answered pretty well to the Observations on the Motions of the Sun and the Planets, except Mercury and Mars.

But Kepler has shewn that they revolve in ellipses.

From considering the Course of this last Planet, *Kepler* suspected that the Orbits of the Planets might possibly be Ellipses, having the Sun placed in one of the Foci, and this Curve agrees so exactly with all the Phenomena, that it is now universally acknowledged by Astronomers, that the Planets revolve round the Sun in elliptic Orbits, having the Sun in one of the Foci.

VIII.

Assuming this Discovery, *Newton* examines what is the Law of centripetal Force, required to make the Planets describe an Ellipse, and he found (Prop. 11.) that this Force must follow the inverse Ratio of the Planet's Distance from the Focus of this Ellipse. But having found before (cor. 6. Prop. 4.) that if the periodic Times of Bodies revolving in Circles be in the sesquiplicate Ratio of their Rays, the centripetal Forces would be in the inverse Ratio of those same Distances; he had no more to do to invincibly prove that the centripetal Force which directs the celestial Bodies in their Courses, follows the inverse Ratio of the Square of the Distances; but to examine if the periodic Times follow the same Proportion in Ellipses as in Circles.

Newton demonstrates that in ellipses the periodical times are in the same proportion as in circles.

But *Newton* demonstrates (Prop. 15.) that the periodic Times in Ellipses are in the sesquiplicate Ratio of their great Axes; that is, that those Times are in the same Proportion in Ellipses, and Circles whose Diameters are equal to the great Axes of those Ellipses.

Consequently the centripetal force which retains the planets in their orbits decreases as the square of the distance.

This Curve which the Planets describe in their Revolution is endued with this Property, that if small Arcs described in equal Times be taken, the Space bounded by the Line drawn from one of the Extremities of this Arc, and by the Tangent drawn from the other Extremity increases in the same Ratio as the Square of the Distance from the Focus decreases; from whence it follows, that the attractive Power which is proportional to this Space, follows also this same Proportion.

IX.

The centripetal force being in this proportion the planets can only describe conic sections the Sun being placed in one of the foci.

Newton, not content with examining the Law that makes the Planets describe Ellipses; he enquired further weather in consequence of this Law: Bodies might not describe other Curves, and he found (Cor. 1. Prop. 13.) that this Law would only make them describe a conic Section, the Center of the Force being placed in the Focus, let the projectile Force be what it would.

Other Laws, by which Bodies might describe conic Sections, would make them describe them about Points different from the Focus. *Newton* found, for example, (Prop. 10.) that if the Force be as the Distance from the Center, it will make the Body describe a conic Section, whose Center would be the Center of Forces, thus *Newton* has discovered not only the Law which the centripetal Force observes in our planetary System, but he has also shewn that no other Law could subsist in our World in its present State.

X.

Manner of determining the orbit of a planet supposing the law of centripetal force to be given.

Newton afterwards examines (Prop. 17.) the Curve a Body would describe with a centripetal Force decreasing in the inverse Ratio of the Square of the Distance, supposing the Body let go from a given Point, with a Direction and Velocity assumed at Pleasure.

To solve this Problem, he sets out with the Remark he had made, (Prop. 16.) that the Velocities of Bodies describing conic Sections, are in each Point of those Curves, as the Square-Roots of the principal Parameters, divided by the Perpendiculars, let fall from the Foci on the Tangents to those Points.

This Proposition is not only very interesting, considered merely as a geometrical Problem, but also of great use in Astronomy; for finding by Observation the Velocity and Direction of a Planet in any Part of its Orbit, by the Assistance of this Proposition, the Remainder of its Orbit is found out, and the Determination of the Orbits of Comets, may in a great Measure be deduced from this Proposition.

XI.

It is easy to conceive that in consequence of other Laws of centripetal Force different from that of the Square of the Distances Bodies would describe other Curves, that there are some Laws by which notwithstanding the projectile Force, they would descend to the Sun, and others by which notwithstanding the centripetal Force, they would recede in infinitum in the Heavenly Spaces; others would make them describe Spirals, &c. and *Newton* in the 42d Proposition, investigates what are the Curves described in all Sorts of Hypothesis of centripetal Forces.

What Curves in consequence of other Laws of centripetal force would be described.

XII.

It evidently appears from all that has been said that the perpetual Circulation of the Planets in their Orbits depends on the Proportion between the centripetal and the projectile Force, and those who ask why the Planets arriving at their Perihelia, reascend to their Aphelia, are ignorant of this Proportion; for in the higher Apsis the centripetal Force exceeds the Centrifugal Force, since in descending the Body approaches the Centre, and in the lower Apsis on the Contrary, the centrifugal Force surpasses in its turn the centripetal Force, since in reascending the Body recedes from the Centre: A certain Combination between the centripetal Force and the centrifugal Force was therefore requisit, that they might alternately prevail and cause the Body to descend to the lower, and reascend to the higher Apsis perpetually.

The perpetual circulation of the Planets in their Orbits results from the proportion between the centripetal and projectile force.

Another Objection was alledged with regard to the Continuation of the Heavenly Motions, derived from the Resistance they should undergo in the Medium in which they move. This Objection *Newton* has answered in (Prop. 10. B. 3.) where he shews that the Resistance of Mediums diminish in the Ratio of their Weight and their Density; but he proved in the Scholium of (Proposition 22. B. 2.) that at the Height of two hundred Miles above the Surface of the Earth, the Air is more rarified than at the Surface, in the Ratio of 30 to 0,0000000000003998 or nearly as 75000000000000 to 1, from whence he concludes (Prop. 10. B. 3.) supposing the Resistance of the Medium in which Jupiter moves to be of this Density, this Planet describing five of its Semidiameters in 30 days, would from the Resistance of this Medium, in 1000000 years scarcely lose 1000000th Part of its Motion; from hence we see that the Medium in which the Planets move may be so rare and subtile, that its Resistance may be regarded as Void; and the Proportionality constantly observed, between the Areas and the Times, is a convincing Proof that this Resistance is actually insensible.

The medium in which the heavenly Bodies move is void of all resistance.

XIII.

As we have shewn that the Proportionality of the Times and of the Areas which the Planets describe around the Sun, proves that they tend to the Sun as to their Centre, and that the Ratio subsisting between their periodic Times and their Distances, shews that this Force decreases in the inverse

Ratio of the Square of the Diſtances. If the Planets which perform their Revolutions round the Sun be ſurrounded by others which revolve round them, and obſerving the ſame Proportions in their Revolutions, we may conclude that theſe Satellites are urged by a centripetal Force directed to their Primaries, and that this Force decreaſes as that of the Sun in the duplicate Ratio of the Diſtance.

We can diſcover only three Planets attended with Satellites, Jupiter, the Earth, and Saturn; we know that the Satellites of thoſe three Planets deſcribe around them Areas proportional to the Times, and conſequently are urged by a Force tending to thoſe Planets.

XIV.

The comparison of the periodic times and diſtances of the ſatellites of Saturn and Juptier, proves that the centripetal force of thoſe planets is alſo in the inverſe ratio of the ſquare of the diſtances.

Jupiter and Saturn having each ſeveral Satellites whoſe periodic Times and Diſtances are known, it is eaſy to diſcover whether the Times of their Revolution about their Planet, are to their Diſtance in the Proportion diſcovered by *Kepler*; and Obſervations evince that the Satellites of Jupiter and Saturn obſerve alſo this ſecond Law of *Kepler* in revolving round their Primaries, and of conſequence the centripetal Force of Jupiter and of Saturn decreaſe in the Ratio of the Square of the Diſtances of Bodies from the Centre of thoſe Planets.

XV.

As the Earth is attended only by one Satellite, namely the Moon, it appears at firſt View difficult to determine the Proportion in which the Force acts that makes the Moon revolve in her Orbit round the Earth, as in this Caſe we have no Term of Compariſon.

How Newton diſcovered that the attractive force of the Earth follows the ſame proportion.

Newton has found the Means of ſupplying this Defect; his Method is as follows: All Bodies which fall on the Surface of the Earth, deſcribe according to the Progreſſion diſcovered by *Gallileo*, Spaces which are as the Squares of the Times of their Deſcent. We know the mean Diſtance of the Moon from the Earth which in round Numbers is about 60 Semidiameters of the Earth; and all Bodies near the Surface of the Earth are conſidered as equidiſtant from the Centre; therefore if the ſame Force produces the Deſcent of heavy Bodies, and the Revolution of the Moon in her Orbit; and if this Force decreaſes in the Ratio of the Square of the Diſtance, its Action on Bodies near the Surface of the Earth ſhould be 3600 Times greater than what it exerts on the Moon, ſince the Moon is 60 Times remoter from the Centre of the Earth; we know the Moon's Orbit, becauſe we know at preſent the Meaſure of the Earth, we know that the Moon deſcribes this Orbit in 27 Days, 7 Hours, 43 Minutes, hence we know the Arc ſhe deſcribes in one Minute; now by (Cor. 9 Prop. 4.) the Arc deſcribed in a given Time by a Body revolving uniformly in a Circle with a given centripetal Force, is a mean Proportional between the Diameter of this Circle and the right Line deſcribed in the Body's deſcent during that Time.

It is true that the Moon does not revolve round the Earth in an exact Circle, but we may ſuppoſe it ſuch in the preſent Caſe without any ſenſible Error; and in this Hypotheſis, the Line expreſſing the Quantity of the Moon's deſcent in one Minute, produced by the centripetal Force, is found to be nearly 15 Feet.

But the Moon according to the Progreſſion diſcovered by *Gallileo*, at her preſent Diſtance would deſcribe a Space 3600 Times leſs in a Second than in a Minute, and Bodies near the Surface of the Earth deſcribe, according to the Experiments of Pendulums, for which we are indebted to *Hughens*, about 15 feet in a Second, that is, 3600 Times more Space than the Moon deſcribes in the ſame Time; therefore the Force cauſing their Deſcent acts 3600 Times more powerfully on them than it does on the Moon; but this is exactly the inverſe Proportion of the Squares of their Diſtances.

By this Example we ſee the Advantage of knowing the Meaſure of the Earth; for in order to compare the Verſe Sine which expreſſes the Quantity of the Moon's deſcent towards the Earth, with the cotemporary Space deſcribed by Bodies falling by the Force of Gravity near the Earth, we muſt know the abſolute Diſtance of the Moon from the Earth, reduced into Feet, as alſo the Length of the Pendulum vibrating Seconds; for in this Caſe it is not ſufficient to know the Ratio of Quantities, but their abſolute Magnitudes.

The meaſure of the Earth was neceſſary for making this diſcovery.

XVI.

Jupiter, Saturn, and our Earth therefore attract Bodies, in the ſame Proportion that the Sun attracts thoſe Planets, and Induction authoriſes us to conclude that Gravity follows the ſame Proportion in Mars, Venus, and Mercury; for by all that we can diſcover of theſe three Planets, they appear to be Bodies of the ſame Nature with the Earth, Jupiter, and Saturn; from whence we may conclude, with the higheſt Probability, that they are endued with the attractive Force, and that this Force decreaſes as the Square of the Diſtances.

Induction authoriſes us to conclude, that attraction follows the ſame proportion in the planets which have no ſatellites.

XVII.

It being proved by Obſervation and Induction that all the Planets are endued with the attractive Power decreaſing as the Square of the Diſtances; and by the ſecond Law of Motion, Action is always equal to Re-action, we ſhould conclude with *Newton*, (Prop. 5. B. 3.) that all the Planets gravitate to one another, and that as the Sun attracts the Planets, he is reciprocally attracted by them; for as the Earth, Jupiter, and Saturn act on their Satellites in the inverſe Ratio of the Square of the Diſtances, there is no Reaſon why this Action is not exerted at all Diſtances in the ſame Proportion; thus the Planets ſhould attract each other mutually, and the Effects of this mutual Attraction are ſenſibly perceived in the Conjunction of Jupiter and Saturn.

From whence Newton concluded the mutual attraction of all the celeſtial bodies.

XVIII.

As Analogy enduces us to believe that the secondary Planets are in all Respects Bodies of the same Nature with the primary Planets, it is highly probable that they are also endued with the attractive Power, and consequently attract their Primaries in the same Manner they are attracted by them, and that they mutually attract each other. This is further confirmed by the Attraction of the Moon exerted on the Earth, the Effects of which are visible in the Tides and the Precession of the Equinoxes, as will appear in the Sequel: We may therefore conclude that the attractive Power belongs to all the Heavenly Bodies, and that it acts in all our planetary System in the inverse Ratio of the Square of the Distances.

XIX.

What cause makes one body revolve round another.

But what is the Cause which makes one Body revolve round another? for instance, the Earth and the Moon attracting each other with Forces decreasing in the duplicate Ratio of their Distances, why should not the Earth revolve round the Moon, instead of causing the Moon to revolve round the Earth; the Law which regulates Attraction does not therefore depend on the Distance alone, it must depend also on some other Element, in order to account for this Determination, for the Distance alone is insufficient, since it is the same for one and the other Globe.

This cause appears to be the mass of the central body.

From examining the Bodies that compose our planetary System, it is natural to conclude that this Law is that of their Masses; the Sun, round whom all the Heavenly Bodies turn, appears much bigger than any of them; Saturn and Jupiter are much bigger than their Satellites, and our *Earth* is much bigger than the Moon which revolves round it.

The knowledge of the masses of the planets necessary to determine this point.

But as the Bulk and Mass are two different things, to be certain that the Gravity of the Celestial Bodies follows the Law of their Masses, it is necessary to determine those Masses.

But how can the Masses of the different Planets be determined? this *Newton* has shewn.

XX.

Road that conducted Newton to this discovery.

To trace the Road that conducted him to this Discovery.

Since the Attraction of all the Celestial Bodies on the Bodies which surround them follows the inverse Ratio of the Square of the Distances, it is highly probable that the Parts of which they are composed attract each other in the same Proportion.

The total attractive Force of a Planet is composed of the attractive Forces of its Parts; for supposing several small Planets to unite and compose a big one, the Force of this big Planet will be composed of the Sum of the Forces of all those small planets; and *Newton* has proved in (*Prop.* 74, 75 and 76,) that if the Parts of which a Sphere is composed, attract each other mutually in the inverse Ratio of the Square of the Distances, those entire Spheres will attract Bodies which are exterior to them, at whatever Distance they are placed in this same inverse Ratio of the Square of Distances; and of all the Laws of Attraction examined by *Newton*, he has found only two, namely, that in the inverse Ratio of the Square of the Distances, and that in the Ratio of the simple Distances, according to which Spheres attract external Bodies in the same Ratio in which their Parts mutually attract each other; from whence we see the Force of the Reasoning which made *Newton* conclude that since it is proved on one Hand from Theory, (Cor. 3. Prop. 74.) that when the Parts of a Sphere attract each other with Forces decreasing in the duplicate Ratio of the Distances, the entire Sphere attracts external Bodies in the same Ratio, and on the other, Observations evince that the Celestial Bodies attract external Bodies in this Ratio, it is obvious that the Parts of which the Heavenly Bodies are composed, attract each other in this same Ratio.

He finds the weight of the same body upon the different planets at the same distance from their centres

Newton examines (in Prop. 8. B. 3.) what the same Body would weigh at the Surfaces of the different Planets, and he found by means of (Cor. 2. Prop. 4.) in which he had demonstrated, that the Weights of equal Bodies revolving in Circles, are as the Diameters of those Circles, divided by the Squares of their periodic Times, therefore the periodic Times of Venus round the Sun, of the Satellites of Jupiter round this Planet, of the Satellites of Saturn round Saturn, and of the Moon round the Earth, and the Distances of those Bodies from the Centres about which they revolve being known, supposing also that they describe Circles, which may be supposed in the present Case, he discovers how much the same Body would weigh transferred successively on the Surfaces of Jupiter, Saturn and of the Earth.

And proves that their quantities of matter are proportional to those weights.

Having thus found the Weights of the same Body on the Surface of the different Planets at the same Distance from their Centres, *Newton* deduces the Quantities of Matter they contain, for Attraction depending on the Mass and the Distance, at equal Distances the attractive Forces are as the Quantities of Matter in the attracting Bodies; therefore the Masses of the different Planets are as the Weights of the same Body at equal Distances from their Centres.

XXI.

From whence he deduces their densities.

We may discover after the same Manner the Density of the Sun and of those Planets which have Satellites, that is, the Proportion of their Bulks and Masses, for *Newton*, (Prop. 72.) has proved, that the Weights of equal Bodies, at the Surfaces of unequal homogeneous Spheres, are as the Diameters of those Spheres; therefore if those Spheres were heterogeneous and equal, the Weights of Bodies at their Surfaces would be as their Density, supposing the Law of Attraction to depend only of the Distance, and the Mass of the attracting Body; therefore the Weights of Bodies at the Surfaces of unequal and heterogeneous Spheres, are in the compound Ratio of their Densities and Diameters; consequently the Densities are as the Weights of the Bodies divided by their Diameters.

XXII.

The smallest and densest planets are nearest the sun.

From hence we find, that the smaller Planets are denser and placed nearer the Sun, for where all the Proportions of our System were laid down, we saw that the Earth, which is less and nearer the Sun than Jupiter and Saturn, is more dense than those Planets.

XXIII.

The reason assigned by Newton.

Newton deduces from thence, the Reason of the Arrangement of the Celestial Bodies of our planetary System, which is adapted to the Density of their Matter, in order that each might receive a Degree of Heat more or less according to its Density and Distance; for Experience shews us that the denser any Body is, the more difficultly does it receive Heat; from whence *Newton* concludes that the Matter of which Mercury is composed should be seven Times denser than the Earth, in order that Vegetation might take place; for Illumination, to which, ceteris paribus, Heat is proportional, is inversely as the Square of the Distance; but we know the Proportion of the Distances of the Earth and Mercury from the Sun, and from this Proportion we discover that Mercury is seven Times more illuminated, and consequently seven Times more heated than the Earth; and *Newton* discovered, from his Experiments on the Thermometer, that the Heat of our Summer Sun, seven Times augmented, would make Water boil; therefore if the Earth was placed at the Distance of Mercury from the Sun, our Ocean would be dissipated into Vapour; removed to the Distance of Saturn from the Sun, the Ocean would be perpetually frozen, and in both Cases all Vegetation would cease, and Plants and Animals would perish.

XXIV.

The densities of the planets which have satellites only can be discovered, the moon excepted.

It easily appears, that the Masses and Densities of such Planets only as are attended by Satellites can be discovered, since to arrive at this Discovery we must compare the periodic Times of the Bodies revolving round those Planets, the Moon alone is to be excepted, of which mention will be made hereafter.

XXV.

Why the sun is the centre of the celestial revolutions.

Having determined the Masses of the Planets, we find that those Bodies which have less Mass, revolve round those which have a greater, and the greater Mass a Body has the greater is, ceteris paribus, its attractive Force; thus all the Planets revolve round the Sun, because the Sun has a much greater Mass than any of the Planets, for the Masses of the Sun, Jupiter, and Saturn are respectively as 1, 1100 and 3000; since therefore the Masses of these Planets exceed those of any other in our System, it follows that the Sun should be the Centre of the Motions of our planetary System.

XXVI.

If Attraction be proportional to the Masses, the Alteration caused by the Action of Jupiter on the Orbit of Saturn in their Conjunction, ought much to exceed that produced in the Orbit of Jupiter by the Action of Saturn, since the Mass of Jupiter is much greater than that of Saturn, and this Observation evinces; the Alteration in the Orbit of Jupiter in its Conjunction with Saturn, though sensible is considerably less than what is observed in the Orbit of Saturn.

The alterations which the planets mutually produce in their courses follow the ratio of their masses.

XXVII.

But if the Effect of Attraction, or the Space described by the attracted Body, depends on the Mass of the attracting Body, why should it not also depend on the Mass of the attracted Body? This Point surely deserves to be examined.

Experiment proves that all Bodies near the Surface of the Earth, when the Resistance of the Air is removed, descend with equal Velocities; for in the Air-pump, after exhausting the Air, Gold and Feathers fall to the Bottom in the same Time.

Newton has confirmed this Experiment by another, in which the smallest Difference becomes obvious to our Senses. He relates (Prop. 24. B. 2. and Prop. 6. B. 3.) that he composed several Pendulums of Materials entirely different; for instance of Water, Wood, Gold, Glass, &c. and having suspended them by Threads of equal Length, for a considerable Time their Oscillations were Synchronal.

XXVIII.

It admits therefore of no Doubt, that the attractive Force of our Earth is proportioned to the Masses of the Bodies it attracts, and at equal Distances it depends solely on their Masses, that is on their Quantities of Matter; hence if the terrestrial Bodies were transferred to the Orbit of the Moon, it having been proved already that the same Force acts on the Moon and on those Bodies, and that it decreases as the Square of the Distances. The Distances being supposed equal, it follows, that supposing the Moon deprived of her projectile Force, those Bodies and the Moon would fall in the same Time to the Surface of the Earth, and would describe equal Spaces in equal Times, the Resistance of the Air being taken away.

Attraction is proportional to the masses without any respect being had to the form or species of the attracting bodies.

XXIX.

The same Thing is proved of all the Planets having Satellites, for instance, of Jupiter and Saturn; if the Satellites of Jupiter, for example, were all placed at the same Distance from the Centre of this Planet, and deprived of their projectile Force, they would descend towards it and reach its Surface in the same Time; this follows from the Proportion between the Distances of the Satellites and their periodic Times.

XXX.

From the Proportion between the periodic Times and Distances of the primary Planets from the Sun, it may be proved in like Manner, that the Sun acts on each of them proportionally to its Mass, for at equal Distances their periodic Times would be equal, in which Case, supposing their projectile Force destroyed, they would all reach the Sun at the same Time; therefore the Sun attracts each Planet in the direct Ratio of its Mass.

XXXI.

This Truth is further confirmed by the Regularity of the Orbits which the Satellites of Jupiter describe round this Planet, for *Newton* has proved (Cor. 3. Prop. 65.) that when a System of Bodies move in Circles or regular Ellipses, these Bodies cannot be acted upon by any sensible Force but the attractive Force which makes them describe those Curves; now the Satellites of Jupiter describe round that Planet circular Orbits, sensibly regular and concentric to Jupiter, the Distances of these Moons and of Jupiter from the Sun should be considered as equal, the Difference of their Distances bearing no Proportion to the entire Distance; therefore if any of the Satellites of Jupiter, or Jupiter himself, were more attracted by the Sun in Proportion to its Mass than any other Satellite, then this stronger Attraction of the Sun would disturb the Orbit of this Satellite; and *Newton* says, (Prop. 6. B. 3.) that if this Action of the Sun on one of the Satellites of Jupiter was greater or less in Proportion to its Mass than that which it exerts on Jupiter in Proportion to his, only one thousandth part of its total Gravity, the Distance of the Centre of the Orbit of this Satellite from the Sun would be greater or less than the Distance of the Centre of Jupiter from the Sun, by the two thousandth part of its whole Distance, that is by a fifth Part of the Distance of the outermost Satellite of Jupiter from Jupiter, which would render its Orbit sensibly excentric; since then those Orbits are sensibly concentric to Jupiter, the accelerating Gravities of the Sun on Jupiter and on its Satellites, are proportional to their Quantities of Matter.

The same Reasoning may be applied to Saturn and its Satellites, whose Orbits are sensibly concentric to Saturn.

Attraction is always reciprocal.

Experience and Observation therefore leads us to conclude, that the Attraction of the Celestial Bodies is proportional to the Masses, as well in the attracting Body, as in the Body attracted; that it is the Mass which determines a Body to revolve round another, that every Body may be considered indifferently, either as attracting or attracted; in fine, that Attraction is always mutual and reciprocal between two Bodies, and that it is the Proportion between their Masses which decides when this double Attraction shall or shall not be sensible.

XXXII.

Attraction acts uniformly & continually whether the Bodies be at rest or in motion.

There is another Property of Attraction, by which it acts equally on Bodies whether at Rest or in Motion, and produces equal Accelerations in equal Times, from whence it follows that its Action is continued and uniform. Which sufficiently appears from the Manner gravity accelerates falling Bodies, and from the Motion of the Planets, which as we have shewn before, are only greater Projectiles regulated by the same Laws.

XXXIII.

Effect of the Attraction of the planets on the sun

Since the Proportion subsisting between the Masses of Bodies which attract each other determines how much one approaches towards the other, it is evident that the Sun having a much greater Mass than the Planets, their Action on him should be insensible. However the Action of the Planets upon the Sun, tho' too inconsiderable to be sensible, produces its Effect; and on Examination we find that the center round which each Planet revolves is not the center of the Sun, but the Point which is the common center of Gravity of the Sun and Planet, whose revolution is considered. Thus the Mass of the Sun being to that of Jupiter as 1 to $\frac{1}{1067}$ and the distance of Jupiter from the Sun being to the Sun's semi diameter in a Ratio somewhat greater, it follows that the common Center of Gravity of Jupiter and the Sun is not far distant from the Surface of the Sun.

By the same way of reasoning we find that the common Center of Gravity of Saturn and the Sun falls within the Surface of the Sun, and making the same Calculation for all the Planets, *Newton* says (Prop. 12, B, 3.) that if the Earth and all the Planets were placed on the same Side of the Sun, the common Center of Gravity of the Sun and all the Planets would scarce be one of his Diameters distant from his Center. For tho' we cannot determine the Masses of Mercury, Venus and Mars, yet as these Planets are still less than Saturn and Jupiter, which have infinitly less Mass than the Sun, we may conclude that their Masses do not alter this Proportion.

XXXIV.

This effect consists in making the sun oscilliate round the common center of gravity of our planetary system

It is about this common Center of Gravity that the Planets revolve, and the Sun himself oscillates round this Center of Gravity in Proportion to the Actions of the Planets exerted on him. When therefore we consider the Motion of two Bodies whereof one revolves round the other, rigorously speaking we should not regard the central Body as fixed. The two Bodies, viz the central Body and that which revolves round it, both revolve round their common center of Gravity, but the spaces they describe round this common Center being in the inverse ratio of their Masses, the Curve described by the Body which has the least Mass is almost insensible: For this Reason the Curve described by the Body whose revolution is sensible is only considered, and the small Motion of the central Body, which is regarded as fixed, is neglected.

XXXV.

The Earth and the Moon therefore revolve round their common Center of Gravity, and this Center revolves round the Center of Gravity of the Earth and the Sun. The Case is the same with Jupiter and his Moons, Saturn and his Satellites, and with the Sun and all the Planets. Hence the Sun according to the different Positions of the Planets should move successively on every Side around the common Center of Gravity of our planetary System.

XXXVI.

This common center of gravity is at rest.

This common Center of Gravity is at rest, for the different Parts of this System constantly corresponds to the same fixed Stars; now, if this Center was not at rest but moves uniformly in a straight Line, during so many thousand Years that the Heavens have been observed, there must have been remarked some Alteration in the Relation that the different Parts of our planetary System bear to the fixed Stars; but as no Alteration has been observed; it is natural to conclude that the common center of Gravity of our System is at rest. This Center is the Point where all the Bodies of our planetary System would meet if their projectile Forces were destroy'd.

Hence this center cannot be the center of the sun, which moves perpetually.

As the Center of Gravity of our planetary System is at rest, the Center of the Sun cannot be this Center of Gravity since it moves according to the different Positions of the Planets, though on Account of the small Distance between the Center of the Sun and the common Center of gravity of our planetary World it never sensibly recedes from its Place.

XXXVII.

Since Attraction is proportional to the Mass of the attracting Body, and that of the Body attracted, we should conclude that it belongs to every Particle of Matter, and that all the Particles of which a Body is composed attract each other; for if Attraction was not inherent in every Particle of Matter it would not be proportional to the Mass.

XXXVIII.

Answer to the objection founded on the attraction of terestrial bodies not being sensible.

This Property of Attraction, of being proportional to the Masses, supplys us with an Answer to an Objection which has been alledged against the mutual Attraction of Bodies. If all Bodies it is said are endued with this Property of mutually attracting each other, why is not the Attraction which terestrial Bodies exert on each other sensible? but it is easly perceived that Attraction being proportional to the Masses of the Attracting Bodies, the Attraction exerted by the Earth on terestrial Bodies is far more intense than what they exert on each other, and of Consequence these partial Atractions are absorbed and rendered insensible by that of the Earth.

XXXIX.

It is sensible in some cases, as in the deviation of the

The Academicians who measured a Degree of the Meridian in Peru, imagined they perceived a sensible Deviation in the plumb Line occasioned by the Attraction of the Mountain Chimboraco the highest of the Cordiliers it is certain from Theory that the Attraction of this Mountain should affect the

Plumb Line and all Bodies in its Neighberhood: but it remains to know whether the quantity of the observed Deviation corresponds with that which should result from the Bulk of the Mountain for besides that these Observations do not determine the precise Quantity of the Devitation, on account of the errors inseperable from practice, Theory does not furnish any Method of estimating exactly the quantity of this Devitation, as the entire Magnitude, Density &c. of the Mountain are unknown.

plumb line at the foot of chimboraco.

XL.

The same reason that hinders us from perceiving the mutual Attraction of Bodies on the surface of the Earth, renders allso the mutual Attraction of the heavenly Bodies very seldom sensible. For the more powerful Action that the Sun exerts on them, prevents this mutual Attraction from appearing. However in some cases it is perceivable, for instance in the conjunction of Saturn and Jupiter their Orbits are sensibly disturbed, the Attraction of those two Planets being too strong to be absorbed by that of the Sun.

As to the sensible Attractions of certain terestrial Bodies, such as Magnetism and Electricity, they follow other Laws and probably arise from Causes different from the universal Attraction of Matter.

Magnetism and electricity have different causes from the univer sal attraction of bodies

Newton demonstrates (Prop. 66.) that the mutual Attractions of two Bodies revolving round a Third, disturb less the Regularity of their motions when the Body round which they revolve is agitated by their Attractions, than if it was at rest; hence the inconsiderable Irregularities observed in the planetary Motions, is a further Proof of the mutual attraction of the celestial Bodies.

XLI.

The Irregularities in the Motion of any Planet arising from the Actions of the rest, are more or less considerable, in Proportion as the Sum of the Fractions composed each of the Mass and Square of the Distance of each of the other Planets, is more or less considerable with respect to the Mass of the Sun divided by the Square of its distance from the Planet, but as the Planes in which the Planets describe their Orbs are differently situated with respect to each other, the Directions of the Central Forces of which the Planets are the Origin, are each in different Planes, and they cannot be all reduced to fewer than Three, by the Rules of the Composition of Forces; each Planet therefore should be considered as actuated every instant by three Forces at the same Time, the first is a tangential Force, or a Force acting in the Direction of the Tangent of the Planets Orb, which is the Result of the Composition of all the Motions which the Planet was affected with the precedent Instant. The second is an accelerating Force, compounded of all the central Forces of the Planets, reduced to one in a right Line in a Plane whose Position is determined by the Center of the Sun, and by the Direction of the tangential Force; the Difference between this

Manner of determining the irregularities in the motion of the planets arising from their mutual attractions compounded Force and the simple central Force which has no other Source but the Sun, is called the perturbating Force. The third Force is the deturbating Force, compounded of all the same central Forces of the Planets reduced to one in a Direction perpendicular to the Planes of their Orbits; this Force is very small in comparison of the two others, on account of the small Inclination of those Planes to one another, and because the Sun placed in the Intersection of all those Planes does no way contribute to the Production of this deturbating Force. If the Planets were only actuated by the two first Forces their Combination would serve to determine their Trajectories which would be each in a constant Plane, and if the perturbating Force vanished then they would be regular Ellipses, and consequently the Aphelia and Nodes of the Planets would be fixed (Prop. 14. B. 3. & Prop. 1. & 11. B. 1.) if not; these Trajectories might be considered as moveable Ellipses on account of the prodigious excess of the central Force of the Sun over the perturbating Force, it is thus *Newton* investigated the quantity and direction of the Motion of the Line of the Apsides of the Planets occasioned by the Action of Jupiter and Saturn, which according to his Determination follows the Sesquiplicate Proportion of the distances of the Planets from the Sun, from whence he concludes (Prop. 14. B. 3.) that supposing the Motion of the line of the Apsides of Mars in which this Motion is the most sensible to advance in a 100 Years $33^m\ 20^s$ in consequentia, the Aphelia of the Earth, Venus and Mercury would advance $17^m\ 40^s$, $10^m\ 53^s$ & $4^m\ 16^s$ respectively in the same Time.

Abstracting from the mutual attraction of the planets their aphelia are at rest.

The slow motion of the aphelia of the planets is a new proof that attraction acts in the inverse ratio of the square of the distances

This slow Motion of the Aphelia confirms the Law of universal Gravitation, for *Newton* has demonstrated (Cor. 1. Prop. 45.) that if the Proportion of the centripetal Force would recede from the Duplicate to approach to the Triplicate only the 60th Part, the Apsides would advance 3 Degrees in a Revolution, therefore since the Motion of the Apsides is almost insensible, Gravity follows the inverse duplicate Proportion of the distances.

But the deturbating Force which acts at the same Time causes the Planes of those moveable Ellipses to Change continually their Position; let there be supposed in the Heavens an immoveable Plane, in a mean Position between all those the Trajectory of the Earth would take in consequence of the deturbating Force, which may be called the true Plane of the Ecliptic, it is manifest that this Plane being very little enclined to the Plane of the Orbit of Each Planet, it is almost parallel to it, and consequently the Direction of the deturbating Force is always sensibly perpendicular to the true Plane of the Ecliptic, and it is easy to conceive that the effect of this Force produced in the Direction in which it acts, is either to remove the Planets from or to make it approach the true Plane of the Ecliptick, consequently to cause a Variation in the Inclination of the small Arc which the Planet describes that instant with the true Plane of the Ecliptick, the Position of the Planes of the Trajectories of the Planets varies therefore in Proportion of the Intensity of the deturbating Force, and in the Direction in which this Force acts; if for Example the Force tends to make the Planet approach the true Plane of the Ecliptic the Node advances towards the Planet with a Velocity, which tho' small increases diminishes or vanishes according as the intensity of the deturbating Force increases diminishes or vanishes, but in this Case the Node cannot advance or go meet the Planet without moving in an opposite Direction to that of the Planet, if therefore the heliocentric Motion is retrograde as in a great Number of Comets, that of the Nodes will be direct, the contrary would arrive if the deturbating Force tended to remove the Planet from the true Plane of the Ecliptic. *Newton* says that supposing the Plane of the Ecliptic to be fixed the Regression of the Nodes is to the Motion of the Aphelium in any Orbit of a Planet as 10 to 21 nearly (c).

Retrogradation of the nodes of the planets according to Newton.

It is therefore only by this Composition of Forces that all the Irregularities of the celestial Motions can be investigated, it is in discerning the particular Effects of each of those compounded Forces, and afterwards uniting them, that not only those Irregularities that have been observed can be determined, but those which will be remarked hereafter will be foretold. But it is easy to perceive how much sagacity and address to handle the sublimest Analysis these Reschearches require, and as it is almost impossible to combine at once the central Forces of more than three Bodies placed in different Planes, in order to discover the irregularities of the Motions of a Planet or Comet it is necessary to calculate succesively the Variations that each Planet taken seperately can cause in the central Force of which the Sun is the Focus. The Succes that has attended the united Efforts of the first Mathematicians in Europe shall be explained hereafter,

Theory of the Figure of the Planets.

I.

The cause of the rotary motion of the planets has not as yet been discovered.

The Planets have another Motion viz. their Rotation round their Axes, we have seen already, that this Motion of Rotation has only been discovered in the Sun, the Earth, Mars, Jupiter and Venus, and that Astronomers do not agree about the Time in which Venus turns round tho' they are unanimous with respect to its Rotation. But tho' it has not been discovered from Observation that Mercury, Saturn and the Satellities of Jupiter and Saturn turn round their Axes, from the uniformity that Nature Observes in her Operations, it tis highly probable that those Planets revolve round their Axes, and that all the celestial Bodies partake of this Motion.

(c) De Systemate mundi Page 36 Edition, 1731.

from the Center; therefore the Gravities in each of the Canals correspond-ing to the Equator and to the Pole will be as the Distances from the Center of the Bodies, which are placed in those Canals; therefore supposing these Canals to be divided into Parts, proportional to the Wholes, consequently at Distances from the Center proportional to each other, by Transverse Planes, which pass at Distances proportional to those Canals. The Weights of each Part in one of those Canals, will be to the Weights of each correspondent Part in the other Canal, in a constant Ratio, consequently these Weights will be to each other in a constant Ratio of each Part, and their accelerative Gravities Conjointly, that is as 101 to 100, and 500 to 501, that is, as 505 to 501; therefore if the centrifugal Force of any Part of the Equatorial Canal be to the absolute Weight of the same Part as 4 to 505, that is, if the centrifugal Force detracts from the Weight of any Part of the Equatorial Canal $\frac{4}{501}$ Parts, the Weights of the Correspondent Parts of each Canal will become equal, and the Fluid will be in Equilibrio. But we have seen that the Centrifugal Force of any Part under the Equator, is to its Weight as 1 to 289, and not as 4 to 505; the Proportion of the Axes therefore must be different from that of 100 to 101, and such a Proportion must be found as will give the Centrifugal Force under the Equator, only the 289th Part of Gravity.

From whence he concludes the ratio of the axes of the earth to be that of 229 to 230.

But this is easly found by the Rule of Three; for if the Proportion of 100 to 101 in the Axes has given that of 4 to 505 for the Proportion of the Centrifugal Force to Gravity, it is manifest that the Proportion of 229 to 230 is requisite to give the Proportion 1 to 289 of the Centrifugal Force to Gravity.

V.

The flatness or the earth towards the poles would always result from the theory of centrifugal forces and that of fluids what hypothesis of gravity is assumed.

This Conclusion of *Newton*, that is, the Quantity of the Depression of the Earth towards the Poles, which he has determin'd is grounded on his Principle of the mutual Attractions of the Parts of Matter. But this Depression towards the Poles would also result from the Theory of Fluids, and that of Centrifugal Forces, tho' *Newton*'s Discoveries concerning Gravity were rejected, unless very improbable Hypotheses concerning the Nature of primitive Gravity were adopted.

VI.

The measure of the degrees of the meridian in France

Notwithstanding the Authority of *Newton*, and although *Hughens* in assuming a different Hypothesis of Gravity arrived, at the same Conclusion of the Depression of the Earth towards the Poles; and tho' all the Experiments made on Pendulums in the different Regions of the Earth, confirmed the decrease of Gravity towards the Equator, and consequently favoured the opinion of the Flatness of the Earth towards the Poles, yet the Measures of Degrees in France, which seemed to decrease as the Latitude increased still rendered the Figure of the Earth uncertain. Hypotheses were formed on the Nature of primitive Gravity, which gave to the Earth, supposed at rest, a Figure whose Alteration agreed with the Theory of centrifugal Forces, and with the oblong Figure towards the Poles resulting from the actual Measures. occasioned doubts with regard to the figure of the Earth.

For the Question of the Figure of the Earth depends on the Law according to which primitive Gravity acts, and it is certain, for Example, that if this Force depended on a Cause which would make it draw sometimes to one Side and at other Times to another, and which increased or diminished without any constant Law, neither Theory nor Observation ever could determine this Figure.

VII.

To decide this Question finally it was Necessary to Measure a Degree under the Equator, and another within the polar Circle; if the French Astronomers gave Occasion to the Doubts raised concerning the true Figure of the Earth, yet in Justice to them it must be acknowledged, that it is to their indefatigable Industry we are indebted for the Confirmation of the Theory of *Newton*, with Respect to the Figure of the Earth, whose Depression towards the Poles is now universally allowed. The measures of the meridian taken at the polar circle and at the equator confirms the theory of Newton.

VIII.

In determining the Ratio of the Axes of the Earth, *Newton* besides the mutual Attraction of the Parts of Matter supposes the Earth to be an Elliptic Spheroid, and that its Matter is Homogeneous; *Maclaurin* in his excellent Piece on the Tides which carried the Prise of the royal Academy of Sciences in 1740, was the first who demonstrated that the Earth supposed Fluid and Homogeneous, whose Parts attract each other mutually and are besides Attracted by the Sun and Moon, revolving about its Axis, would necessarily assume the Form of an Elliptic Spheroid, and demonstrated further, that in this Spheroid not only the Direction of Gravity was perpendicular to the Surface, and the Central Columns in Equilibrio, but that any Point whatsoever within the Spheroid was equally pressed on every Side; which last Point was no less Necessary to be proved than the two first, in Order to be assured that the Fluid was in Equilibrio, yet had been neglected by all those who before treated of the Figure of the Earth. Two suppositions made by Newton in determining the figure of the earth. Maclaurin verified the first.

The Case is not the same with regard to the second Supposition viz. the Homogeneity of the Matter of the Earth, for it is very possible (and *Newton* himself was of Opinion Prop. 20 B. 3) that the Density of the Earth increases in approaching the Center, now, the different Densities of the Strata of Matter composing the Earth should change the Law according to which the Bodies of which it is composed Gravitate, and of Consequence should alter the Proportion of its Axes. It is probable that the second is false.

IX.

The ratio of the axes of the earth decreases in proportion as gravity increases at the poles.

Clairaut improving on the Researches of Maclaurin has shewn that among all the most probable Hypotheses that can be framed concerning the Density of the interior Parts of the Earth cosidered as an Elliptic Spheroid, that adopting Attraction, there always subsists such a Connexion between the Fraction expressing the Difference of the Axes, and that which expresses the Decrease of Gravity from the Pole to the Equator, that if one of those two Fractions exceeds $\frac{1}{230}$ by any Quantity, the other will be exactly so much less; so that supposing, for Instance, that the excess of the equatorial Diameter above the Axe is $\frac{1}{178}$, a Supposition conformable with the actual Measures, we shall have $\frac{1}{178} - \frac{1}{230}$ or $\frac{1}{898}$ for the Quantity to be subtracted from $\frac{1}{230}$ in Order to obtain the total Abreviation of the Pendulum in advancing from the Pole to the Equator, that is to say, that this Abreviation or what comes to the same the total Diminution of Gravity, will be $\frac{1}{230} - \frac{1}{898}$, or $\frac{1}{141}$ nearly.

Now, as all the Experiments on Pendulums shew that the Diminution of Gravity from the Pole to the Equator, far from being less than $\frac{1}{230}$ as this Theory requires, is much greater, it follows, that the actual Measures in this Point are inconsistant with the Theory of the Earth considered as an Elliptic Spheroid.

It follows from the Theory of *Clairaut*, that admitting, the Suppositions the most natural we can conceive or imagine with regard to the internal Structure of the Earth considered as an oblate Elliptic Spheroid, that the Ratio of the Axes cannot exceed that of 229 to 230 since this Ratio is what arises from the Supposition of the Homogeneity of the Earth, and that it results from this Theory, that in every other Case Gravity increasing, the Depression towards the Poles is less.

Tho' the Earth supposed Fluid and Heterogeneous whose Parts attract each other mutually, assumes an Elliptic Form consistent with the Laws of Hydrostaticks, yet it might equally assume an infinite Number of other Forms consistent with the same Laws, as Dalambert has demonstrated, and as a Variation in the Form would necessarily produce one in the Decrease of Gravity from the Pole to the Equator, and consequently in the Ratio of the Axes, it is highly probable that a Figure will be found that will conduct to a Result such as will reconcile Theory with Observation. The Recherches of this eminent Mathematician shall be explained hereafter.

Newton having computed the Ratio of the Axes of the Earth, determines the Excess of its Height, at the Equator above its Height at the Poles, in the following Manner. The Semidiameter (b + c) at the Equator being to the Semidiameter (b) at the Poles, as 230 to 229, $c = \frac{b}{229}$ and $2b = 458\,c$. and the Mean Semidiameter according to Picart's mensuration, being 19615800 Paris Feet, or 3923, 16 Miles, (reckoning 5000 Feet for a Mile,) $2 \times 19615800 = 2b + c$. consequently $459. c. = 2 \times 19615800$ and the Excess (c) of the Height of the Earth at the Equator, above its Height at the Poles, is 85472 Feet or 17 Miles $\frac{1}{10}$, and Substituting in the Equation $2 \times 19615800 = 2b + c$. for c its Value, there will result $459b = 2 \times 19615800 \times 229$, wherefore the Height (b) at the Poles will be 19573064 and the Height (b+c) at the Equator 19658536 Feet.

X.

After determining the Relation of the Axes of the Earth supposed Homogeneous, *Newton* investigates after the following Manner (Prop. 20 B. 3) what Bodies weigh in the different Regions of the Earth. Since he had proved that the Polar and Equatorial Columns, were in Equilibrio when their Lengths were to each other as 229 to 230 it follows that if a Body (B) be to another (b) as 229 to 230, and the one (B) be placed at the Pole, and the other (b) at the Equator, the Weight (W) of the Body (B) will be equal to the Weight (w) of the Body (b). but if those two Bodies be placed at the Equator the Weight (*W*) of the Body (B) will be to the Weight (w) of the Body (b) as 229 to 230, wherefore the Weight [W] of the Body [B] at the Pole will be to the Weight [*W*] of the same or of an equal Body at the Equator, as 230 to 229, that is reciprocally as those Columns. we see by the same reasoning, that on all the Columns of Matter composing the Spheroid, the Weights of Bodies should be inversely as these Columns, that is as their Distances from the Center: therefore supposing the Distance, of any Place on the Surface of the Earth, from the Center to be known, the Weight of a Body in this Place will be known, and consequently the Quantity of the Increase or Decrease of Gravity, in advancing towards the Poles or the Equator: but as the Distance of any Place from the Center decreases nearly as the Square of the Sine of the Latitude, or as the Verse Sine of double the Latitude as may easily be proved by Calculation, we see how *Newton* formed the Table given (Prop. 20 B. 3) where he lays down the Decrease of Gravity in advancing from the Pole to the Equator.

What are the weights of bodies in the different regions of the earth.

Example. The Latitude of Paris being 48^d 50^m that of Places under the Equator 00^s 00^m and that of Places under the Poles 90^d, the verse Sines of double those Latitudes are 1134, 00000, and 20000, and the Force of Gravity (g) at the Poles being to the Force of Gravity (G) at the Equator as 230 to 229, the Excess (g — G or E) of the Force of Gravity at the Pole, is to the Force of Gravity (G) at the Equator as 230 — 229 to 229, or as 1 to 229 but the Excess (e) of the Force of Gravity in the Latitude of Paris is to the Excess (E) of the Force of Gravity at the Poles as 11334 to 20000, wherefore by the Composition of Ratios, $e \times E$ is to $E \times G$, or the Excess [e] of the Force of Gravity in the Latitude of Paris is to the Force of Gravity [G] at the Equator as 1×11334 to 229×20000, that is, as 5667 to 2290000, and the Force of Gravity [e+G] in the Latitude of Paris is to the Force of Gravity [G] at the Equator as 5667+2290000, that is, as 2295667 to 2290000. By a like Calculus the Force of Gravity in any other Latitude is determined.

They are proportional to the lengths of synchronal pendulums.

As Gravity is the sole Cause of the Oscillations of Pendulums, the slackning of these Oscillations proves the Diminution of Gravity, and their Acceleration proves that Gravity acts more powerfully; but it is demonstrated that the Celerity of the Oscillations of Pendulums is inversely as the Length of the Thread to which they are suspended, therefore when in Order to render the Vibrations of a Pendulum in a certain Latitude synchronal with its Vibrations in another Latitude, it must be shortened or lengthned, we should conclude that Gravity is less or greater in this Region than in the other; *Hughens* has determined the Relation which subsists between the Quantity a Pendulum is lengthned or shortened and the Diminution or Augmentation of Gravity; so that this Quantity being proportional to the Augmentation or Diminution of the Weight, *Newton* has given in his Table the Length of Pendulums instead of the Weights.

Example. The Length of the Pendulum in the Latitude of Paris being 3f. 8l, 561, the Gravity in the Latitude of Paris [2295667] is to the Gravity at the Equator [2290000] as the Length of the Pendulum in the Latitude of Paris [3f. 8l, 561] to the Length of the Pendulum at the Equator [3f. 7l,684] By a like Calculus the Length of the Pendulum in any other Latitude is determined.

XI.

The degrees of latitude are in the same proportion.

The Degrees of Latitude decreasing in the Spheroid of *Newton* in the same Proportion as the Weights, the same Table gives the Quantity of the Degrees in Latitude commencing from the Equator where the Latitude is 0^d to the Pole where it is 90^d

Example. The Length of a Degree [d] at the Poles, being to the Length of a Degree [D] at the Equator, as the Ray of the Circle which has the same Curviture as the Arc of the Meridian at the Pole, is to the Ray of the Circle which has the same Curviture as the Arc of the Meridian at the Equator of the Earth, that is, by the Property of the Ellipsis, as the Cube of 230 to the Cube of 229, that is, as 12167000 to 12008989, the Excess [d—D or E] of the Degree at the Pole is to the Degree [D] at the Equator, as 158011 to 12008989; but the Excess [e] of a Degree in the Latitude of Paris, is to the Excess [E] of the Degree at the Pole, as 11334 to 20000 verse Sines of Double of those Latitudes. Wherefore by the Composition of Ratios e×E is to E×D, or the Excess [e] of a Degree in the Latitude of Paris is to the Length of the Degree [D] at the Equator, as 895448337 is to 12008989000; and the Length [e+D] of a Degree in the Latitude of Paris is to the Length of a

Degree [D] at the Equator, as 120985338337 to 120089890000; but the Length of a Degree in the Latitude of Paris, according to *Picard's*, Mensuration is 57061 Toises, wherefore the Length of a Degree at the Equator is 56637. By a like Calculus the Length of a Degree in any other Latitude is Determined.

XII.

Latitude of the Place.	*Length of the Pendulum.*	*Measure of one Degree in the Meridian.*
Deg.	Feet. Lines.	Toises.
0	3 . 7,468	56637
5	3 . 7,482	56642
10	3 . 7,526	56659
15	3 . 7,596	56687
20	3 . 7,692	56724
25	3 . 7,812	56769
30	3 . 7,948	56823
35	3 . 8,099	56882
40	3 . 8,261	56945
1	3 . 8,294	56958
2	3 . 8,327	56971
3	3 . 8,361	56984
4	3 . 8,394	56997
45	3 . 8,428	57010
6	3 . 8,461	57022
7	3 . 8,494	57035
8	3 . 8,528	57048
9	3 . 8,561	57061
50	3 . 8,594	57074
55	3 . 8,756	57137
60	3 . 8,987	57196
65	3 . 9,044	57250
70	3 . 9,162	57295
80	3 . 9,329	57360
85	3 . 9,372	57377
90	3 . 9,387	57382

XIII.

Newton's Table gives the decrease of Gravity from the Pole to the Equator somewhat less than what results from actual Measures, but this Table is only calculated for the Case of Homogeneity; and he informs us at the End of the Propofition where he gives this Table, that fuppofing the Denfity of the Parts of the Earth to increafe from the Circumference to the Center, the Diminution of Gravity from the Pole to the Equator would alfo increafe.

XIV.

He attributes this difference to the heat at the equator which lengthens the pendulum in thofe regions but leater experiments have fhewn that thofe differences cannot arife from the lengthing of the pendulum produced by the heat in thofe regions.

Altho *Newton* feems inclined to believe, from the Obfervations he relates in Prop. 20 on the lengthning of the Pendulum occafioned by the Heat in the Regions of the Equator, that thefe Differences arrife from the different Temparature of the Places in which the Obfervations have been made, the great Care and Attention employ'd in preferving the fame Degree of Heat by means of the Thermometer in the experiments made fince *Newton's* Time on the Length of Pendulums in the different regions of the Earth proves that thefe Differences do not arife from this Caufe, and that the Decreafe of Gravity from the Pole to the Equator exceeds the Proportion affign'd by *Newton* in his Table.

In Effect the Lengths of the Pendulum Corrected by the Barometer and reduced to that of a Pendulum ofcillating in a Medium without Refiftance

are under the Equator,		439, 21 *Lines*,	
At Portobello Latitude,	9 Degrees,	439, 30	0, 09 *Differences*.
At litle Goave Latitude,	18 Degrees,	439, 47	0, 26
At Paris Latitude,	$48^d\ 50^m$	440, 67	1, 46
At Pello Latitude,	$66^d\ 48^m$	441, 27	2, 06

Now the differences proportional to the Squares of the Sines of the Latitude, are 7, 24, 138, 206, which are lefs than what refults from Experiment.

XV.

Method given by Newton for finding the ratio of the axes of any planet.

At the End of Prop. 19. B. 3. *Newton* fhews how to find the Proportion of the Axes of a Planet whofe Denfity and diurnal Rotation are known, employing for Term of Comparifon the Ratio difcovered between the Axes of the Earth; for whether the Bulk or Ray (r) of a Planet be greater or lefs than the Bulk or Ray (R) of the Earth, if its Denfity (d) be equal to the Denfity (D) of the Earth, and the Time (t) of its diurnal Rotation be equal to the Time (T) of the diurnal Rotation of the Earth, the fame Proportion will fubfift between the centrifugal Force and Gravity, and confequently between its Diameters as was found between the Axes of the Earth: But if its diurnal Rotation is more or lefs rapid than that of the Earth, the centrifugal Force of the Planet will be greater or lefs than the centrifugal Force of the Earth and confequently the Difference of the Axes of the Planet will be greater or lefs than the difference of the Axes of the Earth in the Ratio of $\frac{r}{tt}$ to $\frac{R}{TT}$ (Cor. 2. Prop. 4.) and if the Denfity of the Planet be greater or lefs than the Denfity of the Earth, the Gravity on this Planet will be greater or lefs than the Gravity on the Earth, in the Ratio of d r to DR, and the Difference of the Axes of the Planet will be greater or lefs than the Difference of the Axes of the Earth, in Proportion as the Gravity on the Planet is less or greater than the Gravity on the Earth consequently in the Ratio $\frac{1}{dr}$ to $\frac{1}{DR}$ wherefore if the Time of Rotation and Density of a Planet be different from that of the Earth, the Difference of the Axes of this Planet compared with its lesser Axis, is to $\frac{1}{229}$ the difference of the Axis of the Earth compared with its lesser Axis, as $\frac{r}{tt \times dr}$ to $\frac{R}{TT \times DR}$ which gives $\frac{1}{229} \times \frac{D \times TT}{d \times tt}$ for the expression of the Difference of the Axes of the Planet.

XVI.

Hence the Difference of the diameters of Jupiter, for instance whose diurnal Revolution and Density are known will be to its lesser Axis in the compound Ratio of the Squares of the Times of the diurnal Revolution of the Earth and Jupiter of the Densities of the Earth and Jupiter, and the Difference of the Axes of the Earth compared with its lesser Axis, that is, as $\frac{29}{5} \times \frac{400}{49\frac{1}{4}} \times \frac{1}{229}$ to 1. that is, as 1. to $9\frac{1}{3}$ neerly: Therefore the Diameter of Jupiter from East to West is to its Diameter passing thro' the Poles as $10\frac{1}{3}$ to $9\frac{1}{3}$ neerly. *Newton* adds that in this Determination he has supposed that the Matter of Jupiter was Homogeneous, but as it is probable on account of the Heat of the Sun that Jupiter may be denser towards the Regions of the Equator than towards the Poles, these Diameters may be to each other as 12 to 11, 13 to 12, or even as 14 to 13, and that thus Theory agrees with Observation, since Observation evinces that Jupiter is depressed towards the Poles, and that the Ratio of his Axes is less than that of $10\frac{1}{3}$ to $9\frac{1}{3}$ and is confined between the ratios of 11 to 12 and 13 to 14.

Determination of the ratio of the axes of Jupiter according to this method.

XVII.

This Method that *Newton* takes to explain a Depression towards the Poles of Jupiter less than that which results in the Case of Homogenity seems very improbable, it is surprising that in Order to explain the flatness of the Figure of Jupiter, he has had recourse to a Cause whose Effect would be much more sensibly perceived on the Earth than in Jupiter, since the Earth is much nearer the Sun than Jupiter.

The Proposition of *Clairaut* that the Flatness diminishes as the Density increases towards the Center, furnishes a natural Explication of this Phenomenon in supposing Jupiter denser towards the Center than at the Surface, an Hypothesis entirely consistent with the Laws of Mechanicks.

A very improbable reason assigned by Newton why the flatness of the figure of Jupiter is less than what results from theory.

XVIII.

As the two Principles necessary for determining the Axes namely the diurnal Revolution and the Density, are known only in Jupiter, the Earth, and the Sun, these are the only celestial Bodies the Proportion of whose Axes can be discovered. How this Proportion has been discovered in the Earth

Why the ratio of the axes only of Jupiter the earth and the sun can be found.

The proportion of the axes of the sun is too inconsiderable to be observed.

and Jupiter has been already shewn; the Difference of the Axes of the Sun is to its lesser Axis in the compounded Ratio of the Square of 1 to $27\frac{1}{2}$ diurnal Revolution of the Earth to that of the Sun, of 400 to 100 Density of the Earth to that of the Sun, and $\frac{1}{229}$, Difference of the Diameters of the Earth compared to its lesser Axe, to 1, that is, as $\frac{1}{41175}$ to 1, a Difference too inconsiderable to be observed.

Theory of the Precession of the Equinoxes.

I.

It was thought for a long time that the axis of the earth always preserved its parallelism.

For many Ages it had been thought that the Axis of the Earth during its annual Revolution preserved the same Position, and this Supposition was very natural. For Theory shews that this Parallelism should result from the two known Motions of the Earth, the annual and diurnal Motion; and in Fact for a Number of Years this Parallelism is sensibly preserved. But from the Continuance, and accuracy of Astronomical Observations it has been discovered that the Poles of the Earth are not always directed to the same fixed Stars, and of Consequence that the Axis of the Earth does not always remain parallel to itself.

II.

Hyparchus was the first who perceived the revolution of the poles of the earth.

Ptolemey fixed the duration of this revolution which was called the great year.

Ullughbeig corrected the time assigned by Ptolomey for this revolution.

This Motion of the Axis of the Earth was first perceived by *Hypparchus*; and afterwards established by *Ptolomey* who fixed this Motion to a Degree in a hundred Years, so that the entire Revolution of the Sphere of the fixed Stars from whence *Ptolomey* derived this appearance, was compleated in 36000 Years; and it was generally believed in his Time that at the Expiration of this Revolution called *the great Year*, the celestial Bodies would return to their primitive Position.

The Arabs discovered that *Ptolomey* had made this Motion too slow, *Ullugbbeig* fixed it to a Degree in 72 Years, and Modern Astronomers by fixing it to 51″ annually have confirmed the Discovery of *Ullugbbeig*; so that the Revolution of the Poles of the Earth is compleated in 25920 Years.

III.

This regression causes an apparent motion in the fixed stars.

The equinoctial Points change their Places in the same Time and by the same Quantity as the Poles of the World, and it is this Motion of the Equinoctial Points which is called the Precession of the Equinoxes.

Tho' the fixed Stars are immovable, at least in respect of us, yet as the common intersection of the Equator and Ecliptic Recedes, it is necessary that the Stars which correspond to those Points should continually appear to change their Places, and that they should seem to advance estward, from whence it arrives, that their Longitudes, which is reckoned on the Ecliptic from the Beginning of *Aries*, or the vernal Intersection of the Equator and Ecliptic, continually increases, and the fixed Stars appear to move in *Consequentia*; but this Motion is only apparent and arises from the Regression of the Equinoctial Points in a contrary Direction.

It is the cause why the intersection of the equator and the ecliptic does not correspond to the same star it did formerly, & that the constellations of the zodiac have changed their places.

V.

In Consequence of this Regression, all the Constellations of the Zodiac have changed their Places since the Observations of the first Astronomers; For the Constellation *Aries*, for Example, which in the Time of Hipparchus corresponded to the vernal Intersection of the Equator and Ecliptic, is now advanced into the Sign *Taurus*, and Taurus has passed into *Gemini*, &c. and thus they have taken the Place of each other, but the twelve Portions of the Ecliptic where these Constellations were formerly placed, still retain the same Names they had in the Time of *Hipparchus*.

VI.

Before *Newton* the physical Cause of the Precession of the Equinoxes was utterly unkown, and we shall now proceed to shew how he deduced this Motion from his Principle of universal Gravitation.

We have seen that the Figure of the Earth is that of an oblate Spheroid, Flat towards the Poles and elevated towards the Equator. In Order to explain the Precession of the Equinoxes, *Newton* premises 3 Lemmas, from whence he deduces (Prop. 39. B. 3.) that this Revolution of the equinoctial Points is produced by the combined Actions of the Sun and Moon on the protuberant Matter about the Earth's Equator.

Lemmas with which Newton sets out to deduce this motion from the principle of universal gravitation.

VII.

In the first Lemma he supposes all the Matter by which the Earth considered as a Spheroid would exceed an inscribed Sphere, to be reduced to a Ring investing the Equator, and collects the Sum of all the Efforts of the Sun, on this Ring, to make it Revolve round its Axis which is the common Section of the Plane of the Ecliptic with the Plane passing thro' the Center of the Earth, and Perpendicular to the straight Line connecting the Centers of the Earth and the Sun. In the second Lemma he investigates the Ratio between the Sum of all those Forces, and the Sum of the Forces exerted by the Sun on all the protuberant Parts of the Earth, exterior to the inscribed Sphere. In the third Lemma he compares the Quantity of the Motion of this Ring, placed at the Equator, with that of all the Parts of the Earth taken as a Sphere.

VIII.

To determine the Force of the Sun upon this Protuberant Matter about the Equator of the Earth, *Newton* assumes for *Hypothesis*, that if the Earth was anihilated, and that only this Ring remained, describing round the Sun the annual Orb, and revolving at the same Time by its diurnal Motion, round its Axe, inclined to the Ecliptic in an Angle of 23^d 30^m, the Motion of the Equinoctial Points would be the same, whether the Ring was fluid or composed of solid Matter.

Newton after having investigated the Ratio of the Matter of this supposed Ring, that is, of the Protuberant Matter about the Equator, to the Matter of the Earth taken as a Sphere, and having found it [assuming the Ratio of the Axes of the Earth] to be as 459 to 52441, he proves that if the Earth and this Ring revolved together about the Diameter of this Ring, the Motion (R) of the Ring would be to the Motion (T), of the interior Globe, or to the Motion of the Earth round its Axis, in a Proportion compounded of the Proportion 459 to 52441 of the Matter in the Ring to the Matter in the Earth, and of the Number 1000000 to the Number 800000, or as 4590 to 419528, (a) and consequently that the Motion (R) of the Ring would be to the Motion (R+T) of the Ring and the Globe, in the Ratio of 4590 to 424118.

He found (Prop. 32. B. 3) that the mean Motion of the Nodes of the Moon in a Circular Orbit, is 20^d, 11^m, 46^s, *in Antecedentia*, in a Sydereal Year; and he proved (Cor. 16 Prop. 66) that if several Moons revolved round the Earth, the Motion of the Nodes of each of those Moons would be as their periodic Times. from whence he concludes that the Motion (n) of the Nodes of a Moon revolving near the Surface of the Earth in 23^h, 56^m, would be to 20^d 11^m 46^s, Motion (N) of the Nodes of our Moon in a Year, as 23^h 56^m, the Time of the Earth's diurnal Rotation, to 27^d 7^h 43^s, the periodic Time of the Moon, that is, as 1436 to 39343; and by the Cor. of Prop. 66 the same Proportions hold for the Motion of the Nodes of an Assemblage of Moons surrounding the Earth, whether these Moons were separate, and detached from each other, or if they coalesced supposing them liquified and forming a fluid Ring, or that the Ring became hard and inflexible.

Newton considers the protuberant matter about the equator of the earth as a ring of moons adhering to the globe of the earth.

He deduces from this supposition the manner that the attraction of the sun on the elevation at the equator causes the precession of the equinoxes.

Therefore, the protuberant Matter about the Equator of the Earth being considered as a Ring of Moons adhering to the Earth, and revolving along with it, since the Revolution (n) of the Nodes of such a Ring, is to the Revolution (N) of the Nodes of the Moon, as 1436 to 39343, (according to Cor. 16. Prop. 66.) and that the Motion (R) of the Ring is to the Sum of the Motions (T+R) of the Ring and the Globe to which it adheres, as 4590 to 424118; $n \times R$ is to $N \times \overline{T+R}$, as 1436×4590 to 39343×424118, or $\frac{n \times R}{T+R}$ is to N, as 1436×4590 to 39343×424118; but it is demonstrated that the Sum of the Motions T+R of the Ring and the Globe to which it adheres is to the Motion (R) of the Ring as the Revolution (n) of the Nodes of this Ring to half the annual Motion [$\frac{1}{2}$P.] of the Equinoctial Points of the Body composed of the Ring and Globe to which it ad-

(a) The ratio of the motion of the ring to the motion of the interior globe assigned by Newton, is 4590 to 485223. which is erroneous as shall be shewn hereafter.

heres, (b) wherefore the annual Motion (P.) of the equinoctial Points of the Body composed of the Ring and Globe to which it adheres, will be to the annual Motion of the Nodes (N) of the Moon, in the compounded Ratio of $1436 \times 4590 \times 2$ to 39343×424118.

But *Newton* found (Lem. 2. B. 3.), that if the Matter of the supposed Ring was spread all over the Surface of the Sphere so as to produce towards the Equator, the same Elevation as that at the Equator of the Earth, the Force of the Matter thus spread to move the Earth, would be less than the Force of the equatoral Ring in the Ratio of 2 to 5; therefore the annual Regress of the equinoctial Points is to the annual Regress of the Lunar Nodes, as 1436 $4590 \times 2 \times 2$ to $39343 \times 424118 \times 5$, and consequently in a Sydereal Year it will be 22^s, 58^t, 33^f without any Regard being had to the Inclination of the Axis of the Ring, which Consideration causes still a Diminution in this Motion in the Ratio of the Cosine [91706] of this Inclination (which is 23 $\frac{1}{2}$) to the Radius (100000.)

The mean annual Precession of the Equinoxes produced by the Action of the Sun will be therefore 21^s 6^t nearly, supposing the Earth Homogeneous and the Depression towards the Poles $\frac{1}{217}$.

Simpson found from his Theory 21^s 6^t *(Miscellaneous Tracts)* D'Alambert 23^s nearly *(Recherches Sur la Precession des Equinoxes)* Euler 22^s *(Mem. de Berlin Tom. 5. 1749).* And if this Quantity is greater by a third than what Observation indicates, it probably arises from the Earth's not being Homogeneous, as was supposed, the Researches of Simpson, Euler, and D'Alambert relative to this Object shall be explained hereafter.

IX.

Irregularities in the motion of the equinoctial points produced by the action of the sun.

In this Manner *Newton* determined the mean Quantity of the Motion of the equinoctial Points. But not without examining the different Varieties of the Action of the Sun on the protuberant Matter about the Equator supposed to be reduced to a Ring.

He shews in Cor. 18, 19 and 20 of Prop. 66 that by the Action of the Sun the Nodes of a Ring, supposed to encompas a Globe as the Earth, would rest in the Syfigies, in every other Place they would move in *Antecedentia,* they would move swiftest in the Quadratures, that the Inclination of this Ring, would vary, that during each annual Revolution of the Earth, its Axe would Oscillate, and at the end of each Revolution would return to its former Position, but that the Nodes would not return to their former Places, but would still continue to move in *Antecedentia.*

(b) Newton supposes that the Sum of the Motions of the Ring and the Globe to which it adheres is to the Motion of the Ring, as the Revolution of the Nodes of this Ring is to the annual Motion of the Equinoctial Points of the Body composed of the Ring and Globe to which it adheres, in which he is mistaken as shall be shewn hereafter.

X.

The action of the sun on the protuberant matter about the equator causes an annual nutation of the axis of the earth.

The greatest Inclination of the Ring should happen when its Nodes are in the Sysigies, afterwards in the Passage of the Nodes to the Quadratures, this Inclination should diminish, and the Ring by its Effort to change its Inclination, impresses a Motion on the Globe, and the Globe retains this Motion, till the Ring, or the protuberant Matter about the Equator, (for it is the same Thing according to *Newton*) by a contrary Effort destroys this Motion, and impresses a new Motion in a contrary Direction.

Hence we see that the Axis of the Earth should change its Inclination with Respect to the Ecliptic, twice in its annual Course and return twice to its former Position.

If the earth was elevated towards the poles and depressed towards the equator the equinoctial points would advance in stead of retrograding.

Newton has shewn in Cor. 21 of Prop. 66 that the protuberant Matter about the Equator making the Nodes retrograde, the Quantity of this Matter increasing, this Regression, would increase, and would diminish when this Matter diminished; hence if there was no Elevation towards the Equator, there would be no Regression of the Nodes, and the Nodes of a Globe, which instead of been Elevated towards the Equator was depressed, and consequently would have its protuberant Matter about its Poles, would move in *Consequentia*.

Which proves the depression of the earth towards the poles.

And he adds, (Cor. 22 of Prop. 66) that as the Form of the Globe enables us to judge of the Motion of the Nodes, so from the Motion of the Nodes we may infer the Form of the Globe; and consequently if the Nodes move in *Antecedentia*, the Globe will be elevated towards the Equator, but on the Contrary depressed, if the Nodes move in *Consequentia*, which is a further Proof of the Flatness of the Earth towards the Poles.

XI.

The moon contributes to the production of the motion of the equinoctial points

That the action of the moon on the protuberant matter about the equator is more powerful than that of the sun.

We have hitherto considered only the Action of the Sun in explaining the Precession of the Equinoxes, and we have seen that in Consequence of this Action the equinoctial Points would receede annually 21° 6'. But the Moon by her Attraction Acts on the Earth and influence very sensibly this Phenomenon, its Action being to that of the Sun as 2½ to 1 (c) if the Inclination of its Orbit to the Equator was always the same as that of the Ecliptic to the Equator, the Regression thence resulting would be to that arising from the Sun's Action as 2 ½ to 1. But because its Nodes shift continually their Places, it happens that the Inclination of its Orbit to the Equator, on which depends its Effect varies continually, so that when the ascending Node is in Aries, the Inclination of the Moon's Orbit to the Equator a-

(c) The Proportion of the Force of the Sun to that of the Moon, assigned by Newton is 1 to 4, 4815. which he also assigns for the Proportion of the Precession of the Equinoxes produced by the Sun to that produced by the Moon but this Proportion does not agree with the Theories which depend on the Determination of the Mass of the Moon, and it appears from Computation as shall be shewn hereafter, that the Precession of the Equinoxes produced by the Sun and that produced by the Moon are not in the same Proportion as the Forces of those Luminaries.

amounts to $28^d \frac{1}{2}$, but when the ascending Node nine Years after, is in Libra it scarce amounts to 18. $\frac{1}{2}$ in each Revolution, which renders the Precession arising from the Action of the Moon very unequal during the Space of 18 Years, and Causes a Nutation in the Axis of the Earth, whereby its Inclination to the Ecliptic varies during the Revolution of the Nodes of the Moon; after which it returns to its former Position. This Nutation from Theory, amounts to 19^s, agreable to Observation, the mean Precession arising from the Action of the Moon, to 35^s, 5, consequently the Precession arising from the Action of the Sun to 14^s, 5, and the greatest Difference between the true Precession arising from the Action of the Moon, and the mean Precession amounts to 17^s, 8.

Nutation of the axis of the earth produced by the action of the moon.

Theory of the Ebbing and Flowing of the Sea.

I.

It is very easy to perceive the Connection between the Ebbing and Flowing of the Sea and the Precession of the Equinoxes. *Newton* deduces his Explication of the Ebbing and Flowing of the Sea, from the same Corollaries of Prop. 66, from whence we have seen he drew his Explication of the Precession of the Equinoxes; those two Phenomena are both one and the other a necessary Consequence of the Attractions of the Sun and Moon on the Parts which compose the Earth.

The explication of the ebbing and flowing of the sea, is deduced from prop. 66 & its cor. as is that of the precession of the equinoxes.

II.

Galileo imagined that the Phenomena of the Tides might be accounted for, from the Motion of Rotation of the Earth, and its Motion of translation round the Sun. But if this great Man had more attentively examined the Circumstances attending the Ebbing and Flowing of the Sea, he would have perceived that in Consequence of the diurnal Motion of the Earth, the Sea indeed would rise towards the Equator, and that the Earth would assume the Form of a Spheroid depressed towards the Poles, but this Motion of Rotation would never produce in the Waters of the Sea a Motion of Flux and Reflux, as *Newton* has demonstrated Cor. 19. Prop. 66. *Newton* Proves in this same Corollary, applying what he had demonstrated in Cor. 5 and 6 of the Laws of Motion, that the Translation of the Earth round the Sun has no Effect on the Motion of Bodies at its Surface, and consequently the Motion of Translation of the Earth round the Sun, cannot Produce the Motion of Flux and Reflux of the Sea.

Error of Galileo concerning the ebbing and flowing of the sea.

III.

On examining the Circumstances which attend the Ebbing and Flowing of the Sea, it was easy to perceive that those Phenomena depended on the Position of the Earth with Respect to the Sun and Moon; but it was not so, to discover the Manner those two Luminaries Produce those Phenomena and

The ebbing and flowing of the sea arises from the action of the sun and moon on the waters.

the Quantity that each contributes to their Production: we see but the Effects in which the Actions of those two Luminaries are so confounded, that it is only by the Assistance of *Newton*'s Principles we are enabled to distinguish one from the other, and assign their Quantity. It was reserved for this great Man, to discover the true Cause of the Ebbing and Flowing of the Sea, and to reduce those Causes to Computation; we shall now trace the Road which conducted him to those Discoveries.

IV.

Road which conducted Newton to assign the quantity that each of those luminaries contribute to produce those phenomena.

He begins by examining in Prop. 66. the Principle Phenomena which should Result from the Motion of three Bodies which attract each other mutually in the inverse Ratio of the Squares of the Distances, the small ones Revolving round the greater.

After having shewn in the first 17. Corollaries of this Prop. the Irregularities which the greater Body would Cause in the Motion of the lesser, which itself revolves round the third, and by this Means having laid the Foundation of the Theory of the Moon, he considers in Cor. 18 several fluid Bodies which revolve round a third, he afterwards supposes that those fluid Bodies all become contiguous so as to form a Ring revolving round the central Body, and proves that the Action of the greatest Body would produce in the Motions of this Ring the same Irregularities as in those of the solitary Body in whose Place the Ring was substituted; infine Cor. 19. he supposes the Body round which this Ring Revolves to be extended on every Side as far as this Ring, that this Body which is solid contains the Water of this Ring in a Channel cut all round its Circumference, and that it revolves uniformly round its Axis, he then proves that the Motion of the Water in this Channel will be accelerated and retarded alternately by the Action of the greater Body and that this Motion will be swifter in the Sysigies of this Water, and slower in its Quadratures, and finally that this Water will Ebb and Flow after the Manner of the Sea.

Newton applies this Prop. 66 and its Cor. to the Phenomena of the Sea (Prop. 24. B. 3.) and proves that they are a necessary Consequence of the combined Actions of the Sun and Moon on the Parts which compose the Earth.

V.

He afterwards investigates the Quantity, each of those Luminaries contribute, to the Production of those Phenomena. As this Quantity depends on their Distances from the Earth, the nearer they are to the Earth, the greater the Tides should be, Cæteris Paribus, when their Actions, conspire together: and according to Cor 14. Prop. 66, those Effects are in the Inverse Ratio of the Cubes of their Distances from the Earth and the simple Ratio of their Masses.

Newton examines first the Action of the Sun on the Waters of the Sea, because its Quantity of Matter with Respect to that of the Earth is known. He observes that the Attraction of the Sun on the Earth is counterbalanced as to the Totality by the centrifugal Force arising from the annual Motion of the Earth, which he considers as uniform and circular: But what is true as to the Totality is not so as to each particle of the Earth, that is, that the centrifugal Force of each of those Particles cannot be supposed equal to the Force with which the same Particle is Attracted by the Sun, since each Particle has the same centrifugal Force, and the Particles of the Earth which are nearer the Sun are more attracted than those which are remoter. Thus the Distance of the Earth from the Sun, being 22000 Semidiameters of the Earth, and the Law of Attraction, the inverse Ratio of the Squares of the Distances, the Attractive Force corresponding to the Point of the Earth nearest the Sun, to the Center of the Earth, and to the Point of the Earth remotest from the Sun, will be nearly as 11001, 11000 and 10999, and as the Sun's Attraction balances the centrifugal Force of each Particle of the Earth, this Force will be Proportional to 11000; if from the attractive Force of the Sun on each of those three Points, the centrifugal Force be Subducted, there will remain 1, 0,—1; which proves that the Center of the Earth is at Rest with Respect to the Motions of the Waters of the Sea, and that the two Extremities of the Diameter of the Earth directed towards the Sun, are actuated by equal Forces with opposite Directions, whereby the Parts tend to recede from the Center of the Earth.

First source of the ebbing and flowing of the sea.

If in the same Diameter there be taken two Points equally distant from the Center, those two Points will be likewise actuated by equal Forces with opposite Directions, whereby they tend to recede from the Center; but this Force will decrease as the Distance from the Center of the Earth. this Diameter of the Earth directed to the Center of the Sun may be called the Solar *Axis* of the Earth, if we now consider the Equator corresponding to this Axe, it is evident that each Point taken in the Plane of this Equator may be supposed equally distant from the Center of the Sun, and consequently that none of the Points of this Plane are affected by the Inequality between the centrifugal Force and attractive Force, and consequently their Gravity towards the Center of the Earth will not be diminished, therefore if we conceive two Canals full of Water the one passing thro' the demi solar Axe, and the other thro' a Ray at its Equator, which communicate at the Center of the Earth, the Water will ascend in the first and descend in the other, this will happen both in the one and the other demi solar Axe, and is the first Source of the Ebbing and Flowing of the Sea.

Second source of the ebbing and flowing of the sea.

Each Particle of Water in the Canal of the demi solar Axe is attracted towards the Sun in the Direction of the Canal, but this Force acts on the Particles of Water in the other Canal, obliquely, it therefore should be resolved into two, one perpendicular to the Canal, and the other parallel to it. The first may be considered as perfectly destroied by the centrifugal Force; but the other Force adds to the Gravity of each Particle in this Canal, this small Force does not exist in the Canal of the demi solar Axe, and for this Reason the Water will descend in the Canal of the solar Equator, and will sustain that of the solar Axis to a greater Height. This is the second Source of the Ebbing and Flowing of the Sea.

From whence it appears that the Ascent of the Waters of the Sea does not arise from the total Action of the Sun, but from the Inequalities in that Action on the Parts of the Earth. *Newton* observes that in Consequence of this Action the Figure of the Earth (abstracting from its diurnal Motion) ought to be an elliptic Spheroid having for greater and lesser Axes the solar Axe and the Diameter of its Equator, and determines in the following Manner the Force of the Sun which produces the difference of those Axes.

Determination of the force of the sun, producing the elevation or depression of the waters of the sea in two points diametrially opposite.

He considers the Figure of the Earth (abstracting from its diurnal Motion) rendered Elliptic by the Action of the Sun, as a similar Effect to the Figure of the Orbit of the Moon, (abstracting from its excentricity) which he had shewn (Prop. 66. Cor. 5) to be rendered Elliptic and to have its Center in the Center of the Earth, by the same Action. He demonstrated (Prop. 25 B. 3) that the Force (F) which draws the Moon towards the Sun, is to the centripetal Force (g) which draws the Moon towards the Earth, as the Square of the periodic Time (tt) of the Moon round the Earth, to the Square of the periodic Time (TT) of the Earth round the Sun, according to Cor. 17 of Prop. 66; but the Inequality (V) in the Action of the Sun on the Parts of the Earth being to its Action (G), as the Ray (r) of the Earth, to the Ray (R) of its Orbit, and the Force (G) of the Sun which retains the Earth in its Orbit, being to the Force (g) which retains the Moon in its Orbit, as $\frac{R}{TT}$ Ray of the Earth's Orbit divided by the Square of its periodic Time, to $\frac{b}{tt}$ Ray of the Moon's Orbit divided by the Square of its periodic Time (Cor. 2 Prop. 4), $V \times G$ is to $G \times g$, or the Inequality (V) in the Action of the Sun on the Parts of the Earth, is to the centripetal Force (g) of the Moon towards the Earth as $\frac{r \times R}{TT}$ to $\frac{R \times b}{tt}$, that is, as the Ray of the Earth divided by the Square of its periodic time round the Sun ($\frac{r}{TT}$) to the Ray of the Moon's Orbit, divided by the Square of its periodic Time round the Earth ($\frac{b}{tt}$)

Wherefore by the Composition of Ratios, $g \times V$ is to $F \times g$, or the Force (V) of the Sun disturbing the Motion of Bodies on the Surface of the Earth, is to its Force (F) with which it disturbs the Motion of the Moon, as $\frac{TT \times r}{TT}$ to $\frac{tt \times b}{tt}$ or as the Ray (r) of the Earth, to the Ray (b) of the Moon's Orbit, that is, as 1 to 60 ¼.

To compare now those two Forces with the Force of Gravity at the Surface of the Earth. Since the Force (F) which draws the Moon towards the Sun, is to the centripetal Force (g), which would retain the Moon in an Orbit, described about the Earth quiescent at its present Distance ($60\frac{1}{2}$ Semidiameters of the Earth) as the Square of $27^d. 7^h. 43^m.$ to $365^d. 6^h. 9^m.$ or as 1000 to 178725, or as 1 to $178\frac{29}{40}$; and that the Force which retains the Moon in its Orbit, is equal to the Force (γ) which would retain it in an Orbit described about the Earth quiescent in the same periodic Time, at the Distance of 60 Semidiameters, according to Prop. 60, in which it has been demonstrated that the actual Distance ($60\frac{1}{2}$ Semidiameters) of the Centres of the Moon and Earth, both revolving about the Sun, and at the same Time about their common Centre of Gravity, is to the Distance (60 Semidiameters) of their Centres, if the Moon revolved about the Earth quiescent in the same periodic Time, as the Sum ($1+42$) of the Masses of the Moon and Earth, to the first of two mean Proportionals ($42\frac{1}{2}$) between that Sum and the Mass of the Earth. Consequently that the Force (γ) which retains the Moon in its Orbit is less than the Force (g) which would retain it in an Orbit desccribed in the same periodic Time, about the Earth quiescent at the Distance $60\frac{1}{2}$ Semidiameters, in the Ratio of 60 to $60\frac{1}{2}$, (Cor. 2, P. 4); by the Composition of Ratios $F \times g$ is to $g \times \gamma$ or the Force (F) which draws the Moon towards the Sun, is to the centripetal Force (γ) which retains the Moon in its Orbit, as $1 \times 60\frac{1}{2}$ to $178\frac{29}{40} \times 60$. but this Force ($\gamma$) which retains the Moon in its Orbit, (in approaching the Earth) increasing in the inverse Ratio of the Square of the Distance, is to the Force (G) of Gravity as 1 to 60×60, wherefore $\gamma \times F$ is to $\gamma \times G$, or the Force (F) which draws the Moon towords the Sun, is to the Force (G) of Gravity as $1 \times 60\frac{1}{2}$ to $60 \times 60 \times 60 \times 178\frac{29}{40}$ or as 1 to 638092,6.

From whence *Newton* concludes [Prop. 36. B. 3.] that since the Ascent of the Waters of the Sea, and the Elliptic Figure of the Lunar Orbit [abstracting from its Excentricity] are similar Phenomena arising from the Solar Force, and that in descending towards the Surface of the Earth this Force decreases in the Ratio of $60\frac{1}{2}$ to 1. the Force of the Sun which depresses the Waters of the Sea in the Quadratures, or at the Solar Equator, is to the Force of Gravity as 1 to $638092,6 \times 60\frac{1}{2}$ or as 1 to 38604600. But this Force is double in the Sysiges, or in the Direction of the Solar Axis of what it is in the Quadratures, and acts in a contrary Direction [Cor. 6. Prop. 66], wherefore the Sum of the two Forces of the Sun on the Waters of the Sea, in the Quadratures and Sysigies, will be to the Force of Gravity as 3 to 38604600 or as 1 to 12868200. those two Forces united Compose the total Force which raises the Waters of the Sea in the Solar Canal, their Effect

Proportion of the action of the sun on the waters of the sea to the force of gravity.

being the same as if they were wholy employ'd in raising the Waters in the Sysigies, and had no Effect in the Quadratures.

VI.

Newton concludes from his theory that the sun raises the water of the sea to 2 feet.

Newton after having investigated the Force of the Sun which produces the Elevation of the Waters in the Solar Canal, determines in the following Manner the Quantity of this Elevation. He considers the Elevation of the Waters of the Sea arising from the Action of the Sun, as an Effect similar to the Elevation of the Equatorial Parts above the Polar Parts of the Earth, arising from the centrifugal Force at the Equator. Now the centrifugal Force (C) at the Equator being to the Force of Gravity (G) at the Surface of the Earth as 1 to 289, and the Force of the Sun (F) exerted on the Waters of the Sea being to the Force of Gravity (G), as 1 to 12868200, by the Composition of Ratios, F×G is to C×G, or the Force (F) of the Sun exerted on the Waters of the Sea, is to the centrifugal Force (C) at the Equator, as 1×289 to 1×12868200 or as 1 to 44527; consequently the Elevation (85472 Feet) at the Equator produced by the centrifugal Force, is to the Elevation of the Waters in the Solar Canal produced by the Action of Sun, as 1 to 44527. which shews that the Elevation of the Waters in the Places directly under the Sun and in those which are directly opposite to them is 1 Foot, 11, $\frac{1}{10}$ Inches.

VII.

The ebbing and flowing of the sea arises from the motion of rotation of the earth and from the actions of the sun and moon.

The fluid Earth would preserve a Spheroidal Form its longest Diameter pointing to the Sun without any Ebbing or Flowing of its Waters, if it had no Motion of Rotation. It is therefore the Rotation of the Earth round its Axis joined to its oblong Figure which causes alternatly a Depression and Elevation of the Waters of the Sea. If the Axis of Rotation and the Solar Axis were the same, the Waters of the Sea would have no Motion of reciprocation, because each Point during the Rotation of the Earth would be constantly at the same Distance from the Solar Poles. But as those two Axes form an Angle, it is easy to perceive that each Point of the Surface of the Earth approaches and recedes alternatly from the Solar Poles and that twice in a Revolution, and the Waters will continually rise in this Point during its Approach to, and will fall continually during its Recess from those Poles. *Newton* investigated the Relation which subsists between the Elevation of the Waters in any Place above that at the Solar Equator and their Elevation in the Solar Canal; and found that the Square of the Radius [1] is to the Square of the Sine [ss] of the Altitude of the Sun in any Place, as the Elevation [S] of the Waters in the Solar Canal to their Elevation [ssS] in that Place.

Method of estimating the action of the sun on the waters of the sea in any place.

VIII.

It is Manifest that what has been said with Respect to the Sun should be applied without Restriction to the Moon and all the Phenomena of the Tides prove evidently that the Action of this Luminary on the Waters is considerably greater than that of the Sun, which at first View should seem the more surprising, as the Attractive Force of the Sun arising from its immense Bulk is so powerful as to Force the Earth to Revolve round it, whilst the Irregularities produced in its Orbit by the Action of the Moon are scarce sensible, but if we consider that the Motion of the Sea proceedes from its Parts being differently attracted from those of the rest of the Earth, because their Fluidity makes them receive more easily the Impressions of the Forces which Act on them, it will appear, that the Action of the Sun which is very powerful on the whole Earth attracts all its Parts almost equally on Account of its great Distance; but the Moon being much nearer the Earth Acts more unequally on the different Parts of our Globe, and that this Inequality should be much more sensible than that of the Sun; these inequalities being in the Inverse Ratio of the Cubes of the Distances of the Luminaries from the Earth, and in the simple Ratio of their Quantities of Matter.

How is it possible that the attraction of the moon can have such i. fluence on the waters of the sea & cause so little alterations in the motion of the earth.

The Elevation of the Waters of the Sea arising from the Action of the Moon, in the Direction of the lunar Axis, above their Height at the lunar Equator, being once determined, the Elevation of the Waters of the Sea in any Place above their Height at the lunar Equator, will be found, for in this Case, as in that of the Sun, the Square of the Radius (1) is to the Square of the Sine [tt] of the Altitude of the Moon in any Place, as the Elevation [L] of the Waters in the Direction of the lunar Axis, above their Height at the lunar Equator, to their Elevation [tt L] above the same Height, in that Place.

IX.

From the Combination of the Actions of the Sun and Moon on the Waters of the Sea there result two Tides, viz. the *solar Tides* and *lunar Tides*, which are produced independently of each other. Those two Tides by being confounded with each other appear to Form but one, but subject to great Variations, for in the Sysigies the Waters are elevated and depressed at the same Time by both one and the other Luminary, and in the Quadratures the Sun raises the Waters where the Moon depresses them, and reciprocally the Sun depresses the Waters where the Moon raises them, [one being in the Horison when the other is at the Meridian] so that from the Actions of those Luminaries sometimes conspiring and at other Times opposed, there result very sensible Variations both with respect to the Height of the Tides and their Time.

The variations in the tides arise from the conjoint actions of the sun and moon.

X.

It is demonstrated that the Elevation of the Waters, produced by the conjoint Actions of the Sun and Moon, is sensibly equal to the Sum of the Elevations produced by the Actions of each seperately, wherefore the whole Elevation produced by the united Actions of the two Luminaries will be Expressed by $ssS+ttL$; which shews that the Elevation of the Waters in any Place will continually increase until they attain their greatest Height, and then it is high Water, after which it will continually decrease during six Hours, and then it will be low Water; the Difference between those two Heights is called *the Height of the Tide*; from whence it appears that the Height of the Tides depends upon a great Number of Circumstances, viz. the Declination of each Luminary, the Age of the Moon, the Latitudes of Places and the Distance of the two Luminaries from the Centre of the Earth.

XI.

How Newton came to estimate the action of the moon on the waters of the sea.

To examine the Variations in the Height of the Tides according to all those Circumstances, let us first suppose the Orbit of the Moon and that of the Sun in the Plane of the Equator, and let us further suppose them perfectly Circular, and let a Place be chosen at the Equator; in which Case we may suppose $s=1$ and $t=1$, which will happen at the appulse of the Luminaries to the Meridian in the Sysiges, and the whole Elevation will be expressed by $S+L$; about six Hours after $s=0$ and $t=0$ nearly and the Waters will have no Elevation consequently the Height of the Tides in the Sysigies will be expressed by $S+L$; but in the Quadratures at the appulse of the Moon to the Meridian $t=1$ and $s=0$, and the Elevation of the Waters will be expressed by L, about six Hours after $s=1$ and $t=0$ nearly, and the Elevation of the Waters will be evpressed by S and the Height of the Tide will be expressed by $L-S$, consequently the Height of the Tides in the Sysigies and Quadratures will be as $S+L$ to $L-S$. if therefore the Height of the Tides in the Sysigies and Quadratures at the Time of the Equinoxes was determined from Observation, on the Coast of an Island situated near the Equator, in a deep Sea, and open on every Side to a great extent, the Ratio of L to S, the Effects of the Forces of the Sun and Moon, or the Ratio of those Forces which are proportional to those Effects, would be found.

As no such Observations have been made, *Newton* employs for determining the Ratio of those Forces the Observations made by Sturmy three Miles below Bristol. this Author relates that the Height of the Ascent of the Waters in the vernal and autumnal Conjunction and Opposition of the Sun and Moon, amounts to about 45 Feet, but in the Quadratures to 25 only, wherefore $L+S$ is to $L-S$ as 45 to 25 or as 9 to 5, consequently $5L+5S=9L-9S$, or $14S=4L$ and S is to L as 2 to 7.

To reduce this Determination to the mean State of the variable Circumstances; it is to be observed 1° that in the Sysigies the conjoint Forces of the Sun and Moon being the greatest, it has been supposed that the corresponding Tide is also the greatest, but the Force impressed at that Time on the Sea being increased by a new Though a less Force still acting on it until it becomes too weak to raise it any more, the Tides do not rise to their greatest Height but some Time after the Moon has passed the Sysigies, *Newton* from the Obſervations of Sturmy concludes that the greateſt Tide follows next after the Appulſe of the Moon to the Meridian when the Moon is diſtant from the Sun about 18d ½. the Sun's Force in this Diſtance of the Moon from Syſigies being to the Force [S] in the Syſigies, as the Coſine [7986355] of double this Diſtance, or of an Angle of 37 Degrees, to the Radius [10000000] in the Place of L+S in the preceding Analogy L+o, 7986355 S is to be Subſtituted. In the Quadratures the conjoint Forces of the Sun and Moon being leaſt, it was alſo ſuppoſed that the leaſt Tide happens at that Time, but the Sea looſes its Motion by the ſame Degrees that it acquired it, ſo that the Tides are not at their leaſt Height until ſome Time after the Moon has paſſed the Quadratures, and *Newton* from the ſame Obſervations of Sturmy concluded that the leaſt Tide follows next after the Appulſe of the Moon to the Meridian when the Moon is diſtant from the Quadratures 18d ½. Now the Sun's Force in this Diſtance of the Moon from the Quadratures being to the Force [S] in the Quadratures, as the Coſine (7986355) of double this Diſtance or of an Angle of 37 Degrees, to Radius (10000000) in the Place of L—S in the preceding Analogy, L—o, 7986355S is to be Subſtituted. Reduction of this eſtimation to the mean ſtate of the variable circumſtances.

It is to be obſerved 2° that the Orbit of the Moon was ſuppoſed to Coinſide with the Plane of the Equator, but the Moon in the Quadratures, or rather 18d ½ paſt the Quadratures, declines from the Equator by about 22d 13m, now the Force of the Moon in this diſtance from the Equator being to its Force (L) in the Equator, as the Square of the Coſine (8570327) of its Declination 22d 13m, to Radius (10000000) in the Place of L—o, 7986355S in the preceding Analogy 0,8570327L—0,7986355S is to be Subſtituted.

It is to be obſerved 3° that the Orbits of the Sun and Moon were ſuppoſed to be perfectly Circular, and conſequently thoſe Luminaries to be in their mean Diſtances from the Earth, But *Newton* demonſtrated that the lunar Orbit (abſtracting from its Excentricity) ought to be an Elliptic Figure, having its Centre in the Centre of the Earth and the ſhorteſt Diameter directed to the Sun; and determined (Prop. 28. B. 3.) the Ratio of this ſhorteſt Diameter to the longeſt or the Diſtance of the Moon from the Earth in the Syſigies and Quadratures to be as 69 to 70. To find its Diſtance when 18 ½ Degrees advanced beyond the Syſigies, and when 18 ½ Degrees paſſed by the Quadratures, it is to be obſerved that in an Ellipſis if the longeſt Semidiameter be expreſſed by (a) its ſhorteſt by [b] and the Difference of the Squares of the longeſt and ſhorteſt Semidiameters by [cc] and the Sine of the Angle which any Diameter [y] makes with the longeſt Semidiameter by [s] $yy = \frac{aabb}{aa-ssc c}$ wherefore ſubſtituting ſucceſſively in this Expreſſion 69 for [a] 70 for [b] for [s] 3173047 and 9483236 the Sines of 18 ½ Degrees and 71 ½ Degrees: thoſe Diſtances will be 69,098747 and 69,897345 and the mean Diſtance will be 69 ½ as equal to half the Sum

Computation founded on the Laws of Equilibrium, wherefore the great Spring Tides and Neap Tides are in a greater Ratio according to the Laws of Equilibrium than that of 9 to 5.

Force of the moon according to Bernoully. Bernoully supposes them to be to each other as 7 to 3, consequently that the Force (L) of the Moon is to the Force (S) of the Sun as 5 to 2. A proportion which answers better to the Observed Variations in the duration and interval of the Tides (Variations which receive no Alteration from the above mentioned secondary Causes) and to the other Theories which depend on a Determination of the Force of the Moon. Hence the Density of the Moon is to the Density of the Earth as 7 to 10, the Quantity of Matter in the Moon is to the Quantity of Matter in the Earth as 1 to 70, and finally the accelerative Gravity at the Surface of the Moon is to the accelerative Gravity on the Surface of the Earth as 1 to 5.

XIV.

Singular figure of the moon. If the Moon's Body were Fluid like our Sea it would be elevated by the Action of the Earth in the Parts which are nearest to it and in the Parts opposite to these, and *Newton* enquires into the Quantity of this Elevation. He observes that the Elevation ($8\frac{3}{5}$) of the Earth produced by the Action of the Moon would be to the Elevation (E) of the Moon (if it had the same Diameter as the Earth) produced by the Action of the Earth as the Quantity of Matter in the Moon to the Quantity of Matter in the Earth, or as 1 to 39,788. and the Elevation (E) produced by the Action of the Earth in the Moon if it had the same Diameter as the Earth, is to the real Elevation (x) produced in the Moon by the Action of the Earth, as the Diameter of the Earth to the Diameter of the Moon or as 365 to 100. wherefore by the Composition of Ratios $8\frac{3}{5} \times E$ is to $E \times x$ or the Elevation of the Earth ($8\frac{3}{5}$) produced by the Action of the Moon is to the real Elevation of the Moon produced by the Action of the Earth as 1×365 to $39,788 \times 100$ or as 1081 to 100 or $x = 93$ Feet. consequently the Diameter of the Moon that passes through the Centre of the Earth, must exceed the Diameter which is perpendicular to it by 186 Feet. Hence it is, that the Moon always turns the same Side towards the Earth.

Effect of the oblong spheroidal figure of the moon. In Effect La Grange in his Piece which carried the Prize of the royal Academy of Sciences in the Year 1764, supposing with *Newton* that the Moon is a Spheroid having its longest Diameter directed towards the Earth, has found that this Planet should have a libratory or oscillatory Motion about its Axis, whereby its Velocity of Rotation is sometimes accelerated and other Times retarded, and that then the Moon should always turn the same Side nearly towards the Earth, though it did not receive in the Beginning a Motion of Rotation whose Duration was equal to that of its Revolution. La Grange has demonstrated also that the Figure of the Moon might be such that the Precession of its equinoctial Points or the Retrogradation of the

Nodes of the lunar Equator, would be equal to the retrograde Motion of the Nodes of the lunar Orbit; and in this Case he found that the lunar Axis would have no sensible Nutation. The Action of the Sun in all those Inquiries, is almost insensible with respect to that of the Earth; it is the Earth which produces the Motion of the Nodes of the lunar Equator, by acting more or less obliquely on the lunar Spheroid; hence the Precession of the lunar Equator, and the Law of the Motion produced in the lunar Spheroid, differ very much from that which is observed in the Equator of the Earth. The Researches of this eminent Mathematician of *Turin*, shall be explained hereafter.

XV.

Newton having shewn that the Tides proceed from the combined Actions of the Sun and Moon, and determined the Quantity that each of those Luminaries contribute to their Production, enters into an Explanation of the Circumstances which attend the Phenomena of the Tides.

Three kinds of variations have been observed in the motion of the sea.

There has been observed in all Times, three Kinds of Motions in the Sea, its diurnal Motion, whereby it ebbs and flows twice a Day, the regular Alterations which this Motion receives every Month, and which follow the Position of the Moon with respect to the Sun, and those which arrive every Year and which depend on the Position of the Earth with respect to the Sun.

Diurnal variations.

To deduce those Motions from their Cause, we are to observe that the Sea yielding to the Force of the Sun and Moon impressed on it in Proportion to their Quantity, acquires its greatest Height by a Force compounded of those two Forces; hence this greatest Height (even abstracting from the Force of *Inertia* of the Waters) should not be immediately under the Moon, nor immediately under the Sun, but in an intermediate Point, which corresponds more exactly to the Motion of the Moon than to that of the Sun, because the Force of the Moon on the Sea is greater than that of the Sun. To determine the Position of this Point, it is manifest that at High-Water in any Place, $ssS+ttL$ is a *Maximum*, and at Low-Water a *Minimum* or $Ssds+Ltdt=0$. But the instantaneous Increment (ds) of the Sine of the Altitude of the Sun, is to the corresponding Increment (dz) of the Sun's diurnal Arc, as the Cosine $(\sqrt{1-ss})$ of the Altitude of the Sun to Radius (1), or $ds=\sqrt{1-ss}\times dz$ and the corresponding Decrement $(-dt)$ of the Sine of the Moon's Altitude, is to the corresponding Increment (dx) of the Moon's diurnal Arc, as the Cosine $(\sqrt{1-tt})$ of its Altitude to Radius (1), or $-dt=dx\times\sqrt{1-tt}=\frac{29}{30}dz\times\sqrt{1-tt}$, dx being to dz as 29 to 30, on account of the Motion of the Moon. Substituting those Values of ds and dt in the Expression $Ssds+Ltdt=0$, we will have $Ss\sqrt{1-ss}=\frac{29}{30}\times L\times t\sqrt{1-tt}$, or $\frac{s\sqrt{1-tt}}{t\sqrt{1-tt}}=\frac{29\,L}{30\,S}$ from whence it appears that at the Time of high and low Water the Quantities $s\sqrt{1-ss}$ and $t\sqrt{1-tt}$ are always in the constant Ratio of 29 L to 30 S, or of 20 × 5 to 30 × 2; but the Quantity $s\sqrt{1-ss}$ can never exceed $\frac{1}{2}$; consequently $t\sqrt{1-tt}$ can never exceed $\frac{30 \times 1}{29 \times 5}$ or $\frac{6}{29}$; and of course one of the Factors t or $\sqrt{1-tt}$ must be always very small, which proves that the Moon is near the Meridian at High-Water, and near the Horizon at Low-Water.

The waters of the Sea ought twice to rise and twice to fall every day.

The Waters of the Sea therefore should be elevated and depressed twice in the Space of a lunar Day, that is in the Interval of Time elapsed between the Passage of the Moon at the Meridian of any Place, and its Return to the same Meridian; for the conjoint Force of the Sun and Moon on the Sea, being greatest when the Moon is near the Meridian, it should be equal twice in 24 Hours 49 Minutes (a), when the Moon is near the Meridian of the Place above and below the Horizon; wherefore in each diurnal Revolution of the Moon about the Earth, there should be two Tides distant from each other, by the same Interval that the Moon employs to pass from the Meridian above the Horizon to that below it, which Interval is about 12h. 24m. hence the Time of High-Water will be later and later every Day.

XVI.

High-water does not immediately follow the Appulse of the Moon to the Meridian.

Since $t\sqrt{1-tt}$ can never exceed $\frac{6}{29}$, and consequently the Distance of the Moon from the Meridian 12 Degrees, the greatest Elevation of the Waters in any Place can never happen later than 48 lunar Minutes, or 50 solar Minutes after the Appulse of the Moon to the Meridian, if the Waters had no *Inertia*, and their Motion were not retarded by their Friction against the Bottom of the Sea. But from those two Causes this Elevation still happens two Hours and a Half or three Hours later

(a) Whilst the Heavens seem to carry the Sun and Moon round from East to West every Day, those Luminaries move in a contrary Direction, the Sun 59 m. 8s.,3 the Moon 13 d. 10 m. 35 s. in a Day, consequently after their Conjunction the Moon continually recedes 12 d. 11 m. 26 s.,7 from the Sun towards the East each Day, until she is 180 Degrees from the Sun, or in Opposition, after which being to the West of the Sun, she continually approaches, and at length overtakes him in 29 Days and an Half. From whence it appears that this Planet, the Day of the new Moon, rises, passes at the Meridian and sets about the same Time as the Sun; the following Days she rises, passes at the Meridian, and sets later and later than the Sun, so that the mean Quantity of the Retardation of one rising compared with the following, of one Appulse to the Meridian compared with the following, &c. is about 49 Minutes. Seven Days and One third after the Conjunction, the Moon being 90 Degrees to the East of the Sun, or in its first Quarter, she rises when the Sun is in the Meridian, passes at the Meridian when the Sun sets, and sets at Midnight. The following Days she comes sooner to the Meridian than the Sun to the opposite Meridian, but the Difference continually decreases to the Opposition, and then she rises when the Sun sets, passes at the Meridian at Midnight, and sets when the Sun rises. The following Days she comes later and later to the Meridian than the Sun to the opposite Meridian, the Difference increasing to the last Quarter when the Moon being 90 Degrees to the West of the Sun, rises at Midnight, passes at the Meridian at Six of the Clock in the Morning and sets at Noon. The following Days she rises, passes at the Meridian, and sets sooner than the Sun, the Interval decreasing to the Conjunction.

in the Ports of the Ocean where the Sea is open; for the Waters in consequence of their Force of *Inertia* receiving but by Degrees their Motion, and retaining for some Time the Motion they have acquired, the Motion of the Sea is perpetually accelerated during the six Hours which precedes the Appulse of the Moon to the Meridian, by the combined Actions of the Sun and Moon on the Waters, which increases in proportion as the Moon rises above the Horizon, and by the diurnal Motion of the Earth which then conspires with that of the Moon. This Motion impressed on the Waters retains during some Time its Acceleration, so that the Sea rises higher and higher until the diurnal Motion of the Earth which becomes contrary after the Appulse of the Moon to the Meridian, as also the combined Actions of the Luminaries which becomes weaker and weaker, diminishes gradually the Velocity of the Waters, in consequence of which they sink. It is easy to perceive that the Friction of the Waters against the Bottom of the Sea should also contribute to retard the Tides. What are the Causes which retard the Tides.

In the Regions where the Sea has no Communication with the Ocean, the Tides are much more retarded, in some Places even 12 Hours, and it is usual to say in those Places, that the Tides precede the Appulse of the Moon to the Meridian. In the Port of *Havre-de-grace*, for Example, where the Tide retards 9 Hours, it is imagined that it precedes by 3 Hours the Appulse of the Moon to the Meridian; but in Reality, this Tide is the Effect of the precedent Culmination.

The Waters falling to the lowest when the Moon is near the Horizon, her Action on the Sea being then most oblique, it is manifest that Low-water does not exactly fall between the two High-waters which immediately succeed each other, but is so much nearer to that which follows, as the Elevation of the Pole in the proposed Place is greater, and the Moon has a greater Declination; that is, in proportion to the Interval between the rising and setting of the Moon and the horary Circle of six Hours after her Culmination. Low-water does not exactly fall between the two Elevations which immediately succeed each other, and why.

XVII.

These are the principal Phenomena which attend the Tides depending on the Position of the different Parts of the Earth in its diurnal Revolution with respect to the Sun and Moon. We shall now proceed to explain the Variations in the Tides which happen every Month, and which depend on the Change of Position of the Moon with Respect to the Sun and the Earth. The menstrual Variations.

XVIII.

In the Conjunction of the Sun and Moon, those Luminaries coming to the Meridian at the same Time, and in the Opposition when one comes to the Meridian the other coming to the opposite Meridian, they conspire to raise the Waters of the Sea. In the Quadratures on the The greatest Tides happen at the new and full Moon.

The least in the Quadratures.

contrary the Waters raised by the Sun, are depressed by the Moon, the Waters under the Moon being 90 Degrees from those under the Sun; consequently the greatest Tides happen at full and new Moon, and the least at first and last Quarter.

XIX.

The greatest and least Tides do not precisely happen at that Time, and why.

The greatest and least Tides do not happen in the Sysigies and Quadratures, but are the Third or the Fourth in Order after the Sysigies and Quadratures, because as in other Cases so in this, the Effect is not the greatest or the least when the immediate Influence of the Cause is greatest or least. If the Sea was perfectly at Rest when the Sun and Moon act on it in the Sysigies, it would not instantly attain its greatest Velocity, nor consequently its greatest Height, but would acquire it by Degrees. Now as the Tides which precede the Sysigies are not the greatest, they increase gradually, and the Waters have not acquired their greatest Height until some Time after the Moon has passed the Sysigies, and she begins to counteract the Sun's Force and depress the Waters where the Sun raises them. Likewise the Tides which precede the Quadratures are not the least, they decrease gradually and do not come to their least Height until some Time after the Moon has passed the Quadratures.

XX.

The greatest Elevation of the Waters happens sooner after the Appulse of the Moon to the Meridian whilst she passes from the Sysigies to the Quadratures, and later whilst the Moon passes from the Quadratures to the Sysigies.

The greatest Height of the Waters which by the single Force of the Moon would happen at the Moon's Appulse to the Meridian, and by the single Force of the Sun at the Sun's Appulse to the Meridian, abstracting from the external Causes which retard it; by the combined Forces of both must fall out in an intermediate Time, which corresponds more exactly to the Motion of the Moon than to that of the Sun, wherefore when the Moon passes from Conjunction or Opposition to Quadrature, this greatest Height answers more to the setting of the Moon. The Sun in the first Case coming sooner to the Meridian than the Moon, and in the latter the Moon coming later to the Meridian than the Sun to the opposite Meridian; and when the Moon passes from Quadrature to Opposition or Conjunction, this greatest Elevation answers more to the rising of the Moon. In the first Case, the Moon coming sooner to the Meridian than the Sun to the opposite Meridian, and in the latter, the Moon coming sooner to the Meridian than the Sun (b). To calculate those Variations in the Time of High-water which arise from the respective Positions of the Sun and Moon, let us suppose on a certain Day, the Sun and Moon to be in Conjunction at the Appulse of the Moon to the Meridian of any Place, and consequently that it is High-Water there at that Instant. The following Day at the

(b) See preceding Note

Time of High-Water in said Place, the Sum of the Distances $(z'+x')$ of the Sun and Moon from the Meridian will be 12d. 30m. and the Interval between the two Tides will be expressed in solar Hours by 360d.+Arc z'. Since the Arcs z' and x' are very small, they may be supposed without any sensible Error to coincide with their Sines $(\sqrt{1-ss})$ $(\sqrt{1-tt})$ and $\sqrt{1-ss}+\sqrt{1-tt}$ may be supposed equal to Sin. 12d. 30m. $=0,21643$, and consequently $\sqrt{1-tt}=0,21643-\sqrt{1-ss}$, we may suppose also $s=1$ and $t=1$: after those Substitutions the Equation $\frac{s\sqrt{1-ss}}{t\sqrt{1-tt}}=\frac{29}{30}\times\frac{L}{S}$ will be transformed into $\frac{\sqrt{1-ss}}{0,21643-\sqrt{1-ss}}=\frac{29}{30}\times\frac{L}{S}$; and substituting $\frac{5}{2}$ for $\frac{L}{S}$ we will have $\frac{\sqrt{1-ss}}{0,21643-\sqrt{1-ss}}=\frac{29}{12}$ which gives for $\sqrt{1-ss}$ or for the Sine of the Arc z' required $\frac{29}{41}\times 0,21643=0,15308$ or $z'=8$d. 48m. or $35\frac{1}{5}$ solar Minutes, so that the whole Interval is 24h. 35m. $\frac{1}{5}$.

Let us now suppose on a certain Day, the Sun and Moon to be in Quadrature at the Appulse of the Moon to the Meridian at the above mentioned Place, and consequently that it is High-Water there at that Instant; the following Day at the Time of High-water the Sum of the Distances $(z'+x')$ of the Sun and Moon from the Meridian (if it be the last Quadrature) will be $77\frac{1}{2}$ Degrees, and the Sum of the Distances $(z+z')$ of the Sun from the Horizon and Meridian being 90 Degrees, $z-x'=12$d. 30m, that is, $s-\sqrt{1-tt}=0,21643$ and $\sqrt{1-tt}=s-0,21643$. But in this Case $\sqrt{1-ss}$ may be supposed $=1$ and $t=1$, wherefore $\frac{s\sqrt{1-ss}}{t\sqrt{1-tt}}=\frac{s}{\sqrt{1-tt}}=\frac{s}{s-0,21643}=\frac{29}{12}$ which gives $s=0,36920$ answering to 21d. 40m. or to 1h. $26\frac{2}{3}$ Minutes, so that the whole Interval (360d.+Arc z) is 25 Hours, $26\frac{2}{3}$ Minutes.

From whence it appears that High-Water should precede the Appulse of the Moon to the Meridian whilst she is passing from the Sysigies to the Quadratures, and should follow the Appulse of the Moon to the Meridian whilst she is passing from the Quadratures to the Sysigies; that the greatest Anticipation or Retardation should be about 50 solar Minutes, and that the Distance of the Sun and Moon from each other at the Time of the greatest Anticipation or Retardation is about 57 Degrees. But from external Causes High-Water happens in the Ports of the Ocean three Hours later, consequently in those Ports it should precede the third lunar Hour, and that by the greatest Interval the ninth Tide after the Sysigies, and this greatest Anticipation being repaired in the five subsequent Tides, it should follow by like Intervals the third lunar Hour, whilst the Moon is passing from the Quadratures to the Sysigies.

XXI.

The Tides are greater ceteris paribus, when the Moon is in Perigee than when ſhe is in Apogee.

Finally, all other Circumſtances being alike, the Tides are greateſt in the ſame Aſpects of the Sun and Moon, when they have the ſame Declination, when the Moon is in Perigee than when ſhe is in Apogee. The Force of the Moon on the Waters of the Sea decreaſing in the triplicate Ratio of her Diſtance from the Earth.

XXII.

The annual Variations, the Tides are greater in Winter than in Summer.

The annual Variations of the Tides depend on the Diſtance of the Earth from the Sun, hence it is that in Winter the Tides are greater, all other Circumſtances being alike, in the Syſigies, and leſs in the Quadratures than in Summer, the Sun being nearer to the Earth in Winter than in Summer.

XXIII.

The Tides depend on the Declination of the Sun and Moon.

The Effects of the Sun and Moon upon the Waters of the Sea depend upon the Declination of the Luminaries, for if either the Sun or Moon was in the Pole, any Place of the Earth in deſcribing its Parallel to the Equator, would not meet in its Courſe with any Part of the Water more elevated than another, ſo that there would be no Tide in any Place; therefore the Actions of the Sun and Moon on the Waters of the Sea become weaker as they decline from the Equator, and *Newton* found (Prop. 37. B. 3.) that the Force of each Luminary on the Sea decreaſes in the duplicate Ratio of the Coſine of its Declination; hence it is, that the Tides in the ſolſticial Syſigies are leſs than in the equinoctial Syſigies, and are greater in the ſolſticial Quadratures than in the equinoctial Quadratures, becauſe in the ſolſticial Quadratures the Moon is in the Equator, and in the other the Moon is in one of the Tropics, and the Tide depends more on the Action of the Moon than that of the Sun, and is therefore greateſt when the Moon's Action is greateſt.

The Spring Tides therefore ought to be the greateſt, and the Neap Tides the leaſt at the Equinoxes, and becauſe the Sun is nearer the Earth in Winter than in Summer, the Spring Tides are greateſt and the Neap Tides the leaſt in Winter; hence it is, that the greateſt Spring and leaſt Neap Tides are after the autumnal and before the vernal Equinox.

Two great Spring Tides never follow each other in the two neareſt Syſigies, becauſe if the Moon in one of the Syſigies be in her Perigee, ſhe will in the following Syſigie be in her Apogee. In the firſt Caſe her Action being greateſt and conſpiring with that of the Sun, the Waters will be raiſed to their greateſt Height; but in the latter Caſe her Action being leaſt, the Tide will be leſs.

XXIV.

The Time and Height of the Tides depend up-

The ebbing and flowing of the Sea depends alſo upon the Latitude of the Place; for the conjoint Actions of the Sun and Moon changing the Water upon the Earth's Surface into an oblong Spheroid, one of the

Vertices of its longer Axis describing nearly, the Parallel on the Earth's Surface, which the Moon, because of the diurnal Motion, seems to describe, and the other a Parallel as far on the other Side of the Equator. The whole Sea is divided into two opposite hemispheroidal Floods, one on the North Side of the Equator, the other on the South Side of it, which come by Turns to the Meridian of each Place after an Interval of 12 Hours. Now the Vertex of the hemispheroidal Flood which moves on the same Side of the Equator with any Place, will come nearer to it than the Vertex of the opposite hemispheroidal Flood which moves in a Parallel on the other Side of the Equator; and therefore the Tides in all Places without the Equator, will be alternately greater and less; the greatest Tide when the Declination of the Moon is on the same Side of the Equator with the Place, will happen about three Hours after the Appulse of the Moon to the Meridian above the Horizon, and the least Tide about three Hours after the Appulse of the Moon to the Meridian below the Horizon, the Height of the Tide in the first Case, being expressed by a Semidiameter of the elliptic Section of the Spheroid nearer the transverse Axe than in the latter Case, and consequently is greater; and the Tide, when the Moon changes her Declination, which was the greatest will be changed into the least, for then the hemispheroidal Flood which is opposite to the Moon, moves on the same Side of the Equator with the Place, and therefore its Vertex comes nearer to it than the Vertex of the hemispheroidal Flood under it. And the greatest Difference of those Tides will be in the Solstices, because the Vertices of the two hemispheroidal Floods in that Case describe the opposite Tropics, which are the farthest from each other of any two parallel Circles they can describe. Thus it is found by Observation, that the Evening Tides in the Summer exceed the Morning Tides, and the Morning Tides in Winter exceed the Evening Tides; and we learn (Pro. 24. B. 3.) that at *Plymouth*, according to the Observations of *Colepress* this Difference amounts to one Foot, and at *Bristol*, according to those of *Sturmy* to 15 Inches. *Newton (de Mundi Systemate*, page 58.) found, that the Height of the Tides decreases in each Place, in the duplicate Ratio of the Cosine of the Latitude of this Place. Now we have seen, that at the Equator, they decrease in the duplicate Ratio of the Cosine of the Declination of each Luminary; therefore in all Places without the Equator, half the Sum of the Heights of the Tides Morning and Evening, that is, their mean Height decreases nearly in the same Ratio. Hence the Diminution of the Tides arising from the Latitude of Places, and the Declination of the Luminaries may be determined.

on the Latitude of Places.

The Height of the Tides decreases in the duplicate Ratio of the cosine of the Latitude.

XXV.

The Height of the Tides depend likewise upon the Extent of the Sea in which they are produced, whether the Seas be entirely sepa-

The Height of the Tides depend on

the Extent of the Seas.

rated from the Ocean, or communicate with it by a narrow Channel; for if the Seas be extended from East to West 90 Degrees, the Tides will be the same as if they came from the Ocean, because this Extent is sufficient that the Sun and Moon may thereby produce on the Waters of the Sea their greatest and least Effect; but if those Seas be so narrow, that each of their Parts are raised and depressed with the same Force, there can be no sensible Effect, for the Water cannot rise in any one Place without sinking in another; hence it is, that in the *Baltick-Sea*, the *Black Sea*, the *Caspian-Sea*, and other Seas or Lakes of less Extent, there is neither Flood nor Ebb.

XXVI.

The Tides in the Mediterranean are scarce sensible.

In the *Mediterranean-Sea*, which is extended from East to West only 60 Degrees, the Flood and Ebb are scarce sensible, and *Euler* has given a Method for determining their Quantity. Those small Tides are still rendered less by the Winds and Currents which are very great in this Sea; hence it is, that in most of those Ports, there are scarce any regular Tides, except in those of the *Adriatick* Sea, which having a greater Depth, the Elevation of the Waters are rendered more sensible; hence it is, that the *Venetians* were the first who made Observations on the Tides of the *Mediterranean*.

XXVII.

Causes which influence the Tides that are indeterminable.

Besides the assignable Causes which serve to account for the Phenomena of the Tides, there are several others which produce Irregularities in those Motions which cannot be reduced to any Law, because they depend on Circumstances which are peculiar to each Place; such are the Shores on which the Waters flow, the Straits, the different Depths of the Sea, their Extent, the Bays, the Winds, &c. so many Causes which alter the Motion of the Waters, and consequently retard, increase, or diminish the Tides, and are not reducible to Calculation. Hence it is, that in some Places, the Flood falls out the third Hour after the Culmination of the Moon, and in other Places the 12th Hour; and in general, the greater the Tides are, the later they happen, because the Causes which retard them act so much longer.

If the Tides were very small, they would immediately follow the Culmination of the Moon, because the Action of the Obstacles which retard them would be rendered almost insensible; this is partly the Reason why the great Tides which happen about the new and full Moon, follow later the Appulse of the Moon to the Meridian, than those which happen about the Quadratures; the latter being less than the former.

XXVIII.

Velocity of the Waters of the Sea.

Euler relates that at *St. Malos*, at the Time of the Sysigies, it is High-Water the sixth Hour after the Appulse of the Moon to the Meridian, and the Retardation increases more and more until at *Dun-*

kerk and *Ostend*, it happens at Midnight. From this Retardation the Velocity of the Waters may be determined, and *Euler* concludes from those, and other Observations, that they move at the Rate of eight Miles an Hour; but it is easy to perceive, that this Determination cannot be general.

XXIX.

The Tides are greater towards the Coasts, and why.

The Tides are always greater towards the Coasts than in the open Sea, and that for several Reasons; first the Waters beat against the Shores, and by the Re-action, are raised to a greater Height. Secondly, they come with the Velocity they had in the Ocean where their Depth was very considerable, and they come in great Quantity, consequently meet with great Resistance whilst they flow on the Shores; from which Circumstance, their Height is still encreased. Finally, when they pass over Shoals, and run through Straights, their Height is greatly encreased, because being beat back by the Shores, they return with the Force they had acquired from the Effort they had made to overflow them. Hence it is, that at *Bristol*, the Waters are raised to so great a Height at the Time of the Sysigies, for the Shores on this Coast, are full of Windings and Sand-Banks, against which the Waters beat with great Violence, and are much impeded in their Motion.

XXX.

Explication of several Phenomena of the Tides.

Those Principles serve to account for the extraordinary great Tides which are observed in some Places, as at *Plymouth*, *Mount St. Michael*, the Town of *Avranches* in *Normandy*, &c. where *Newton* says, the Waters rise to 40 or 50 Feet, and some Times higher.

It may happen, that the Tide propagated from the Ocean, arrives at the same Port by different Ways, and that it passes quicker through some of those Ways than through the others; in this Case, the Tide will appear to be divided into several Tides, succeeding one another, having very different Motions, and no ways resembling the ordinary Tides. Let us suppose, for Example, that the Tides propagated from the Ocean, arrive at the same Port by two different Ways, one of which is a readier and easier Passage, so that a Tide arrives at this Port through one of those Inlets at the third Hour after the Appulse of the Moon to the Meridian, and another through the other Inlet, six Hours after, at the 9th Hour of the Moon. When the Moon is in the Equator, the Morning and Evening Tides in the Ocean being equal, in the Space of 24 Hours, there will arrive four equal Tides to this Port, but one flowing in as the other ebbs out, the Water must stagnate. When the Moon declines from the Equator, the Tides in the Ocean are alternately greater and less, consequently two greater and two lesser Tides would arrive at this Port by Turns, in the Space of 24 Hours. The two greatest Tides would make the Water acquire its greatest Height at a mean Time betwixt them, and the two lesser would make it fall lowest, at a mean Time between those two least Tides, and the Water would acquire at a mean Time betwixt its greatest and least Height, a mean Height; thus in the Space of 24 Hours, the Waters would rise, not twice, as usual, but once only to their greatest Height, and fall lowest only once.

If the Moon declines towards the Pole elevated above the Horizon, its greatest Height would happen the third, the sixth, or the 9th Hour after the Appulse of the Moon to the Meridian; and if the Moon declines towards the opposite Pole, the Flood would be changed into Ebb.

XXXI.

Explication of the Circumstances attending the Tides at Batsham in the Kingdom of Tunquin.

All which happens at *Batsham* in the Kingdom of *Tonquin*, in the Latitude of $20^{d}. 50^{m}$. North. The Day in which the Moon passes the Equator, the Waters have no Motion of flux and reflux: as the Moon removes from the Equator, the Waters rise and fall once a Day, and come to their greatest Height when the Moon is near the Tropics; with this Difference, that when the Moon declines towards the North-Pole, the Waters flow in whilst the Moon is above the Horizon, and ebb out whilst she is under the Horizon, so that it is High-Water at the setting of the Moon, and Low-Water at her rising. But when the Moon declines towards the South-Pole, it is High-Water at the rising, and Low-Water at the setting of the Moon; the Waters ebbing out during the whole Time the Moon is above the Horizon.

The Tide arrives at this Port by two Inlets, one from the *Chinese* Ocean, by a readier and shorter Passage between the Island of *Leuconia* and the Coast of *Canton*, and the other from the Indian Ocean, between the Coast of *Cochin-China* and the Island of *Borneo*, by a longer and less readier Passage; but the Waters arrive sooner by the readiest and shortest Passage; hence they arrive from the *Chinese* Ocean in six Hours, and from the *Indian* Ocean in 12 Hours, consequently the Tide arriving the third and ninth Hour after the Appulse of the Moon to the Meridian, there result the above Phenomena.

XXXII.

At the Entrance of Rivers the Ebb lasts longer than the Flood, and why.

At the Entrance of Rivers, there is a Difference in the Time of the Tides flowing in and ebbing out, arising from the Current of the River, which running into the Sea, retards its Motion of flux, and accelerates its Motion of reflux, consequently makes the Ebb last longer than the Flood, which is confirmed by Experience; for *Sturmius* relates, that above *Bristol*, at the Entrance of the River *Oundal*, the Tide is five Hours flowing in, and seven Hours ebbing out. Hence it is also, that all other Circumstances being alike, the greatest Floods arrive later at the Mouths of Rivers than elsewhere.

XXXIII.

It has been found, as has been already mentioned, that the Tides depend on the Declination of the Luminaries, and the Latitude of the Place; hence at the Poles there is no diurnal ebbing and flowing of the Waters of the Sea; for the Moon being at the same Height above the Horizon during 24 Hours, cannot raise the Waters; but in those Regions, the Sea has a Motion of flux and reflux depending on the Revolution of the Moon about the Earth every Month; in consequence of which the Waters are at the lowest when the Moon is in the Equator, because she is then always in the Horizon with respect to the Poles; and as the Moon declines either towards the North or South Pole, the Sea begins to ebb and flow, and when her Declination is greatest, the Waters are raised to their greatest Height at the Pole towards which she declines; and as this Elevation, which does not exceed ten Inches, is produced but by a very slow Motion, the Force of *Inertia* increases it very little, consequently is scarce sensible.

At the Poles there is no diurnal Tides but such as depend on the Revolution of the Moon about the Earth.

XXXIV.

It is only at the Poles that the Waters have no diurnal Motion; in the Frigid-Zone, there is one Tide every Day instead of two, as in the Torid-Zone, and in our Temperate-Zones; and it is easy to shew, that this Passage of two Tides to one, is not effected suddenly, but like all other Effects of Nature, is produced gradually. For we have seen, that the Morning and Evening Tides in our Temperate-Zones are unequal, not only as to their Height, but also as to the Time of their Duration; that the remoter the Place is from the Equator, the greater is this Inequality between the two Tides which immediately succeed each other, both as to their Height and the Time of their Duration, for the greatest Tide should last longer than the least; and notwithstanding which they both cease in $12^h. 24^m.$ nearly; therefore, in those Regions where the Moon after her Appulse to the Meridian above or below the Horizon, returns to it in this Interval, the least Tide will entirely vanish, and there will remain but the greatest Tide, which alone will fill up the Interval of $12^h. 24^m.$

But it is only at the Poles that there is no diurnal Tides, for in the Frigid-Zone there is one and why there are not two as in the other Regions of the Earth.

XXXV.

The Force of the Sun and Moon are sufficient to produce the Tides, but are incapable of producing any other sensible Effects here below; for the Force (S) of the Sun in its mean Distance, being to the Force (G) of Gravity, as 1 to 12868200, and the Force (S) of the Sun being to the Force (L) of the Moon, as 1 to 4,4815, by the Composition of Ratios $L \times S$ is to $S \times G$, or the Force (L) of the Moon in her mean Distance, is to the Force (G) of Gravity, as 4,4815 to 12868200, or as 1 to 2871400. And since $S+L$ is to L as 5,4815 to 4,4815, $S+L \times L$ is to $L \times G$ or the Sum of the Forces $(S+L)$ of the Sun and Moon

Why the Sun and Moon producing the Tides, produce no other sensible Effects here below.

when they conspire together, and in their mean Distances from the Earth, is to the Force *(G)* of Gravity as 5,4815×1 to 4,4815×2871400, or as 1 to 2347565, and the Sum of the greatest Forces of the Luminaries, or at their least Distance from the Earth, is to the Force of Gravity, as 1 to 2032890. From whence it appears, that those Forces united, cannot deflect the Direction of Gravity, nor consequently the Pendulum, from the true Vertical the 10th Part of a Second, nor cause a Variation in the Length of the Pendulum beating Seconds, which would exceed the $\frac{1}{500}$ of a Line, *&c.*

THEORY *of the* REFRACTION *of* LIGHT.

I.

Explication of the Refraction of Light derived from the Principle of Attraction.

THE Effects which Bodies exert on each other by their Attraction, become sensible only when it is not absorbed by the Attraction of the Earth, and it appears that this mutual Attraction of Bodies becomes sensible only when they are almost contiguous, and that then it acts in a Ratio greater than the inverse Triplicate of the Distances. Now the Atmosphere, or the Mass of Air encompassing the Earth, acting on Light in a very sensible Manner, it is certain, that if Attraction be the Cause, it should follow this Ratio.

The Advantage of the Principle of Attraction consists in having no Need of any Supposition but only the Knowledge of the Phenomena, and the more accurate are the Observations and Experiments, the easier it is to apply this Principle to their Explication.

II.

The Sine of Incidence and Refraction are always in a constant Ratio.

It is well known, that Light traversing Mediums of different Densities, changes its Direction. *Snellius*, and after him *Descartes*, found from Experiment, that the Sine of Incidence and that of Refraction are always in a constant Ratio; and *Newton* employs the 14th and last Section of the first Book of the *Principia* in explaining the Reason why those Sines should be in a constant Ratio, and proving that this Ratio depends on the Principle of Attraction. It is in this Explication we shall follow *Newton*.

Every Ray of Light which enters obliquely into any Medium, is to be considered as a Body acted on at the same Time by two Forces, in order to apply to the Explication of their Effects the Principles of Mechanicks. *Descartes* and *Fermat* considered Light as a Body of a sensible Magnitude on which the Mediums act after the same Manner as they appear to do on other Bodies: and finding that the Mediums which Light traverses, produce in them Effects quite contrary to those which should result from the Principles of Mechanicks, they invented each an Hypothesis in order to reconcile, in this Case, the Laws of Mechanicks, which are incontestable, and the phisicial Effects which are almost as certain.

III.

It is well known, that the denser the Mediums are, the greater Resistance Bodies which penetrate them meet with in separating their Parts. Now, in this Case, the Angle of Refraction is greater than the Angle of Incidence, because the vertical Velocity of the Body being diminished by the Resistance of the Mediums, the horizontal Velocity influences more the Direction of the Diagonal which the Body in obeying the two Forces into which its Motion is resolved, describes; hence it is, that when the Resistance of the Medium is insurmountable, the Body, instead of penetrating the Medium, returns back by its Elasticity, and the Proportion between this Resistance and the vertical Velocity of the Body may be such, that the Body would lose all its vertical Velocity, and would slide on the Surface of the Medium if it had no Elasticity, and if the Surface of the Medium was a perfectly smooth Plane.

The Laws of Refraction of Bodies of a sensible Magnitude.

IV.

Now quite the contrary happens to the Rays of Light, the denser the Medium is which they traverse, the more the Sine of Incidence exceeds that of Refraction; therefore the vertical Velocity of the Rays is increased in this Case, which is quite the Reverse of what the Laws of Mechanicks seem to indicate.

The Laws of Refraction of Light different from that of Bodies of a sensible Magnitude.

Descartes, in order to reconcile them with Experiment, which he could not evade, maintained, that the denser the Mediums were, the easier Passage they opened to Light; but this Manner of accounting for this Phenomenon was rather rendering it doubtful than explaining it.

Fermat, finding the Explication of *Descartes* impossible, thought it more advisable to have Recourse to Metaphisicks, and the final Causes. He asserted, that since Light does not arrive to us by the shortest Passage, which is the straight Line, it was becoming the Divine Wisdom, it should arrive in the shortest Time; this Principle, once allowed, it followed, that the Sines of Incidence and Refraction are to each other as the Facilities of the Medium to be penetrated.

Hypotheses of Descartes and Fermat.

V.

It is easy to see how Attraction solves this Difficulty; for this Principle evinces, that the progressive Motion of Light, not only is not less retarded in the more dense Medium, as *Descartes* pretended, but is really accelerated, and that by the Attraction of the more dense Medium when it penetrates it. It is not only when the Ray has arrived at the refracting Medium and at the Point of Incidence that it acts on it; the Incurvation of the Ray commences some Time before, and it increases in proportion as it approaches the refracting Medium, and even within this Medium to a certain Depth.

Attraction accounts for every Circumstance attending the Refraction of Light.

Attraction accounts for every Circumstance attending Light in its Passage through one Medium into another; for the vertical Velocity of the Ray is increased in the more dense Medium, which it traverses until it arrives at the Point where the superior and inferior Parts of this Medium act with equal Force on it, then it continues to advance with the acquired Velocity until being on the Point of quitting it, the superior Parts of this Medium attract it with a greater Force than the inferior Parts. The vertical Velocity of the Ray is diminished thereby, and the Curve it describes at its Emersion, is perfectly equal and similar to the one it described at its Incidence, (the Surfaces which bound the refracting Medium being supposed parallel) and the Position of this Curve is directly opposite to that of the first. In fine, the Ray passes through Degrees of Retardation which are in the same Ratio, and in the same inverse Order as the Degrees of Acceleration which it passed through at its Incidence.

VI.

Experiments of Newton which prove that the Refraction of Light depend on the Density of the Mediums thro' which it passes.

Newton, who was as superior in the Art of making Experiments as in that of employing them, found on examining the Deviation of the Rays of Light in different Mediums, that the Attraction exerted on the Particles of Light follows the Ratio of the Density of those Mediums, if we except those which are greasy and sulphurous. Since then the different Densities of those Mediums is the Cause of the Refraction of Light, the more homogeneous Bodies are, the more transparent they will be; and those which are most heterogeneous will be least so, for the Light in traversing them, being perpetually reflected in different Directions within those Bodies, the Quantity of Light which arrives to us is thereby diminished; hence it is, that when the Sky is clear, the Stars are so distinctly perceived, but when clouded, the Rays are intercepted, and cannot reach the Earth.

VII.

The Rays of Light have not all the same Degree of Refrangibility.

Newton also found, that every Ray of Light, however small, is composed of seven Rays, which as long as they are united continue white, but resume their natural Colour when they are separated, and that those Rays have not all the same Degree of Refrangibility, that is, in passing through one Medium into another of different Density, are inflected some more and others less; so that when they pass through a Lens, those Rays do not all meet the Axe at the same Distance, but some nearer and others farther off, and thus form as many distinct Pictures of the Object as there are Colours. The Eye only perceives the most vivid, but as the Pictures are not equal, the greatest form round those several coloured Circles, which is called the Crown of Aberration. This Aberration is quite distinct from that which arises from the Defect of Reunion of the Rays caused by the spherical Figure of the Lenses.

The Aberration of Refrangibility in the Rays of Light is not sensible when their Refraction is inconsiderable; now the Rays parallel to the optic Axe of a Lens, and those at a small Distance from this Axe, are very little inflected, and the Picture they form may be considered as simple, as not being surrounded by any coloured Circles. Hence it is, that Artists are under the Necessity of giving to the objective Glass an Aperture of a very small Number of Degrees of the Sphere of which this Glass forms a Part, and consequently of increasing the focal Distance of this Glass, and the Length of the Telescope, as often as they change the Proportion of the objective and ocular Glasses, in order to increase its magnifying Power. Those Obstacles to the Perfection of refracting Telescopes arising from the Nature of Light, and the Laws of Refraction, *Newton* was on the Point of removing; an Experiment he made opened the Way which leads to this Discovery, but he did not pursue it: the Experiment is as follows: *As often as Light, traversing different Mediums, is so corrected by contrary Refractions, that it emergeth in Lines parallel to those in which it was Incident, continues ever after white.* Optics, First B. Part II. Exp. 8.

How the Method for correcting the Aberration arising from the different Refrangibility of the Rays was discovered.

Euler in 1747, meditating on this Subject, demonstrated, that this Assertion was false, and consequently that the Experiment was ill made. Mr. *Dolond*, an eminent *English* Optician, well versed in the Theory and Practice of his Art, repeated this Experiment after the same Manner that *Newton* described it; he constructed for this Purpose, with two Plates of Glass, a Kind of Port-folio, which being filled with Water, formed a Prism of Water, that by closing or opening the Glasses, was susceptible of all Kinds of Angles; he plunged into the Water of this Prism, whose Angle was turned downwards, another Prism of Chrystal, whose Angle was turned upwards, and by moving the Plates of Glass, he found that Inclination which was necessary to make the Objects observed through the two Prisms of Water and Glass appear exactly at the same Height as they did to the naked Eye; it was then manifest, that the Refraction of one Prism was destroyed by the Refraction of the other, yet the Objects were tinged with various Colours, which was quite contrary to what *Newton* had asserted. Mr. *Dolond* afterwards tried, by moving the Plates of his Prism of Water, whether there was not some possible Proportion between the Angles of the two Prisms capable of destroying the Colours, and found that there was such a Proportion, which widely differed from that which destroys the absolute Refraction. The Objects not coloured viewed through the Prisms thus combined, not appearing at the same Height as when viewed by the naked Eye. From whence it was easy to conclude, that the Aberration of the Rays arising from their different Degrees of Refrangibility, might be corrected by employing transparent Mediums of different Densities, and that the Rays would be refracted, but in a different Manner from what they would be in passing through one Medium. Mr. *Dolond* in 1759, discovered a Method answering this Purpose, which he has employed with Success in the Construction of achromatic Telescopes, and the most eminent Mathematicians have since exerted all their Skill in investigating the different Combinations for the focal Distances, and the Quantity of Curviture requisite to correct at once, the Aberration arising as well from the different Degrees of Refrangibility of the Rays, as from the circular Figure of the Lenses. Those Researches shall be explained hereafter.

VIII.

The Principle of Attraction serves to explain how Refraction is changed into Reflection.

The Principle of Attraction serves to explain why the Refraction is changed into Reflection at a certain Obliquity of Incidence, when the Rays of Light pass through a more dense Medium into a less dense one; for in the Passage of a Ray through a more dense Medium into another that is less, the Curve it describes is inflected towards the more dense Medium it has passed through; now the Proportion between its Obliquity and the Force which draws it towards this more dense Medium may be such, that its Direction may become parallel to the Surface of the Medium which it quits, before it has passed the Limits within which the Attraction of this Medium is confined; and in this Case, it is very easy to see, that it should return toward the refracting Medium it had quitted, describing a Branch of a Curve equal and similar, to the Curve which it described in passing through this Medium, and reassume after having again entered this Medium the same Inclination it had before it quitted it.

The Action of the Medium which Light traverses, may give the Rays the Obliquity they require in order to be reflected, and as the more the Mediums differ in Density the less is the Obliquity of Incidence requisite that the Rays may be reflected, the Rays will be reflected at the least Obliquity of Incidence when the contiguous Space or refracting Medium will be purged of Air, and when the Vacuum will be most perfect. And so it happens in the Air-Pump, in which the more the Vacuum is increased, the quicker a Ray is reflected at the superior Surface of a Prism placed therein. The Refraction is therefore changed into Reflection at different Incidences, according to the Density of the different Mediums, *Diamond* which is the most brilliant Body known, operates an entire Reflection when the Angle of Incidence is only 30 Degrees, and it is according to this Angle Jewellers cut their Diamonds, that they may lose the least Quantity of the Light they receive.

IX.

It is easy to perceive, that when a Ray of Light passes through a less dense Medium into a more compact one, the Refraction cannot be changed into Reflection let the Obliquity of Incidence be ever so great, for when the Ray is on the Point of quitting the less dense Medium, the other Medium which is contiguous to it, begins to act on it, and increaſes continually its vertical Velocity, the Rays of Light therefore in their Paſſage through the different Couches of the Atmoſphere, whoſe Denſity continually increaſes in approaching the Earth, are more and more curved; in conſequence of which the celeſtial Objects appear more elevated than they really are, and that by how much the more their Rays are curved from their Entrance into the Atmoſphere until they arrive to us, the Eye receiving the Impreſſion of Light in the Direction which the Rays have when they enter the Eye.

Refraction increases the Length of the Day.

This apparent Elevation of the heavenly Bodies above their true Height, is called Aſtronomical Refraction, and is greateſt near the Horizon, where repeated Obſervations prove, that it amounts to 33′; hence it is, that in our Climates, the Sun appears to riſe 3 Minutes ſooner, and ſet 3 Minutes later than it really does, whereby the artificial Day is increaſed 6 Minutes by the Effect of Refraction. This Effect gradually increaſes in advancing towards the Frigid-Zone, and at the Pole, by the Refraction alone, the Day becomes 36 Hours longer; hence it is alſo that the Sun and Moon at their riſing and ſetting appear oval, the inferior Margin of thoſe Luminaries being more refracted than the ſuperior one, or appear higher in Proportion.

Rule for finding the Refraction at any diſtance from the Zenith.

Newton has ſhewn how to determine the Law according to which Refraction varies from the Zenith to the Horizon; from his Theory it reſults, that the Radius *(R)* is to the Sine of 87d. as the Sine of *(z)* the Diſtance from the Zenith, to the Sine of *(z—6r)* of this ſame Diſtance diminiſhed by ſix Times the Refraction at this Diſtance, wherefore *R*—Sine 87 : Sine 87=Sine *z*—Sine *(z—6r)* : Sine *(z—6r)*; and *R*—Sine 87 : Sine *z*—Sine *(z—6r)*= Sine 87 : Sine *(z—6r)*; but *R*—Sine 87 is to Sine *z*—Sine *(z—6r)* as 3d.×Coſ. 88[1] to 6*r*×Coſ. *(z—3r)*, Differences of the Arcs multiplied by the Coſines of the Arcs, which are the arithmetical Means between 90 and 87, and between z and z—*6r*. Therefore the Sine of 88d. $\frac{1}{3}$, that is of 90d. diminiſhed by the Triple of the horizontal Refraction, is to the Sine of the Diſtance z diminiſhed by the Triple of the Refraction at that Diſtance, as the horizontal Refraction, is to the Refraction at the Diſtance z, and as the Coſine of 88d. $\frac{1}{3}$ to the Coſine of the Arc z diminiſhed by the Triple of the Refraction; therefore the Refraction at the Diſtance z, is equal to the horizontal Refraction multiplied by the Tangent of z diminiſhed by the Triple of its Refraction, the whole divided by the Tangent of 88d. 21m. from whence it appears, *that the Refractions are proportional to the Tangents of the Diſtances from the Zenith diminiſhed by three Times the Refraction.*

Example. Let the Refraction at the Diſtance of 45 Degrees from the Zenith be required, which is known to be about 1m. the Tangent of 88d. 21m. is to the Tangent of 44d. 57m. as the horizontal Refraction 33m. is to 57′, the Refraction at 45 Degrees Diſtance from the Zenith. By this Rule the following Table was conſtructed.

X.

Table of Astronomical Refraction.

App. Alt.	Refrac.	App. Alt.	Refrac.	App. Alt.	Refrac.	App. Alt.	Refrac.	App. Alt.	Refrac.	App. Alt.	Refrac.
D. M.	M. S.	D. M.	M. S.	D. M.	M. S.	D. M.	M. S.	D. M.	M. S.	D. M.	M. S.
0 0	33. 0,0	4 0	11.51,1	8.30	6. 8,0	15.30	3.23,7	36 0	1.18,5	63 0	0.29,1
0 5	32.10,4	4 10	11.28,9	8.40	6. 1,3	16. 0	3.16,9	37 0	1.15,7	64 0	0.27,8
0 10	31.22,2	4 20	11. 7,9	8.50	5.54,8	16.30	3.10,5	38 0	1.13,0	65 0	0.26,5
0 15	30.35,4	4 30	10.48,0	9. 0	5.48,5	17. 0	3, 4,5	39 0	1.10,4	66 0	0.25,3
0 20	29.49,7	4 40	10.29,2	9.10	5.42,4	17.30	2.58,9	40 0	1. 7,9	67 0	0.24,1
0 30	28.22,3	4 50	10.11,3	9.20	5.36,5	18. 0	2.53,6	41 0	1. 5,5	68 0	0.22,9
0 32	28. 4,8	5 0	9.54,3	9.30	5.30,9	18.30	2.48,6	42 0	1. 3,3	69 0	0.21,7
0 36	27.30,3	5 10	9.38,2	9.40	5.25,4	19. 0	2.43,9	43 0	1. 1,1	70 0	0.20,6
0 40	26.59,7	5 20	9.22,8	9.50	5.20,0	19.30	2.39,4	44 0	0.59,0	71 0	0.19,5
0 50	25.41,8	5 30	9. 8,0	10. 0	5.14,8	20. 0	2.35,1	45 0	0.57,0	72 0	0.18,4
1 0	24.28,6	5 40	8.54,0	10.15	5. 7,3	20.30	2.31,0	46 0	0.55,0	73 0	0.17,3
1 10	23.19,8	5 50	8.40,6	10.30	5. 0,1	21. 0	2.27,2	47 0	0.53,1	74 0	0.16,2
1 20	22.15,2	6 0	8.27,8	10.45	4.53,2	21.30	2.23,6	48 0	0.51,2	75 0	0.15,1
1 30	21.14,7	6 10	8.14,9	11. 0	4.46,6	22. 0	2.20.3	49 0	0.49,4	76 0	0.14,0
1 40	20.17,9	6 20	8. 2,8	11.15	4.40,3	23. 0	2.13,7	50 0	0.47,6	77 0	0.13,0
1 50	19.24,8	6 30	7.51,1	11.30	4.34,3	24. 0	2. 7,4	51 0	0.45,9	78 0	0.12,0
2 0	18.35,0	6 40	7.40,3	11.45	4.28,6	25. 0	2. 1,6	52 0	0.44,2	79 0	0.11,0
2 10	17.48,4	6 50	7.30,2	12.00	4.23,2	26. 0	1,56,2	53 0	0.42,6	80 0	0.10,0
2 20	17. 4,5	7 0	7.20,5	12.20	4.16,1	27. 0	1.51,2	54 0	0.41,1	81 0	0. 9,0
2 30	16.23,8	7 10	7.11,1	12.40	4. 9,4	28. 0	1.46,6	55 0	0.39,6	82 0	0. 8,0
2 40	15.45,4	7 20	7. 2,1	13. 0	4. 3,0	29. 0	1.42,4	56 0	0.38,2	83 0	0. 7,0
2 50	15. 9,4	7 30	6.53,4	13.20	3.56,9	30. 0	1.38,4	57 0	0.36,8	84 0	0. 6,0
3 0	14.35,6	7 40	6.45,1	13.40	3.51,1	31. 0	1.34,6	58 0	0.35,5	85 0	0. 5,0
3 10	14. 3,9	7 50	6.37,1	14. 0	3.45,5	32. 0	1.31,0	59 0	0.34,2	86 0	0. 4,0
3 20	13.34,1	8 0	6.29,4	14.20	3.40,1	33. 0	1.27,6	60 0	0.33,0	87 0	0. 3,0
3 30	13. 6,2	8 10	6.22,0	14.40	3.34,9	34. 0	1.24,4	61 0	0.31,7	88 0	0. 2,0
3 40	12.39,6	8 20	6.14,8	15. 0	3.29,9	35. 0	1.21,4	62 0	0.30,4	89 0	0. 1,0
3 50	12.14,6	8 30	6. 8,0	15.30	3.23,7	36. 0	1.18,5	63 0	0.29,1	90 0	0. 0,0

THEORY *of the* SECONDARY PLANETS.

I.

THE first Phenomenon which the Secondary Planets offer to natural Philosophers, is their Tendency towards their Primaries, in observing the same Law as the primary Planets towards the Sun. This Tendency has been sufficiently established in treating of the primary Planets, abstracting at first, as was necessary in order to simplify the Question, from all the Irregularities which the Planets, by their mutual Attractions produce in each others Motions, or which arise from the Action of the Sun. Having afterwards examined the Irregularities in the Motions of the primary Planets; but the Irregularities in the Motions of the secondary Planets deserve particularly to be considered, in order to shew after a more satisfactory Manner, the Universality of the Principle of Attraction, and the Harmony of the System to which it serves as a Basis.

The different Kinds of Motions observed for many Ages in the Moon, and the Laws of those Motions discovered by eminent Astronomers, furnished *Newton* the Means of applying his Theory with Success to this Planet. This great Man, who had made so many Discoveries in the other Parts of the System of the World, was resolved not to leave this Part unexamined; and though the Method he has pursued on this Occasion, is less evident, and less satisfactory than the Method he employed in explaining the other Phenomena; we are however much indebted to him for having made it the Object of his Inquiry.

II.

It is easy to perceive, that if the Distance of the Sun from the Earth and the Moon, was infinite with the respect to their Distance from each other, the Sun would not disturb the Motion of the Moon about the Earth; because equal Forces, whose Directions are parallel, which act on any two Bodies, cannot affect their relative Motions. But as the Angle formed by the Lines drawn from the Moon and the Earth to the Sun, though very small, cannot be esteemed as having no Quantity, from this Angle therefore is to be deduced the Inequality of the Action of the Sun on these two Bodies.

Manner of having regard to the Inequality of the Force of the Sun, on the Earth and the Moon.

Taking therefore, as *Newton* has done, (Propos. 66.) in the straight Line drawn from the Moon to the Sun, a Line to express the Force with which the Sun attracts it; let this Line be considered as the Diagonal of a Parallelogram, one of whose Sides will be in the straight Line drawn from the Moon to the Earth, and the other a Line drawn from the Moon parallel to the straight Line which joins the Sun and the Earth, it is evident, that those two Sides of the same Parallelogram will express two Forces which might be substituted for the Force of the Sun on the Moon; and that the first of those two Forces which urges the Moon towards the Earth, will neither accelerate nor retard the Description of the Areas, nor consequently prevent her from observing the Law of *Kepler*, viz. the Areas proportional to the Times, but will only change the Law of the Force with which the Moon tends towards the Earth, and consequently will alter the Form of her Orbit. As to the second Force, that which acts in a Direction parallel to the Ray of the Orbit of the Earth, if it was equal to the Force with which the Sun acts on the Earth, it is easy to perceive that it would produce no Irregularity in the Motion of the Moon; but those Forces are only equal in those

The Force of the Sun is resolved in two others.

One urges the Moon towards the Earth.

The other acts in the Direction of the Line drawn from the Earth to the Sun.

Points of the Moon's Orbit, where her Diſtance from the Sun becomes equal to the Diſtance of the Earth from the Sun at the ſame Time, which happens in the Quadratures; in every other Point of her Orbit thoſe two Quantities being unequal, their Difference expreſſes the perturbating Force of the Sun on the Moon, not only preventing her from deſcribing equal Areas in equal Times, but alſo from moving always in the ſame Plane.

III.

Meaſure of the perturbating Forces of the Sun.

We find in Prop. 66 of the firſt Book, only the general Expoſition of the Manner of eſtimating the perturbating Forces of the Sun on the Moon: But in Prop. 25 of the third Book, we find the Calculation which determines their Quantity; we learn that the Part of the Force of the Sun which urges the Moon towards the Earth, is in its mean Quantity, the $\frac{1}{178\frac{29}{40}}$ of the Force with which the Earth acts on her when ſhe is in her mean Diſtance. The other Part of the ſame Force of the Sun which acts in a Direction parallel to the Ray of the Orbit of the Earth, is to the firſt Part, as the Triple of the Coſine of the Angle formed by the ſtraight Lines drawn from the Moon and the Earth to the Sun.

IV.

Acceleration of the Areas deſcribed by the Moon produced by this Force.

Newton employs this Determination of the perturbating Forces (Prop. 26, 27, 28, 29.) for computing the monthly Inequality in the Moon's Motion, called her Variation, whereby ſhe moves ſwifter in the firſt and third Quarter, and ſlower in the Second and Fourth, and which becomes moſt ſenſible in the Octants or 45 Degrees from the Syſigies.

Newton, to determine this Inequality, abſtracts from all the reſt; he further ſuppoſes the Moon's Orbit to be circular, if the Sun was away, and he inveſtigates the Acceleration in the Area which the Moon deſcribes, produced by that one of the two perturbating Forces which acts in a Direction parallel to the Ray drawn from the Earth to the Sun. He found that the Area deſcribed by the Moon in ſmall equal Portions of Time, to be nearly as the Sum of the Number 219,46, and the verſed Sine of double of the Moon's Diſtance from the neareſt Quadrature, (the Radius being Unity); ſo that the greateſt Inequality in the Areas deſcribed by the Moon, arrives in the Octants or 45 Degrees from the Syſigies, where this verſed Sine is in its Maximum.

V.

The Action of the Sun renders the Orbit of the Moon more contracted between the

To determine afterwards the Equation or Correction in the mean Motion of the Moon ariſing from this Acceleration of the Area deſcribed by the Moon, he has Regard to the Change in the Form of the lunar Orbit, produced by the perturbating Force. He inveſtigates the Quantity which the perturbating Force would render the Line paſſing through the Quadratures longer than that which traverſes the Syſigies. The

Data which he employs in solving this Problem, are the Velocities of the Moon in those two Points, which he had shewn how to determine in the foregoing Proposition, and the centripetal Forces corresponding to the same Points, which are both one and the other compounded of the Force with which the Moon tends towards the Earth, and the perturbating Forces of the Sun, which in the Sysigies and Quadratures act in the Direction of the Ray of the Orbit of the Moon. Now the Curvatures in those Points, being in the direct Proportion of the Attractions, and in the Inverse of the Squares of the Velocities, by this Means he obtaines the Ratio of the Curvatures, and from thence deduces the Ratio of the Axes of the Orbit, assuming for Hypothesis, that this Curve is an Ellipse, having its Centre in the Centre of the Earth, if the Sun be supposed to have no apparent Motion round the Earth; but when Regard is had to this Motion of the Sun, because the lesser Axe of the Ellipse is also carried about the Earth with the same Motion, as being always directed towards the Sun, that it is a Curve whose Rays are the same as those of the Ellipse, but the Angles they form are increased in the Ratio of the periodic Motion of the Moon to its synodical Motion. The first of those Motions being that in which the Moon is referred to a fixed Point in the Heavens; the other in which she is compared with the Sun. By the Means of those Suppositions, *Newton* found that the Axe which passes through the Quadratures, is greater than that which passes through the Sysigies by $\frac{1}{87}$. Sysigies than between the Quadratures.

VI.

He afterwards computes in the same Hypothesis of the Moon's Orbit being circular, if the Sun was away, by the Principle of the Areas proportional to the Times, the Equation or Correction in the mean Motion of the Moon resulting not only from the Acceleration found in the foregoing Problem, her Orbit being supposed circular, but from the new Form of this Orbit. From the Combination of those two Causes, he finds an Equation or Correction which becomes most considerable in the Octants, and then amounts to 35m. 10′ when the Earth is in its mean Distance; and in the other Points of the Earth's Orbit, is to 35m. 10′, in the inverse Ratio of the Cube of the Distance from the Sun, because the Expression of the perturbating Force of the Sun, which is the Cause of all these Irregularities of the Moon, is divided by the Cube of the Earth's Distance from the Sun. This Correction in the other Points of the Moon's Orbit, is proportional to the Sine of double of the Distance of the Moon from the nearest Quadrature. Computation of the Variation of the Moon.

VII.

Newton passes from the Examination of the Variation of the Moon, to that of the Motion of the Nodes, (Prop. 30, 31.) In this Inquiry he supposes the Moon's Orbit to be circular if the Sun was away, and attributes to the Force of this Luminary no other Effect than to change Computation of the Motion of the Nodes.

this circular Orbit into an Ellipse, whose Centre is in the Centre of the Earth, or rather into the Curve whose Construction we have already given by the Means of an Ellipse. Of the two perturbating Forces of the Sun, that which urges the Moon towards the Earth, acting in the Plane of the Orbit, cannot produce any Motion in this Plane; he therefore only considers that Force which acts parallel to the Line drawn from the Earth to the Sun, which he had shewn to be proportional to the Cosine of the Angle formed by the Lines drawn from the Moon and the Earth to the Sun, and we shall now explain how he employs this Force.

Which of the two perturbating Forces of the Sun he employs.

At the Extremity of the little Arc which the Moon describes in any small Portion of Time, he takes one equal to it, which would be that which the Moon would describe if the perturbating Force of the Moon ceased to act on her; and at the Extremity of this new Arc, he draws a Line parallel to that which joins the Centre of the Earth and the Sun, and he determines the Length of this straight Line, by the Measure already determined of the Force which acts in the same Direction as it; which being done, the Diagonal of the Parallelogram, one of whose Sides is the little Arc which the Moon would describe if the perturbating Force ceased to act, and the other, the Arc the Moon would describe if this Force acted alone, is the real Arc the Moon would describe. There remains therefore no more to be done than to determine, how much the Plane which would pass through this small Arc and the Earth, differs from the Plane which passes through the first Side and the Earth.

The two Sides already mentioned, being produced until they meet the Plane of the Orbit of the Earth, and having drawn from their Points of Concourse with this Plane, two straight Lines to the Centre of the Earth, the Angle which those two straight Lines form, is the Motion of the Node during the small Portion of Time which the Moon employs in describing this small Arc, which we have been considering. And *Newton* finds that the Measure of this Angle, and consequently the Velocity or the instantaneous Motion of the Node, is proportional to the Product of the Sines of three Angles, which express the Distance of the Moon from the Quadrature, of the Moon from the Node, and of the Node from the Sun.

Law of the Motion of the Nodes.

VIII.

Regression and Progression of the Nodes in each Revolution.

It follows from hence, that when one of those three Sines becomes negative, the Motion of the Nodes which before was retrograde, becomes direct. Wherefore when the Moon is between the Quadrature and the nearest Node, the Motion of the Node is direct; in all other Cases, its Motion is retrograde, but the retrograde Motion exceeding the direct Motion, it happens that in each Revolution of the Moon, the Nodes are made to recede.

At the End of every Revolution the Nodes recede.

When the Moon is in the Syzigies, and the Nodes in the Quadratures, that is, 90 Degrees from the Sun, their Motion is 33″ 10‴ 37^{iv} 12^{v}, wherefore the horary Motion of the Nodes in every other Situation, is to 33″ 10‴ 27^{iv} 12^{v}, as the Product of the three Sines already mentioned to the Cube of Radius.

Formula which gives the horary Motion of the Nodes in any Situation.

IX.

Supposing the Sun and the Node to be in the same Situation with respect to the fixed Stars, whilst the Moon passes successively through its several Distances with respect to the Sun. *Newton* investigates (Prop. 32. B. III.) the horary Motion of the Node, which is a Mean between all the different Motions resulting from the foregoing Formula, and this mean Motion of the Node is 16″ 33‴ 16^{iv} 36^{v}, when the Orbit is supposed circular, and the Nodes are in Quadrature with the Sun; in every other Situation of the Nodes, this Motion is to 16″ 33‴ 16^{iv} 36^{v}, as the Square of the Sine of the Distance of the Sun from the Node, is to the Square of the Radius. If the Orbit of the Moon be supposed to be an Ellipse, having its Centre in the Centre of the Earth, the mean Motion of the Nodes in the Quadratures is only 16″ 16‴ 37^{iv} 42^{v}, and in any other Situation of the Nodes, it depends likewise on the Square of the Sine of the Distance from the Sun.

Determination of the mean Motion of the Nodes.

In order to determine for any given Time, the mean Place of the Nodes, *Newton* takes a Medium between all the mean Motions already mentioned. He employs in this Inquiry, the Quadrature of Curves, and the Method of Series. By this Means he finds that the Motion of the Nodes in a sydereal Year, should be 19° 18′ 1″ 23‴, which only differs 3′ from that which results from astronomical Observations.

X.

The same Curve the Quadrature of whose Area determines the mean Velocity of the Nodes, serves also for finding the true Place of the Nodes for any given Time, (Prop. 33. B. III.)

Determination of the true Place of the Nodes for any given Time.

The Result of his Computation is as follows: Having made an Angle equal to the Double of that which expresses the Distance of the Sun from the mean Place of the Nodes, let the Sides of this Angle be to each other, as the mean annual Motion of the Nodes, which is 19° 49′ 3″ 55‴, to the Half of their true mean Motion, when they are in the Quadratures, which is 0° 31′ 2″ 3‴, that is, as 38,3 to 1, which being done, and having completed the Triangle which will be given, since this Angle and its two Sides are given, the Angle of this Triangle opposite to the least of those Sides, will express with sufficient Accuracy, the Equation or Correction in the mean Motion of the Nodes for determining the true Motion required.

XI.

Variation of the Inclination of the Moon's Orbit.

From the Inveſtigation of the Motion of the Nodes, *Newton* paſſes (Prop. 34. B. III.) to the Determination of the Variation in the Inclination of the Orbit of the Moon. By employing that one of the two perturbating Forces of the Sun which does not act in the Plane of the Orbit of the Moon, he obtains the Meaſure of the horary Variation in the Inclination of the Orbit of the Moon; this Variation, when the Orbit is ſuppoſed circular, being to the horary Motion of the Nodes, 33″ 10‴ 3^{iv} 12^{v}, (the Nodes being in the Quadratures, and the Moon in the Syſigies) diminiſhed in the Ratio of the Sine of the Inclination of the Orbit of the Moon to the Radius: as the Product of the Sine of the Diſtance of the Moon from the neareſt Quadrature, the Sine of the Diſtance of the Sun from the Nodes, and the Sine of the Diſtance of the Moon from the Nodes to the Cube of Radius. And this Quantity diminiſhed by $\frac{1}{60}$ is the Variation correſponding to the Orbit rendered elliptic by the perturbating Force of the Sun.

Horary Variation of the Inclination.

XII.

Method for finding the Inclination of the Moon's Orbit for any given Time.

The horary Variation of the Inclination of the Orbit of the Moon being thus determined, *Newton* employing the ſame Method, and the ſame Suppoſitions by which he found the true Place of the Nodes for any given Time, determines (Prop. 35. B. III.) the Inclination of the Orbit for any given Inſtant of Time; the Reſult of his Computation is as follows.

Let there be taken from the ſame Point of a ſtraight Line, aſſumed as a Baſe, three Parts in geometrical Proportion, the firſt expreſſing the leaſt Inclination, the third the greateſt; let there be afterwards drawn through the Extremity of the Second, a Line making with this Baſe an Angle equal to double the Diſtance of the Sun from the Node for the propoſed Motion let this Line be produced until it meets the Semicircle deſcribed on the Difference of the firſt and third Lines in geometrical Proportion; which being done, the Interval compriſed between the firſt Extremity of the Baſe, and the Perpendicular let fall from the common Section of the Circle and the Side of the Angle juſt mentioned, will expreſs the Inclination for the propoſed Time.

Determination of the Latitude of the Moon.

From hence is deduced the Moon's Latitude corrected; for in a Right-angled ſpherical Triangle is given, beſides the Right-angle, the Hypothenuſe, *viz.* the Moon's Diſtance from the Node, the Angle at the Node, *viz.* the Inclination of the Plane of the Moon's Orbit to the Plane of the Ecliptic, conſequently the Side oppoſite to this Angle, which expreſſes the Latitude corrected, will be be alſo given.

But there is a more ſimple Method for finding the Latitude of the Moon corrected. For the mean Latitude being computed, the Inclination of the Moon's Orbit to the Ecliptic being ſuppoſed conſtant and equal to 5°. 9′. 8″. the Equation or Correction of the Latitude will be

8′ 50″ multiplied by the Sine of double the Diſtance of the Moon from the Sun leſs the Diſtance from the Node.

XIII.

What Newton ſays with regard to the other Irregularities of the Moon's Motion.

Newton, after having expoſed the Method by which he calculated that Inequality in the Moon's Motion, called her Variation, and the Method he had followed in determining the Motion of her Nodes, and the Variation of the Inclination of her Orbit to the Ecliptic, gives an Account of what he ſays he deduced from his Theory of Gravitation, with reſpect to the Motion of the Apogee, the Variation of the Excentricity, and all the other Irregularities in the Moon's Motion. It is in the Scholium of Prop. 35. B. III. he delivers thoſe Theorems, which ſerve as a Foundation to the Conſtruction of the Tables of the Moon's Motion. The Subſtance of which is as follows.

XIV.

Annual Equations of the Motion of the Moon, of the Apogee and of the Nodes.

The mean Motion of the Moon ſhould be corrected by an Equation depending on the Diſtance of the Sun from the Earth. This Equation or Correction, called the annual one, is greateſt when the Sun is in his Perigee, and leaſt when in his Apogee. Its Maximum is 11′ 51″, and in the other Caſes, it is proportional to the Equation of the Centre of the Sun. It is to be added to the mean Motion of the Moon in the ſix firſt Signs, counted from the Apogee of the Sun, and to be ſubtracted in the ſix other Signs.

The mean Places of the Apogee and of the Nodes ſhould be alſo each corrected by an Equation of the ſame Kind, depending on the Diſtance of the Sun from the Earth, and proportional to the Equation of the Centre of the Sun. The Equation of the Apogee in its Maximum is 19′ 43″, and is to be added from the Perihelion to the Aphelion of the Earth; the Equation for the Node is to be ſubtracted from the Aphelion to the Perihelion of the Earth, and in its Maximum amounts to 9′ 24″.

XV.

Firſt ſemeſtrial Equation of the mean Motion of the Moon.

The mean Motion of the Moon requires a ſecond Correction, depending at once on the Diſtance of the Sun from the Earth, and on the Situation of the Apogee of the Moon with reſpect to the Sun; this Equation, which is in the direct Ratio of the Sine of double the Angle expreſſing the Diſtance of the Sun from the Apogee of the Moon, and in the inverſe Ratio of the Cube of the Diſtance of the Sun from the Earth, is called the Semeſtrial Equation; it is 3′ 45″ when the Apogee of the Moon is in Octants with the Sun, and the Earth is in its mean Diſtance. It is to be added, when the Apogee of the Moon advances from its Quadrature with the Sun to its Syſigie: and is to be ſubſtracted when the Apogee paſſes from the Syſigie to the Quadrature.

XVI.

Second femeftrial Equation.

The mean Motion of the Moon requires a third Correction, depending on the Situation of the Sun with refpect to the Nodes, as alfo on the Diftance of the Sun from the Earth; this Correction or Equation, which *Newton* calls the fecond Semeftrial Equation, is in the direct Ratio of the Sine of double the Diftance of the Node from the Sun, and in the inverfe Ratio of the Cube of the Diftance of the Earth from the Sun: it amounts to 47″ when the Node is in Octant with the Sun and the Earth in its mean Diftance; it is to be added when the Sun recedes in Antecedentia from the neareft Node, and is to be fubtracted when the Sun advances in Confequentia.

XVII.

After thofe three firft Equations of the Moon's Motion, follows that which is called her Equation of the Centre; but this Equation cannot be obtained as that of the other Planets, by the Help of one Table, becaufe her Excentricity varies every Inftant, and the Motion of her Apogee is very irregular. In order therefore to obtain the Equation of the Centre of the Moon, the Excentricity and the true Place of the Apogee of the Moon is firft to be determined, which is effected by the Help of Tables founded on the following Propofition.

Determination of the Place of the Apogee, and of the Excentricity.

A ftraight Line being taken to exprefs the mean Excentricity of the Orbit of the Moon, which is 5505 Parts of the 100000 into which the mean Diftance of the Moon from the Earth is fuppofed to be divided; at the Extremity of this ftraight Line affumed as a Bafe, an Angle is made equal to double of the annual Argument, or of double the Diftance of the Sun from the mean Place of the Moon once corrected, as has been already directed. The Length of the Side of this Angle is afterwards determined by making it equal to 1172¾, half the Difference between the leaft and greateft Excentricity. The Triangle being then completed, the other Angle at the Bafe, expreffes the Equation or Correction to be made to the Place of the Apogee already once corrected; and the other Side of the Triangle which is oppofite to the Angle made equal to double of the annual Argument, will exprefs the Excentricity correfponding to the propofed Time. The Equation of the Apogee being added to its Place already corrected, if the annual Argument be lefs than 90, or between 180 and 270, or being fubtracted in every other Cafe, the true Place of the Apogee will be obtained, which is to be fubducted from the Place of the Moon corrected by the three Equations already mentioned, in order to have the mean Anomaly of the Moon. With this Anomaly, and the Excentricity, the Equation of the Centre by the ufual Methods will be obtained, and confequently the Place of the Moon corrected for the fourth Time.

Equation of the Centre, or fourth Correction of the Place of the Moon.

The Equation of the Centre may be obtained without fuppofing the Excentricity variable, or a Motion in the Apogee, by applying to double of the Angle at the Moon subtended by the mean Excentricity, or to the mean Equation of the Centre, the Equation 80′ Sin (2 Dif. ☾ ☉ — m. An. ☾) expressing the Variation produc'd by the Change of Excentricity, and Libration of the Apogee.

XVIII.

The Place of the Moon corrected for the fifth Time, is obtained by applying to the Place of the Moon corrected for the fourth Time, the Equation called the Variation which was already found, to be always in the direct Ratio of the Sine of double the Angle expressing the Distance of the Moon from the Sun, and in the inverse Ratio of the Cube of the Distance of the Earth from the Sun; this Equation, which is to be added in the first and third Quadrant (in counting from the Sun) and subtracted in the second and fourth is 35′ 10″ when the Moon is in Octant with the Sun, and the Earth in its mean Distance.

The fifth Equation of the Moon's Motion, is her Variation.

XIX.

The sixth Equation of the Motion of the Moon is proportional to the Sine of the Angle which is obtained by adding the Distance of the Moon from the Sun, to the Distance of the Apogee of the Moon from that of the Sun. Its Maximum is 2′ 20″, and it is positive when this Sum is less than 180 Degrees, and negative if this Sum be greater.

Sixth Equation.

XX.

The seventh and last Equation, which gives the true Place of the Moon in its Orbit, is proportional to the Distance of the Moon from the Sun; it is 2′ 20″ in its Maximum.

Seventh Equation.

XXI.

It is scarce possible to trace the Road which could have conducted *Newton* to all those Equations, except some Corollaries of Prop. 66. where he shews how to estimate the perturbating Forces of the Sun. It is easy to perceive, that of those two Forces, the one which acts in the Direction of the Ray of the Orbit of the Moon, being joined to the Force of the Earth, alters the inverse Proportion of the Square of the Distances, and consequently should change not only the Curvature of the Orbit, but also the Time which the Moon employs in describing it: But how did *Newton* employ those Alterations of the central Force, and what Principles did he make use of to avoid or surmount the extreme Complication and the Difficulties of Computation which occur in this Inquiry is what has not as yet been discovered, at least after a satisfactory Manner.

The Method Newton made use of in investigating the foregoing Corrections has not as yet been discovered.

We find, it is true, in the first Book of the *Principia*, a Proposition concerning the Motion of the Apsides in general, by which we learn, that if to a Force which acts inversely as the Square of the Distance, another Force which is inversely as the Cube of the Distance be joined, the Body will describe an Ellipse whose Plane revolves about the Centre of the Forces. In the Corollaries of this Propofition, *Newton* extends his Conclufion to the Cafe in which the Force, added to the Force which follows the Law of the Square of the Diftance, does not vary in the Triplicate, but in the Ratio of any Power of the Diftance.

If therefore the perturbating Force of the Sun depended on the Diftance of the Moon from the Earth alone, by the Help of this Propofition, the Motion of the Apfides of the Moon could be determined; but as the Diftance of the Moon from the Sun enters into the Expreffion of this Force, it is only by new Artifices, and perhaps as difficult to be found as the Determination of the entire Orbit of the Moon: the Propofition of *Newton* concerning the Motion of the Apfides in general, can be applied to the Moon. Senfible of which, the firft Mathematicians of the prefent Age, have abandoned in this, as in every other Point that regards the Theory of the Moon, the Road purfued by the Commentators of *Newton*, and have refumed the whole Theory from its very Beginning; they have inveftigated in a direct Manner, the Paths and Velocities of any three Bodies which attract each other mutually. The Succefs which has attended their united Efforts fhall be explained hereafter.

XXII.

Theory of the Satellites of Jupiter and thofe of Saturn.

It is manifeft, that the Satellites of Jupiter, confidered feparately, fhould be affected by the three Forces which actuate them, in the fame Manner as the Moon; but their Number introduces a new Source of Inequalities, not only each of them is attracted by Jupiter and the Sun, but they attract each other mutually, and this mutual Attraction fhould produce very confiderable Variations in their Motions; Variations fo much the more difficult to be fubjected to exact Computations, as they depend on their different Pofitions with refpect to each other, which their different Diftances and Velocities continually alter. However, the Laws of their Motions difcovered by *Bradley*, *Wargentin* and *Maraldi*, have enabled the eminent Mathematicians of this Age, to furmount thofe Difficulties, and to apply the Solution of the Problem of the three Bodies to the Inveftigation of the Inequalities of the Motions of thofe Satellites, with almoft the fame Succefs as they had already done to thofe of the Moon.

As to the Satellites of Saturn, Aftronomers have not been able to determine the Phenomena of their Motions with any Degree of Accuracy on Account of their great Diftanee; hence the Theory of thofe Planets is reduced to fhew, that the Forces with which they act on each other, or that with which the Sun acts on them, and difturbs their Motions, are very inconfiderable when compared with the Force with which they tend towards their principal Planet; and that this Attraction is inverfely proportional to the Squares of the Diftances.

THEORY *of the* COMETS.

I.

THOUGH the Comets have in all Ages, drawn the Attention of Philosophers, yet it is only since the last Century and even since *Newton*, they can be said to be known. *Seneca* seemed to have foreseen the Discoveries which one Day would be made concerning those Bodies, but the Germ of the true Principles which he had sown, were stifled by the Doctrine of the Peripateticks, who, transmitting from Age to Age, the Errors of their Master, maintained that the Comets were Meteors or transient Fires.

The Peripateticks regarded the Comets as Meteors.

II.

Several Astronomers, but particularly *Ticho*, proved this Opinion to be erroneous, by shewing by their Observations, that those Bodies were situated far above the Moon, they destroyed at the same Time, the solid Heavens, invented by the scholastic Philosophers, and proposed Views concerning the System of the World, which were much more conformable to Reason and Observation. But their Conjectures were yet very far from that Point, to which the Geometry of *Newton* alone could attain.

Ticho proved that they were situated above the Moon.

III.

Descartes, to whom the Sciences are so much indebted, did not succeed better than his Predecessors in his Enquiries concerning the Comets; he neither thought of employing the Observations which were so easy for him to collect, nor Geometry to which it was so natural to have Recourse, and which he had carried to so great a Point of Perfection; he considered them as Planets wandering through the different Vortices, which, composed according to him, the Universe; and did not imagine that their Motions were regulated by any Law.

Descartes regarded them as Planets wandering from Vortex to Vortex.

IV.

Newton, aided by his Theory of the Planets, and by the Observations which taught him that the Comets descended into our planetary System, soon perceived that those Bodies were of the same Nature with the Planets, and subject to the same Laws.

Every Body placed in our planetary System, should, according to the Theory of *Newton*, be attracted by the Sun, with a Force reciprocally proportional to the Squares of the Distances, which combined with a Force of Projection, would make it describe a Conic Section about the Sun placed in the Focus. According therefore to this Theory, the Comets should revolve in a Conic Section about the Sun, and describe Areas proportional to the Times.

Newton discovered that the Comets revolve about the Sun, and are subjected to the same Laws as the Planets.

V.

Calculation and Observation, the faithful Guides of this great Man, enabled him to verify his Conjecture. He solved this fine Astronomico-geometrical Problem. Three Places of a Comet which is supposed to

He determines the Orbit of a Comet from three Observations.

move in a parabolic Orbit, describing round the Sun Areas proportional to the Times, being given, with the Places of the Earth in the Ecliptic corresponding to those Times, to find the Vertex and Parameter of this Parabola, its Nodes, the Inclination of its Plane to that of the Ecliptic, and the Passage of the Comet at the Perihelion, which are the Elements necessary for determining the Position and Dimensions of the Parabola.

This Problem, already of very great Difficulty in a parabolic Orbit, was so extremely complicated in the Ellipse and Hyperbola, that it was necessary to reduce it to this Degree of Simplicity. Besides the Hypothesis of a parabolic Orbit, answered in Practice, the same End as that of the Ellipse, because the Comets during the Time they are visible, describing but a very small Portion of their Orbit, move in very excentric Ellipses, and it is demonstrated that the Portions of such Curves which are near their Foci, may be considered without any sensible Error as parabolic Arcs.

VI.

Rules for determining the Elements of a Comet.

The Result of his Solution of this important Problem is as follows. From the observed Distances of the Comet from the fixed Stars, whose right Ascensions and Declinations are known, deduce the right Ascension and Declination, and from thence the Longitude of the Comet reduced to the Ecliptic, and its Latitude, corresponding to each Observation: Compute the Longitude of the Sun at the Time of each Observation, take the Difference (A, A′, A″) between the Longitude of the Comet and that of the Sun, corresponding to each Observation, which is the Elongation of the Comet reduced to the Ecliptic. Compute also the Distance (B, B′, B″) of the Earth from the Sun at the Time of each Observation.

Preliminary Computations.

First Hypothesis.

Those preleminary Calculations being performed, assuming by Conjecture, the Distances (Y and Z) of the Comet from the Sun, reduced to the Ecliptic at the Time of the first and second Observation, determine the true Distances by the Means of the two following Proportions, *as the assumed Distance* (Y *or* Z) *of the Comet from the Sun in the first or second Observation, is to the Sine of the observed Elongation,* (A *or* A′) *so is the Distance* (B *or* B′) *of the Earth from the Sun at the Time of the first or second Observation, to the Sine of the Angle* (C *or* C′) *contained by the straight Lines drawn from the Earth and the Sun to the Comet. Add this Angle* (C *or* C′) *to the Elongation* (A *or* A′) *their Sum will be the Supplement of the Angle of Commutation* (D *or* D′). *And then say as the Sine of the Angle of Elongation* (A *or* A′) *is to the Sine of the Angle of Commutation* (D *or* D′), *so is the Tangent of the observed geocentric Latitude of the Comet corresponding to the first or second Observation, to the Tangent of the corresponding heliocentric Latitude of the Comet* (E *or* E′).

Angle at the Comet.

Heliocentric Latitude.

Each of the curt Diſtances Y and Z divided by the Coſine of the correſponding heliocentric Latitude E and E′ gives the true Diſtances of the Comet from the Sun.

Vector Rays.

Find the Angle contained by thoſe Diſtances thus: Add to, or ſubſtract from the Places of the Earth, the correſponding Angles of Commutation (D, D′) which will give the two heliocentric Longitudes of the Comet, whoſe Difference (F) is the heliocentric Motion of the Comet in the Plane of the Ecliptic. Then ſay, *As Radius, is to the Coſine of the Motion* (F) *of the Comet in the Ecliptic, ſo is the Cotangent of the greateſt of the two heliocentric Latitudes, to the Tangent of an Arc* X. Subſtract this Arc X from the Complement of the leaſt heliocentric Latitude, and call the Remainder X′. Then *the Coſine of the firſt Arc* X, *will be to the Coſine of the ſecond Arc* X′, *as the Sine of the greateſt of the two Latitudes, to the Coſine of the Angle contained by the two vector Rays of the Comet.*

Motion of the Comet in its Orbit.

Which being done, determine the Place of the Perihelion by the following Rule: ſubſtract the Logarithm of the leaſt vector Ray from that of the greateſt, take half the Remainder, to whoſe Characteriſtic, 10 being added, it will be the Tangent of an Angle, from which ſubducting 45°, the Logarithm of the Tangent of the Remainder, added to the Log. of the Cotangent of ½ of the Motion of the Comet in its Orbit, will be the Logarithm of the Tangent of an Angle, to which ½ of the Motion of the Comet in its Orbit being added, the Sum will be the Half of the greateſt true Anomaly, and their Difference will be Half the leaſt of the two true Anomalies. Double thoſe Quantities to obtain the two true Anomalies, which will be both on the ſame Side of the Perihelion, when their Difference is the whole Motion of the Comet, but on different Sides of it, when it is their Sum, which is equal to the whole Motion of the Comet.

True Anomalies.

Find the Perihelion Diſtance by adding twice the Logarithm of the Coſine of the greateſt of the Halfs of the two true Anomalies, to that of the greateſt of the two vector Rays, which will be the Logarithm of the Perihelion Diſtance required.

Perihelion Diſtance.

Determine the Time which the Comet ſhould employ in deſcribing the Angle contained by the two vector Rays, by the following Rule: *To the conſtant Logarithm* 1,9149328, *add the Logarithm of the Tangent of half of each true Anomaly. Add the Triple of this ſame Logarithm of the Tangent to the conſtant Logarithm* 1,4378116, *the Sum of the two Numbers correſponding to thoſe two Sums of Logarithms, will be the exact Number of Days correſponding to each true Anomaly in a Parabola whoſe perihelion Diſtance is* 1. *Take the Logarithm of the Difference or Sum of thoſe two Numbers, according as the two Anomalies are ſituated on the ſame Side, or on different Sides of the Perihelion. To this Logarithm add the* $\frac{3}{2}$ *of the Log. of the perihelion Diſtance, the Sum will be Log. of the*

Interval of Time employed in deſcribing the Angle contained by the two vector Rays.

Time the Comet should employ to describe the Angle contained by the two vector Rays.

Second Supposition of the first Hypothesis. If the Time thus found, does not agree with the observed Time, another Value is to be assumed, for the curt Distance *(Z)* corresponding to the second Observation, retaining the assumed Distance (Y) corresponding to the first, and the heliocentric Longitude and Latitude of the Comet from thence deduced, and all the Operations indicated in the foregoing Articles being repeated, another Expression will be found for the Interval of Time between the two Observations. Which if it approaches nearer the observed Time, the second Value assumed for the Distance (Z) is to be preferred to the first; if not, a third Value is to be assumed for this Distance, and by the Increase or Decrease of the Errors, the Value to be assumed for it, so that the Interval of Time calculated may agree with the observed one, will easily be discovered, and consequently a Parabola will be found, which answers the two first Observations, which may be called *first Hypothesis*.

SECOND HYPOTHESIS. This Parabola answering the two first Observations would be the Orbit sought if it answered likewise the third Observation; but as this never happens, another Parabola is to be found which answers the two first Observations, by increasing or diminishing, at will, the curt Distance (Y) preserved constant in the first Hypothesis, and preserving it still constant, but varying the second assumed Distance *(Z)* until this second Parabola is obtained.

The third Observation calculated in those two Parabolas, will shew which of them approaches nearest the true Orbit sought. To calculate this third Observation in each Hypothesis, the Time of the Passage of the Comet at the Perihelion, the Inclination to the Ecliptic, and the Place of the Nodes of each Parabola is first to be determined.

Passage at the Perihelion. To determine the Time of the Passage of the Comet at the Perihelion, find the Number of Days corresponding to one of the two true Anomalies; for Example, to that which corresponds to the first Observation in the Parabola whose perihelion Distance is 1, as before directed, the Logarithm of this Number of Days added to $\frac{3}{2}$ of the Logarithm of the perihelion Distance, will be the Logarithm of the Interval of Time elapsed between the first Observation and the Passage of the Comet at the Perihelion, which is to be added to or subtracted from the Time of the Observation, according, as it was made before or after the Passage of the Comet at the Perihelion.

Place of the Node. To determine the Place of the Node, say, *As the Sine of the second Arc* X′ *is to the Sine of the first Arc* X, *so is the Tangent of the Motion of the Comet in the Ecliptic, to the Tangent of an Angle* (R). *Then the Radius, is to the Sine of the least Latitude, as the Tangent of the Angle* R, *to the Tangent of the Distance from the Node.* By the Means of this Distance from the Node, and the heliocentric Longitude of the Comet, the heliocentric Longitude of the Node is obtained. With which and the Distance measured on the Orbit of the Comet, the Place of the Perihelion is Determined. To find this Distance say, *As the Sine of Angle R, to Radius, so is this Distance measured on the Ecliptic, to the Distance required.*

Inclination.

To determine the Inclination say, *As the Radius is to the Sine of the Angle R, so is the Cosine of the least Latitude, to the Cosine of the Angle of Inclination.*

The Elements of each Parabola being determined, the Place of the Comet seen from the Earth, answering to the third Observation, is computed in each, by the following Rules.

Rules for finding the heliocentric Longitude and Latitude of a Comet.

First, Take the Logarithm of the Difference between the Time of the third Observation, and the Time of the Passage of the Comet at the Perihelion; subtract from it $\frac{3}{2}$ of the Logarithm of the perihelion Distance, the Remainder will be the Logarithm of the Difference between the Time of the third Observation and the Time of the Passage of the Comet at the Perihelion of the Parabola, whose perihelion Distance is 1. Secondly, Find the true Anomaly corresponding to this Time, by solving the Equation $t^3+3t=\frac{b}{27,4038}$ (b) in which t expresses the Tangent of half the true Anomaly, and b the Time employed in describing it. Thirdly, When the Motion of the Comet is direct, add this true Anomaly to the Place of the Perihelion, if the third Observation was made after the Passage of the Comet at the Perihelion; But subtract it from the Place of the Perihelion if the Observation was made before the Passage at the Perihelion. And when the Motion of the Comet is retrograde, add the true Anomaly to the Place of the Perihelion, if the Observation was made before the Passage at the Perihelion; but subtract it from the Place of the Perihelion, if the Observation was made after the Passage at the Perihelion; by this Means, the true heliocentric Longitude of the Comet in its Orbit is obtained. Fourthly, Take the Difference between this Longitude and that of the ascending Node, which will be the true Argument of the Latitude of the Comet. Fifthly, say, *As the Radius is to the Cosine of the Inclination, so is the Tangent of the Argument of Latitude, to the Tangent of this Argument measured on the Ecliptic*; which added to the true Place of the Node, gives the heliocentric Longitude reduced to the Ecliptic. Sixthly, say, *As the Radius is to the Sine of the Argument of Latitude, so is the Sine of the Inclination of the Orbit of the Comet, to the Sine of its heliocentric Latitude*, which, when the Mo-

(b) The Equation $t^3+3t=\frac{b}{27,4038}$ may be solved thus: Make a Right-angled Triangle, one of whose Sides is expressed by 1, and the other by $\frac{b}{54,8077}$, calculate the Hypothenuse (H), find two mean Proportionals between $H+\frac{b}{54,8077}$ and $H-\frac{b}{54,8077}$ and their Difference will be the Value of t.

tion of the Comet is direct, is North or South, according as the Argument of Latitude is less or greater than six Signs; and when the Motion of the Comet is retrograde, it is North or South according as the Argument of Latitude is greater or less than six Signs. Seventhly, Add the Logarithm of the Cosine of the heliocentric Latitude to the Log. of the perihelion Distance, and subtract from this Sum the Log. of double of the Cosine of half the true Anomaly, the Remainder will be the Logarithm of the curt Distance corresponding to the third Observation.

Rule for finding the cu t Distance.

Rules for finding the geocentric Longitude and Latitude.

Eighthly, Take the Difference between the Logarithm of the curt Distance, and that of the Distance of the Earth from the Sun, add 10 to the Characteristic of this Difference, and it will be the Logarithm of the Tangent of an Angle; from which subtract 45d. and to the Logarithm of the Tangent of the Remainder, add the Logarithm of the Tangent of the Complement of half the Angle of Commutation, the Sum will be the Logarithm of the Tangent of an Arc, which add to this Complement, if the curt Distance of the Comet from the Sun exceeds the Distance of the Earth from the Sun, but subtract from this Complement if the Distance of the Comet be less than that of the Earth; in order to obtain the Angle of Elongation, which added to or subtracted from the true Place of the Sun, according as the Comet seen from the Earth, is to the East or to the West of the Sun, will give the geocentric Longitude of the Comet. Ninthly, and lastly say, *As the Sine of the Angle of Commutation, is to the Sine of the Angle of Elongation, so is the Tangent of the heliocentric Latitude of the Comet to the Tangent of its geocentric Latitude.* The Longitude and Latitude thus found ought to agree with the observed ones, if the Parabola obtained was really the Orbit described by the Comet.

VII.

Example. Let it be proposed to find the Elements of the Parabola described by the Comet which was observed in *Europe*; the beginning of *March* 1742, with a very remarkable Tail, coming with extraordinary Rapidity from the southern Hemisphere, and afterwards advancing towards the North Pole, its heliocentric Motion being retrograde, and its Velocity and Splendor decreasing to the 6th of *May*, when it disappeared.

1742. mean Time.	Obs. Long. of the Comet.	Observ. Lat. North of the Comet.	Long. of the Sun calculated.	Log. of the Dis. of the E. from the Sun	Elong. of the Comet from the Sun.
h. m. s.	s ° ′ ″	° ′ ″	s ° ′ ″		° ′ ″
4 March at 16 9 50	9 16 0 40	34 45 37	11 14 27 44	9.996910	58 27 4 W.
28 . at 13 39 0	2 18 52 45	63 3 55	0 8 11 28	9.999840	
24 April at 9 39 0	8 1 5 33	50 32 50	1 4 27 15	0.003092	56 38 27 E.

1 Suppoſition, Y=0,879, Z=0,957 of the mean Diſtance of the Earth from the Sun =1, then Angle C=105° 42′ 48″, C′=61° 31′ 0″, C+A=164° 9′ 52″, and C′+A′=118° 9′ 17″, wherefore Angle D=15° 50′ 8″, and Angle D′=61° 50′ 43″, conſequently the heliocentric Latitudes, E=10° 31′ 42″ North and E′=52° 3′ 38″, and the Log. of the vector Rays, V=9,954455 V′=0,192159. FIRST HYPOTHESIS. Heliocentric Latitude and Longitude of the Comet.

The Angle of Commutation D=15° 50′ 8″, being added to 5ˢ 14°, 27′ 44″, and Angle D′=61° 50′ 43″ ſubtracted from 7ˢ 4° 27′ 16″, the correſponding Longitudes of the Earth, gives the heliocentric Longitudes of the Comet, L=6ˢ 0° 17′ 52″, and L′=5ˢ 2° 36′ 33″; their Difference F=27° 41′ 19″ is the Motion of the Comet in the Ecliptic, the Arc X will be found =34° 37′ 11″, and Arc X′=42° 51′ 7″; conſequently the Angle contained by the two vectorRays =45° 22′ 8″. Angle contained by the two vector Rays.

The Log. of the greateſt vector Ray, 0,192159 leſs the Log. of the leaſt, 9,954455=0,237704, and its Half 10,118852, 10 being added to its Characteriſtic, is the Tangent of 52° 44′ 38″, from which 45° being ſubtracted, and to the Log. of the Tangent of the Remainder 7° 44′ 38″, the Log. of Cotangent of 11° 20′ 32″, the ¼ of the Motion (45° 22′ 8″,) of the Comet in its Orbit being added, the Sum will be the Logarithm of the Tangent of 34° 8′ 5″ ½, whereby the Halfs of the two true Anomalies are found to be 22° 47′ 33″ ½, and 45° 28′ 37″ ½, conſequently the leaſt true Anomaly =45° 35′ 7″, and the greateſt =90°, 57′ 15″; and their Difference being equal to the Motion of the Comet in its Orbit, thoſe two Anomalies are on the ſame Side of the Perihelion. The Log. of the perihelion Diſtance will be found =9,883835. True Anomalies. Perihelion Diſtance.

To determine the Time the Comet employed to deſcribe the Angle contained by the two vector Rays, to the conſtant Log. 1,9149328 adding 0,007233 Log. of the Tangent of 45°. 28′. 37″½, and to the conſtant Log. 1,438112 adding 0,021699 Triple of the Log. of this ſame Tangent. I find 83,592 and 28,808 for the Numbers correſponding to 1,922166 and 1,459512 Sums of thoſe Lagarithms, conſequently 112,400 Days is the Time correſponding to the true Anomaly 90°. 57′ 15″, in a Parabola whoſe perihelion Diſtance is 1. By a like Proceſs, I find the Number of Days 36,579 correſponding to the true Anomaly 45° 35′. 7″, in the ſame Parabola, I take the Difference 75,821 of thoſe Times, becauſe the two Anomalies are ſituated on the ſame Side of the Perihelion, whoſe Logarithm 1,879789 added to 9,825752 the $\frac{3}{2}$ of the Log. of the perihelion Diſtance, is the Log. 1,705541, to which correſponds 50,762 Days, Time employed by the Comet to deſcribe the Angle contained by the two vector Rays. Interval of Time between the two Obſervations calculated.

Comparing this Time with the Interval 50,728 ½ between the two Obſervations, I find it exceeds it by 0,033, I therefore make a Variation of 0,001 in the Diſtance (Z), in order to diſcover which Way, and by how much the Elements of the corresponding Parabola will be changed.

Second Supposition of the first Hypothesis.

II Supposition, Y = 0,879, Z = 0,956, and repeating the same Calculations as in the first Supposition, I find the heliocentric Latitudes E = 12° 31′ 42″, E′ = 52° 1′ 54″ ½, the Log. of the vector Rays, V = 9,954455, V′ = 0,191424, the heliocentric Longitudes, L = 6s 0° 17′ 52″, L′ = 5s 2° 43′ 11″. The Motion of the Comet in the Ecliptic = 27° 34′ 41″, and the Motion of the Comet in its Orbit = 45° 18′ 13″ the true Anomalies 45° 32′ 3″, and 90° 50′ 16″, the corresponding Days 36.529 and 112,056, the Log. of the perihelion Distance = 9,883997; finally the reduced Time employed in describing the Angle contained by the two vector Rays 50,594 Days. From whence I find that by increasing Z by the Quantity 0,001, I diminish the Time by 0,168: And I say, 0,168 : 0,001 : : 0,033½ : 0,0002. I diminish therefore Z by 0,0002 to obtain a Parabola answering the Conditions required.

III Supposition, Y = 0,879, Z = 0,9568, and I find the heliocentric Latitudes, E = 12° 31′ 42″, E′ = 52° 3′ 16″½, the Log. of the vector Rays, V = 9,954455, and V′ = 0,192009; the heliocentric Longitudes, L = 6s 0° 17′ 52″, and L′ = 5s 2° 37′ 53″; the Motion of the Comet in the Ecliptic, 27° 39′ 59″; and the Motion in its Orbit 45° 21′ 22″; the true Anomalies 45° 34′ 28″, and 90° 55′ 50″; the corresponding Times 36,568½, and 112,330 Days: The Log. of the perihelion Distance 9,883870, and the Time reduced employed in describing the Angle contained by the two vector Rays, 50,728½ Days, agreeable to Observation.

Having found a Parabola answering the two first Observations, I search for another, answering the same Observations, by making a Variation in the Distance (Y) preserved constant in the first Hypothesis.

SECOND HYPOTHESIS.

First Supposition of the second Hypothesis.

IV Supposition, Y = 0,878, Z = 0,957, and I find the heliocentric Latitudes, E = 12° 42′ 11″, E′ = 52° 3′ 38″, the Log. of the vector Rays, V = 9,954257, V′ = 0,192159, the heliocentric Longitudes, L = 6s 0° 31′ 54″, and L′ = 5s 2° 36′ 33″; the Motion of the Comet in the Ecliptic = 27° 55′ 21″, the Angle contained by the two vector Rays = 45° 17′ 56″, the true Anomalies 45° 44′ 56″ and 91° 2′ 52″, the corresponding Times 36,743 and 112,680, the Log. of the perihelion Distance 9,883115, the reduced Time employed in describing the Angle formed by the two vector Rays 50,714, which differs by 0,014½ from the observed Interval, consequently by diminishing Y by 0,001, the Time is diminished by 0,048. I say, 0,048 : 0,001 : : 0,014½ : 0,0003.

Second Supposition of the second Hypothesis.

V Supposition, Y = 0,8783 Z = 0,957, I find the heliocentric Latitudes, E = 12° 39′ 2″ E′ = 52° 3′ 38″ the Log. of the vector Rays, V = 9,954316 V′ = 0,192159, the heliocentric Longitudes, L = 6s 0° 27′ 40″, L′ = 5s 2° 36′ 33″, the Motion of the Comet in the Ecliptic 27° 51′ 7″ the Angle contained by the two vector Rays 45° 19′ 20″, the true Anomalies 45° 41′ 45″ and 91° 1′ 5″ the corresponding Times 36,689, and 112,590, the Log. of the perihelion Distance 9,883344, and the Time reduced employed in describing the Angle contained by the two vector Rays =50,729 agreeable to Observation.

Having found two Parabolas answering the two first Observations, we are next to examine which approaches nearest the Orbit of the Comet sought, by calculating the third Observation in each; for which Purpose I calculate the Place of the Perihelion, the Time of the Passage at the Perihelion, the Inclination to the Ecliptic, and the Place of the Nodes of each Parabola.

To determine those Elements in the first Parabola, I find the Angle R=23° 40′ 15″, then the Distance of the Comet reduced to the Ecliptic at the first Observation from the ascending Node 5° 25′ 45″, which added to the heliocentric Longitude of the Comet, the 4th of March, which is 6s 0° 17′ 52″, because its heliocentric Motion is retrograde, gives the Place of the Node, in 6s 5° 43′ 37″. The Distance of the Comet from the Node measured on its Orbit, which I find to be 13° 38 14″, subtracted from the Place of the Node, gives the Place of the Comet in in its Orbit, at the Time of the first Observation: and because it had then 45° 34′ 28″ true Anomaly, I add them to its Place in its Orbit to obtain the Place of the Perihelion in 7s 7° 39′ 51″. I add $\frac{3}{2}$ of the Log. of the perihelion Distance to that of 36,568$\frac{1}{4}$ Days, Time corresponding to the least true Anomaly 45° 34′ 28″, which gives 24,486 Days, for the Interval of Time elapsed between the first Observation, and the Instant of the Passage of the Comet at the Perihelion, which being subtracted from the 4th of March at 16h 9′ 50″, or at 0,673$\frac{1}{2}$, the Time of the first Observation, fixes the Instant of the Passage at the Perihelion to the 8th of February at 0,188. In fine, I find the Angle of Inclination of the Plane of the Ecliptic, and that of the Comet to be 66° 56′ 14″.

Elements of the Comet calculated in the first and second Hypothesis.

The same Elements in the second Parabola are, the ascending Node in 6s 5° 59′ 6″, the Place of the Perihelion in 7s 7° 53′ 42, the Inclination, 66° 47′ 14″, and the Time of the Passage at the Perihelion, February the 8th, 151$\frac{1}{2}$.

From those Elements I calculate the geocentric Longitude for the 28th of March, at 0,569 of the Day, in each Parabola. The Interval of Time elapsed between the Passage at the Perihelion in the first Parabola, and the Time of the Observation 28th March 0,569 is 48,381 Days. The Log. of the perihelion Distance, 9,883870, its Triple is, 9,651610, its Half, 9,825805, which being subtracted from 1,684675, Log. of 48,381 gives 1,858870, Log of 72,255 Days, which corresponds to 73° 11′ 7″, or 2s 13° 11′ 7″ Anomaly, which subtracted from the Place of the Perihelion 7s 7° 39′ 51″, because the Comet being retrograde, the given Instant follows, that of the Passage at the Perihelion, which gives the true heliocentric Place of the Comet in its Orbit, Geocentric Longitude of the Comet calculated in the first and second Hypothesis.

4ˢ 24° 28′ 44″, from 4ˢ 24° 28′ 44″, subtracting 6ˢ 5° 43′ 37″, the Place of the ascending Node, the Argument of Latitude 10ˢ 18° 45′ 7″ is obtained, which measured on the Ecliptic is 11ˢ 11° 2′ 47″; consequently the heliocentric Longitude of the Comet is 5ˢ 16° 46′ 24″, and the heliocentric Latitude, 37° 20′ 41″ North because the Argument of Latitude of the Comet, which is retrograde, is greater than six Signs.

The true Place of the Sun the 28 of March, at 13ʰ 39ᵐ is 0ˢ 8° 11′ 28″, and the Log. of its Distance from the Earth, is 9,999841; therefore the true Place of the Earth seen from the Sun, is 6ˢ 8° 11′ 28″, which exceeds 5ˢ 16° 46′ 24″ by 21° 25′ 4″, which is the Angle of Commutation. I find the Log. of the curt Distance, corresponding to the third Observation =9,974915, I subtract 9,974915 from 9,999841, Log. of the Distance of the Sun from the Eearth: The Remainder is 0,024926, which by adding 10 to its Characterastic, gives 10,024926, Log. of the Tangent of 46° 38′ 42″ $\frac{2}{3}$, from which subtracting 45, the Log. of Tan. of Remainder, 1° 38′ 42″, added to that of the Tangent of 79° 17′ 28″, (Complement of 10° 42′ 32″, half of the Angle of Commutation 21° 25′ 4″) the Sum is the Log. of the Tangent of 8° 37′ 39″, which subtracted from 79° 17′ 28″; because the Distance of the Comet from the Sun, is less than that of the Earth from the Sun, gives 70° 39′ 49″, or 2ˢ 10° 39′ 49″, for the Angle of Elongation. By Means of a Figure representing the Ecliptic divided into 12 Signs, in which I place the Sun, the Earth, and the Comet, according to their Longitudes found by the above Calculations, I perceive that the Comet seen from the Earth, is to the East of the Sun. I therefore add the Angle of Elongation to the true Place of the Sun, which gives the true geocentric Longitude of the Comet, in 2ˢ 18° 51′ 17″, which is less than the observed Longitude 2ˢ 18° 52′ 45″ by 1′ 28″; by a like Process I find the geocentric Longitude of the Comet in the second Parabola, the 28 of March, in 2ˢ 18° 45′ 14″, which is less than the observed Longitude, by 7′ 31″; consequenly neither of the two Parabolas, is the Orbit of the Comet.

THIRD HYPOTHESIS.

But because the Variations of the Orbits, are sensibly proportional to those made in the curt Distances, to obtain the two curt Distances which correspond to the Orbit sought. I make those two Proportions; (c) *As 6′ 3″ Difference of the two Errors —1′ 28″ and —7′ 31″, Is to the least of the two 1′ 28″ : So is 0,0007 and 0,0002, Corrections made to the two curt Distances Y and Z, to obtain two Parabolas answering the two first Observations, to 0,000235 and 0,000065, Corrections to be made to those Distances Y and Z, to obtain the Orbit required.*

To apply those Corrections, I observe, that since Y, supposed = to 0,879, gives an Error of —1′ 28″, and Y supposed = to 0,8783, gives an Error of —7′ 31″, by diminishing Y, the Error is increased; from whence I conclude, that 0,000235 is to be added to 0,879, to obtain

(c) I would have said as the Sum of the Errors &c. if the one was by excess and the other by defect.

the true Value of Y, which consequently will be 0,879235; in like Manner, I find that Z should be supposed =0,956735.

Geocentric Longitude and Latitude of the Comet calculated in the third Hypothesis.

VI Supposition, Y=0,879235, and Z=0,956735, and I find the heliocentric Latitudes, E=12° 29′ 17″ ½, E′=52° 3′ 10″ ½; the Log. of the vector Rays, V=9,954504, and V′=0,191963; the heliocentric Longitudes, L=6ˢ 0° 14′ 37″, and L′=5ˢ 2° 38′ 19″; the true Anomalies, 45° 32′ 0″ and 90° 54′ 4″; the corresponding Times 36,528 and 112,243 Days; the Log. of the perihelion Distance 9,884049; and the Time employed in describing the Angle contained by the two vector Rays, 50,729; the Place of the Node in 6ˢ 5° 38′ 29″; the Place of the Perihelon, 7ˢ 7° 35′ 13″, the Inclination of the Orbit, 66° 59′ 14″; and the Time of the Passage at the Perihelion the 8th of February, at 4ʰ 48′: In fine, from those Elements, I calculate the geocentric Longitude and Latitude the 28th of March, at 13ʰ 39′, which I find, the one in 2ˢ 18° 53′ 18″, the other 63° 3′ 57″ North, agreeable to Observation. By these Rules the following Table was calculated, containing the Elements of all the Comets which have been observed with any Degree of Accuracy.

Table of the Elements of the Comets.

Years.	Place of the ascending Node.	Inclination	Place of the Perihelion.	Perihelion Distance.	Time of the Passage at the Perihelion at Paris.	
	s ° ′ ″	° ′ ″	s ° ′ ″		d h ′	
837	6.26.33.00	12.00. 0	9.19. 3.00	0,5800	March. 11.12.00	retr.
1231	0.13.30.00	6. 5.00	4.14.48.0	0,9478	Jan. 30. 7.00	dir.
1264	7.28.45.00	20.25.00	9. 5.45.00	0,4108	July. 17. 6.10	dir.
1299	3.17. 8.0	68.57.30	0. 3.20.00	0,3179	March. 31. 7.38	retr.
1301	0.16.00.00	70.00.0	9.30.00.00	0,4467	Oct. 22. 0.0	retr.
1337	2. 6.22.00	32.11.00	0.20.00.00	0,6445	June. 1. 1.00	retr.
1472	9,11.46.20	5.20.00	1,15.33.30	0,5427	Feb. 28.22.32	retr.
1532	2.20.27.00	32.36.00	3.21. 7.00	0,5092	Oct. 19.22.21	dir.
1533	4. 7.42.00	46.30.00	5. 6.38.00	0,1525	May. 25.10.32	dir.
1556	5.25.42,00	32. 6.30	9. 8.50.00	0,4639	April. 21.20.12	dir.
1577	0.25.52.00	74.32.45	4. 9.22.00	0,1835	Oct. 26.18.54	retr.
1580	0.18.57.20	64.40.00	3.19. 5.50	0,5963	Novem. 28.15. 9	dir.
1585	1. 7.42.30	6. 4.00	0. 8.51.00	1,1094	Oct. 7.19.29	dir.
1590	5.15,30.40	29.40.40	7. 6.54.30	0,5767	Feb. 8. 3.54	retr.
1593	5.14.15.00	87.58.00	4.26.19,00	0,8911	July. 18.13.47	dir
1596	10.12.12.30	55 12.00	7.18.16.00	0,5130	Aug. 10.20. 4	retr.
~~1618~~	~~9.25.25.00~~	21.28.00	0.18.20.00	0,5131	Aug. 17. 3.12	dir.
1618	2.16. 1.00	37.34.00	00. 2,14.00	0,3798	Novem. 8.12.32	dir.
1652	2.28.10,00	79.28.0	00.28.18.40	0,8475	Novem. 12.15.49	dir.
1661	2.22.30.30	32.35.50	3.25.58.40	0,4486	Jan. 26.23.50	dir.
1664	2.21.14.00	21,18.30	4.10.41.25	1,1026	Decem. 4.12. 3	retr.
1665	7.18. 2.00	76. 5.00	2.11.54.30	0,1065	April. 24. 5.24	retr.
1672	9.27.30.30	83.22.10	1.16.59.30	0,6975	March. 1. 8.46	dir.

Years.	Place of the ascending Node.	Inclination	Place of the Perihelion.	Perihelion Distance.	Time of the Passage at the Perihelion at Paris.	
	s ° ′ ″	° ′ ″	s ° ′ ″		d h ′	
1677	7.26.4;.10	79. 3.15	4.17.37, 5	0,2806	May. 6. 0.46	retr.
1678	5.11.40.00	3. 4.20	10.27.46.00	1,1238	Aug. 26.14.12	dir.
1680	9. 2. 2.00	60.56.00	8.22.39.30	0,0061	Decem. 18,00.15	dir.
1683	5.23.23.00	83.11.00	2.25.29.30	0,5602	July. 13. 2.59	retr.
1684	8.28.15.00	65.48.40	7.28.52.00	0,9601	June. 8.10.25	dir.
1686	11.20.34.40	31.21.40	2.17.00.30	0,3250	Sept. 16.14.42	dir.
1689	10.23.45.20	69.17.00	8.23.44.45	0,0168	Decem. 1.15. 5	retr.
1698	8.27.44.15	11.46.00	9.00.51.15	0,6915	Oct. 18.17. 6	retr.
1699	10.21.45.35	69.20.00	7. 2.31. 6	0,7440	Jan. 13. 8.32	retr.
1702	6. 9.25.15	4.30.00	4.18.41. 3	0,6459	March. 13.14.22	dir.
1706	0.13.11.40	55.14.10	2.12.29.10	0,4258	Jan. 30. 4.50	dir.
1707	1.22.46.35	88.36.00	2.19.54.56	0,8597	Decem. 11.23.39	dir.
1718	4. 8.43.00	30.20.00	4.01.30.00	1,1027	Jan. 14.23.48	retr.
1723	0.14.16.00	49.59.00	1.12,52.2	0,9876	Sept. 27.16.20	retr.
1729	10.10.32.37	76.58. 4	10.22.40.03	1,4261	June. 25.11. 6	dir.
1737	7.16.22.00	18.20.45	10.25.55.00	0,2229	Jan. 30. 8.30	dir.
1739	6.27.25.14	55.42.44	3.12.38.40	0,6736	June. 17.10. 9	retr.
1742	6. 5.38.29	66.59.14	7. 5.35.13	0,7657	Feb. 8. 4.48	retr.
1743	2.18.21.15	2.19.33	[illegible]	0,8350	Jan. 10.20.35	dir.
1743	0. 5.16.25	45.48.20	8. 6.33.52	0,5205	Sept. 20.21.26	retr.
1744	1.15.46.11	47. 5.18	6.17.10.00	0,2225	March. 1. 8.13	dir.
1747	4.27.18.50	79. 6.20	9. 7. 2.00	1,2198	March. 3. 7.21	retr.
1748	7.22.52.16	85.26.57	7. 5. 0.50	0,8407	April. 28.19.34	retr.
1748	1. 4.39.43	56.59. 3	9. 6. 9.24	0,6553	June. 18. 1.33	dir.
1757	7. 4. 5.50	12.39. 6	4. 2.39.00	0,3391	Oct. 21. 9.42	dir.
1758	7.2 .50. 9	68.19.00	8.27.37.45	0,2154	June. 11. 3.27	dir.
1759	4.19.39.24	78.59.22	1.23.24.20	0,7985	Novem. 27. 2.28	dir.
1759	2.19.50.45	4.51.32	4.18.24.35	0,9660	Decem. 16.21.13	ret.
1762	11.19.00.00	85.20.00	3.14.00.00	1,0094	May. 28.00.00	dir.
1763	11.26.17.00	72.42.00	2.24.43.00	0,4991	Novem. 1.18.39	dir.
1764	4. 0. 7.00	52.47.00	0.15.26.00	0,5567	Feb. 12.13.40	retr.
1766	8. 4.19.50	40.5 .23	4.23.15.25	0,5053	Feb. 17. 8.50	retr.
1766	1.17.22.10	8.18.45	6.26. 5.13	0.6368	April. 17.10.26	dir.

Elements of the COMET *of* HALLEY, *in its different Revolutions.*

1456	1.18.30.00	17.56.00	10.1.00.00	0,5856	June 8. 22. 10.	retr.
1531	1.19.25.00	17.56.00	10.1.39.00	0,5670	Aug. 24. 21.27.	retr.
1607	1.20.21.00	17. 2.00	10.2.16.00	0,5868	Oct. 26. 3. 59.	retr.
1682	1.20.48.00	17.42.00	10.1.36.00	0,5825	Sept. 14. 21.31.	retr.
1759	1.23.49.00	17.35.30	10.3.16.00	0,5835	March 12. 13.41.	retr.

VIII.

Newton having thus solved the above-mentioned Problem, and applied it to all the Comets observed, deduced from thence a complete Confirmation of his Conjecture. For all the Places of the Comets calculated in the parabolic Orbits, whose Elements were delivered in the foregoing Table, compared with those immediately deduced from Observation, never differed sensibly, which will appear so much the more surprising, when we consider how difficult it is to attain to Precision in Observations of this Nature.

Newton verifies his Calculation by the Observations of a great Number of Comets.

IX.

As to the Duration of the Periods of the Comets, it cannot be deduced from the same Calculation, because as we have already hinted, their Orbits being so excentric that they may be taken for Parabolas without any sensible Error, very great Differences in their Duration would produce scarce any Alteration in the Arc of their Orbit, which they describe during the Time they are visible. However, it no less confirms the Theory of *Newton*, to have shewn, that in this Portion of their Orbit, they observe the Law of *Kepler*, that of the Areas being proportional to the Times, and that the Sun attracts them in the same Manner as all the other celestial Bodies, in the inverse Ratio of the Squares of the Distances.

The Duration of their Period cannot be deduced but from the History of the Apparitions of the Comets in the same Circumstances, and at equal Intervals.

X.

Halley, on examining the famous Comet of 1680, having found that the Observations of a Comet recorded in History, agreed with it in very remarkable Circumstances, and that they had appeared at the Distance of 575 Years from each other, conjectured, that it might be but one and the same Comet, performing its Revolution about the Sun in this Period, he therefore supposed the Parabola to be changed into an Ellipse described by the Comet in 575 Years, and having the same Focus and Vertex with the Parabola. Calculating afterwards, the Places of the Comet in this elliptic Orbit, he found them to agree perfectly with those where the Comet was observed; so that the Variation did not exceed the Difference found between the calculated Places of the Planets, and what are immediately deduced from Observation, though the Motions of the Planets have been the Object of the Inquiries of Philosophers for thousands of Years.

Halley employs the Period of the Comet of 1680 to rectify its Orbit.

XI.

Besides the Comet of 1680, *Halley* found three others, which nearly agreed, those of 1531, of 1607, and of 1682, the three Parabolas were situated after the same Manner, the perihelion Distances were equal, and the Intervals of Time 75 or 76 Years; he conjectured that it might be but one and the same Comet, and that the Differencce in their Inclinations and Periods, might arise from the Attractions of the superior Planets; for he observed, that the Comet in 1681, passed very near Jupiter; and it is certain, that the Comets receding farther from the Sun than the Planets, their Velocity and Tendency towards the Sun should thereby be considerably lessened in the superior Parts of their Orbits, and consequently should be more susceptible of the Modifications and Impressions of the Attractions, which the Planets in their Approach exert on them; from whence he concluded, that the following Apparition would be retarded, and anounced the Return of this Comet for 1759. But these Considerations were too vague to be depended upon. To attain to Certainty in this Point, it was necessary to calculate the Situations of the Comet, and the Forces with which Jupiter and Saturn attract it during several Revolutions, and by the Help of those Forces, expressed in Numbers, to determine the total Effect of the Attractions of those Planets on the Comet. This *Clairaut*, and after him the first Mathematicians in *Europe* have effected, and have demonstrated that this Comet observed in 1531, 1607, and 1682, should have the unequal Periods of 913¼ and 898½ Months and that the Period after which it would appear again in this Age, would be 919 Months, which the Event has justified. These Researches shall be explained hereafter.

Effect of Attraction on the Comets.

XII.

Different Opinions concerning the Tails of Comets.

The Tails of Comets which formerly occasioned the Apparition of those Bodies to be regarded as portentous Omens, are now ranked in the Number of those ordinary Phenomena which raise the Attention of Philosophers alone. Some would have it, that the Rays of the Sun passing through the Body of the Comet, which they suppose to be transparent, produced the Appearance of their Tails, in the same Manner as we perceive the Space traversed by the Beams of the Sun passing through the Hole of a darkened Room: others imagined that the Tails were the Light of the Comet refracted in their Passage to the Earth, and producing a long Spectrum, as the Sun does by the Refraction of the Prism. *Newton* having mentioned those two Opinions, and refuted them, exposes a Third which he adopted himself: it consists in regarding the Tail of a Comet as a Vapour which rises continually from the Body of the Comet towards the Parts opposite to the Sun, for the same Reason, that Vapours or Smoke rise in the Atmosphere from the Earth, and even in the Void of the Pneumatic Pump. On Account of the Motion of the Body of the Comet, the Tail is incurved towards the Place through which the Comet passed, much in the same Manner as the Smoke proceeding from a burning Cole put in Motion.

Newton is of Opinion that they are Vapours exhaled from the Body of the Comet.

What confirms this Opinion is, that the Tails are found greatest when the Comet has just past the Perihelion or least Distance from the Sun, where its Heat is greatest, and the Atmosphere of the Sun is most dense. The Head appears after this, obscured by the thick Vapour that

Confirmation of this Opinion.

rises plentifully from it, but about the Centre, a Part more luminous than the rest appears, which is called the Nucleus.

A great Part of the Tails of the Comets should be dilated and diffused over the Solar System by this Rarefaction: some of it by its Gravity may fall towards the Planets, mix with their Atmospheres and repair the Fluids, which are consumed in the Operations of Nature. *Use of the Tails of Comets, according to Newton.*

The Resistance which the Comets meet with in traversing the Atmosphere of the Sun when they descend into the lower Parts of their Orbits, will affect them, and their Motion being retarded, their Gravity will bring them nearer the Sun in every Revolution, until at length they are swallowed up in this immense Globe of Fire. *Some of the Comets may fall into the Sun.*

The Comet of 1680, passed at a Distance from the Surface of the Sun which did not exceed the sixth Part of his Diameter, and it is highly probable, that it will approach nearer in the next Revolution, and at length will fall into his Body.

XIII.

Addition to Article XX of the Theory of the primary Planets, where it was shewn how Newton determined the Proportions of the Matter in the Sun, Jupiter, Saturn, and the Earth.

Let the Distance of any one of the primary Planets from the Sun $=1$ its periodic Time $=1$ the Force of the Sun exerted on it $=1$, the Distance of any Satellite from its Primary $=t$, and the periodic Time of the same Satellite $=r$; the Force (F) of the Sun on the Planet being to the Force (f) of any Planet on its Satellite as 1 to $\frac{r}{tt}$ (Cor. 2. Prop. 4.) and the Force (V) of this Planet on its Satellite if it was just as far from it as the Planet is from the Sun, being to its Force (f) exerted on it at its actual Distance from it, as r^2 to 1; by the Composition of Ratios $F\times f$ is to $V\times f$, or the Force (F) of the Sun on the Planet, is to the Force (V) of a Planet on its Satellite just as far from it as the Planet is from the Sun, as 1 to $\frac{r^3}{tt}$.

Example. The Revolution of Venus round the Sun (5393^h) being to that of the fourth Satellite of Jupiter ($400^h\frac{1}{2}$) as 1 to 0,0742716, $t=0,0742716$ and the Distance of Venus from the Sun 72333 being to the Distance of Jupiter from the Sun 520096 as 1 to 7,1903; and Radius being to the Sine of 8′ 16″ Elongation of the Satelite, or its Distance from Jupiter viewed from the Sun, as 7,1903 to 0,01729, $r=0,01729$; wherefore $\frac{r^3}{tt}=0,000937$ or $\frac{1}{1067}$, consequently the Weight of equal Bodies at equal Distances from the Centre of the Sun and Jupiter, are to one another as 1 to $\frac{1}{1067}$.

The Revolution of Venus round the Sun 5393^h being to that of the fourth Satellite of Saturn $362^h\frac{1}{3}$ as 1 to ,0672475, $t=,0672475$, and the

Diſtance of Venus from the Sun 72333, being to the Diſtance of Saturn from the Sun 954006 as 1 to 13,1890, and Radius being to the Sine of the Elongation of the Satellite or its Diſtance from Saturn, as 13,1890 to 0,1144, $r=0,1144$, wherefore $\frac{r^3}{tt}=0,000332$ or $\frac{1}{3021}$, conſequently the Weights of equal Bodies at equal Diſtances from the Centres of the Sun and Saturn are to one another as 1 to $\frac{1}{3021}$.

The Revolution of the Earth round the Sun 365ᵈ, 256 being to that of the Moon 27ᵈ, 3215 as 1 to 0,748008, and the Diſtance of the Earth from the Sun being to that of the Moon from the Earth, as the Sine of the Parallax of the Moon to the Sine of the Parallax of the Sun, wherefore $\frac{r^3}{tt}=\frac{1}{169282}$ conſequently the Weights of equal Bodies at equal Diſtances from the Centres of the Sun and Earth are as 1 to $\frac{1}{169282}$.

Addition to Article xxi. of the Theory of the primary Planets, where it was ſhewn how Newton determined the Proportions of the Denſities of the *Sun, Jupiter, Saturn* and the *Earth.*

To determine the Weights of Bodies on the Surfaces of the Sun, Jupiter, Saturn, and the Earth, or at the Diſtance of their Semidiameters from their Centres, thoſe Semidiameters are to be inveſtigated. Firſt the apparent Diameter of the Sun in its mean Diſtance being found to be 32′ 8″ and that of Jupiter 37″ ¼ (as determined from the Paſſage of thoſe Satellites over its Diſk) and the mean Diſtance of the Sun from Jupiter, being to the mean Diſtance of the Sun from the Earth as 520096 to 100000, and the true Diameters of Spheres, viewed under ſmall Angles, being in the compound Ratio of thoſe Angles, and the Diſtances conjointly, the true Diameter of the Sun will be to the true Diameter of Jupiter as 1928″ ×100000 to 37″×520096, or as 10000 to 997. Secondly, the apparent Diameter of Saturn being found to be 16″, and the mean Diſtance of Saturn from the Sun being to the mean Diſtance of the Earth from the Sun as 954006 to 100000, the true Diameter of the Sun will be to the true Diameter of Saturn as 1928″×100000 to 16″×954006, or as 10000 to 791. Thirdly and laſtly, the apparent Semidiameter of the Earth being found to be 10″ 30‴ as being equal to the Parallax of the Sun, the true Diameter of the Sun will be to the true Diameter of the Earth as 1928 to 21, or as 10000 to 109 nearly.

Now if we ſuppoſe a Body placed at a Diſtance from the Centre of the Sun equal to its Semidiameter, or on its Surface, the Force (F) of the Sun on this Body being to the Force (V) of Jupiter on an equal Body at the ſame Diſtance from its Centre, as 1 to $\frac{1}{1067}$ and the Force (V) of Jupiter on this Body, being to the Force (f), it would exert on it if it was placed on its Surface, inverſely as the Squares of the

Distances, that is, inversely as the Squares of the true Semidiameters of the Sun and Jupiter, or as $\frac{1}{10000^2}$ to $\frac{1}{997^2}$; by the Composition of Ratios F×V is to V×*f* or the Weight (F) of a Body on the Surface of the Sun is to the Weight *(f)* of an equal Body on the Surface of Jupiter, as $\frac{1}{10000^2}$ to $\frac{1}{1067} \times \frac{1}{997^2}$ or as 10000 to 943, and consequently that the Density of the Sun is to the Density of Jupiter (the Densities being in the direct Ratio of the Weights and inversely as the Diameters) as 100 to $94\frac{1}{2}$. In the same Manner it will be found secondly, that the Weight of a Body on the Surface of the Sun is to the Weight of an equal Body on the Surface of Saturn as $\frac{1}{10000^2}$ to $\frac{1}{3021} \times \frac{1}{791^2}$ or as 10000 to 529, consequently that the Density of the Sun is to the Density of Saturn as 100 to 67. Thirdly and lastly, That the Weight of a Body on the Surface of the Sun, is to the Weight of an equal Body on the Surface of the Earth as $\frac{1}{10000^2}$ to $\frac{1}{169282} \times \frac{1}{109^2}$ or as 10000 to 435, consequently that the Density of the Sun is to the Density of the Earth as 100 to 400. Which Determination on examining the Process of the Computation will appear not to depend on the Parallax of the Sun but on the Parallax of the Moon, and is therefore truly defined.

XIV.

CONCLUSION.

Recapitulation of the Improvements the *Principia* have received to this Day.

Such is the Plan of the immortal Discoveries of the most eminent Philosophers, and of Sir *Isaac Newton* in particular, whose Efforts and and Sagacity we cannot sufficiently admire, which shine through the Whole of those Strokes of Genius, which characterise an Inventor, and a Mind fertile in Resources, that no Man possessed in so eminent a Degree. Aided by the Succours that the analitic Art furnishes in greater Abundance, it is not surprizing that some more Steps have been made in a vast and difficult Career that he has opened to us, that all the Irregularities that have been perceived in the Heavens, have been explained and demonstrated; that a great Number of others, which on Account of their Smallness and Complication had escaped the most exact Observers, have been foreseen and unfolded; that it has been proved, that the Return of the Comet which was observed in 1531, 1607, and 1682, ought to have had the unequal Periods of $913\frac{1}{2}$ and $898\frac{1}{2}$ Months, which was found to be so, and that the Period after which it would appear again in this Age, would be 919 Months; which the Event has justified.

That the Course and Laws of the Winds, the ebbing and flowing of the Sea, as far as they depend on the attractive Action of the Sun and

Moon, have been accurately determined. That the Narure and Laws of Magnetiſm, the Theory of Light and Laws of Viſion, the Theory of Sound and Laws of Harmony, &c. have been accurately inveſtigated.

New Edition of the *Principia*, with the Improvements they have received to this Day.

Such is the Plan of the MATHEMATICAL PRINCIPLES OF NATURAL PHYLOSOPHY, which the Nobility and Gentry of the Kingdom of *Ireland* purſuant to their Reſolution of the 4th of February 1768, have ordered to be publiſhed for the Uſe of the Mathematical School eſtabliſhed under their immediate Inſpection. Previous to which, in the Month of November, 1764, a Copy of the Chapter of the Theory of the primary Planets, as a Specimen of the whole Plan, was delivered to Dr. *Hugh Hamilton*, to have his Opinion of the ſame, which he returned in ſix Months after, with this Anſwer, That the above Piece was printing by Subſcription at *Cambridge*, under the Title of *Excerpta quædam ex Newtoni Principiis*, with Referrences to the Doctor's Treatiſe on Conic Sections.

PLAN *of the Art of making Experiments and that of employing them.*

Experimenta rerum naturalium ita ſunt exhibenda, ut in his nobiles adoloſcentes ſtudio ſuaviſſimo atque utiliſſimo humanæ mentis hiſtoriam, preclaraque artium inventa, quibus naturam et ornare et adjuvare, ediſcere poſſunt.

TO illuſtrate Sir *Iſaac Newton*'s *Principia*, and thereby to enable Youth to make a Progreſs in the Knowledge of the Works of Nature, to improve to Advantage its Powers and Forces, and render them ſubſervient to the Purpoſes of Life, they are initiated in the Art of making Experiments and Obſervations. For theſe Purpoſes the School is furniſhed with a complete Collection of the beſt executed Machines adapted for experimental Inquiries; they are inſtructed in the Management and Uſe of theſe Machines; they are taught how to aſcertain the Difference between the Reſult from Theory and from Experiment, and how to employ this Difference, for determing the Alterations ariſing from external Cauſes, in order to ſhew them how Experiment not only ſerves to confirm Theory, but conducts to new Truths, to which we cannot attain by Theory alone. As to the Phenomena for the Diſcovery of whoſe Cauſes Theory affords little or no Aſſiſtance, for Inſtance, thoſe of Chimiſtry, Electricity, &c. they are taught how to examine and conſider them in different Lights, arrange them in Claſſes, and explain the one by the other as far as the Nature of the Subject will allow.

Courſe of Experiments for illuſtrating the *Principia*.

FIRST CLASS.

Machines for making Experiments on Motion, Gravity, and the Equilibrium of ſolid Bodies.

I.

A Machine for demonſtrating the Theory of central Forces.

This Machine is ſo contrived, that by its Aſſiſtance may be ſolved experimentally, the Problems which appear the leaſt ſuſceptible of ſuch a Solution; the Velocities and Maſſes may be varied at will, Friction is ſo diminiſhed *as to cause no sensible Error, the Times are marked by Sounds, and the Spaces described by an Index.* Experiments for illustrating the Theory of central Forces.

A Glass Globe mounted on an Axis so that it may be turned round with any Degree of Velocity.

This Machine shews the Effects of central Forces on Fluids of different specific Gravities, and on Solids, which circulate in the same Medium.

A terrestrial Globe which turns on its Axis with any given Velocity.

The Surface of this Globe is flexible, its Concavity is filled with a Matter somewhat fluid, and is so contrived, that its two Poles are capable of moving towards each other, so that by turning the Globe, the centrifugal Force raises the Equator of the Globe, and shews the Figure which modern Discoveries attribute to the Earth.

A graduated Rule adapted to a Glass Tube within which a small Cylinder is put in Motion. Second, A Plane upon which two Bodies describe in the same Time unequal Spaces. Third, A Globe of Cork of of three Inches Diameter, with a Ball of Lead of the same Weight.

By the Assistance of the three last Articles are explained the Properties of Motion, viz. Direction, Velocity, Quantity of Motion, &c.

A small Cystern divided into two equal Parts by a Partition upon which is mounted a double Pendulum, shewing in what Ratio different Mediums exert their Resistance.

A Machine with which is demonstrated the Doctrine of the Collision of Bodies. Experiments for illustr. ting the Doctrine of the Collision of Bodies.

The Parts of this Machine are made with the utmost Care, the Masses are in given Proportions, and the Effects remain visible after the Experiment by the Means of an Index.

A CHRONOMETER or Instrument for measuring small Intervals of Time.

The Pendulum which constitutes the principal Part of this Instrument may be lengthened or shortened according to a Scale accurately divided for the vibrating Minutes, Seconds, Thirds, and the different Times of Musick.

A small Billiard-Table with its Appendages.

The Appendages of this Machine are Hammers suspended in such a Manner, that the Quantity of Motion may be regulated by the Velocity, or by the Mass, and so as to exhibit the Motion of a Body impelled by Forces acting in different Directions, and known Proportions.

A Machine for shewing the Motion of a Body which receives at the same Time an Impulse in a perpendicular and horizontal Direction. Experiments for illustrating the Composition and Resolution of Forces.

Another Machine for shewing the Motion produced by two Forces acting on a Body in Directions forming an Angle, but which constantly remain in the same Ratio.

A Machine for shewing the Acceleration of Bodies which fall freely. Secondly, a Kind of Balance for making the same Kind of Experiments.

These two last Machines not only shew that the Motion of Bodies is accelerated in their Descent, but also renders sensible the Law of this Acceleration.

Experiments for illustrating the Doctrine of the Motion of heavy Bodies.

A Machine for shewing the Line a Body describes when abandoned to its Weight after having received an Impulsion in an horizontal Direction.

A Machine for shewing the Motion of a Body abandoned to its Weight after having received an Impulsion upwards, but oblique to the Horizon.

As the Curve which results from this Motion depends on the Obliquity of the Direction, the Machine is constructed so that the Degree of Obliquity may be varied at will.

A Machine which serves to compare the Velocity of a Body which in its Discent describes a Cycloyd with that of another tending to the same Point along an inclined Plane.

A Machine for shewing in what Ratio several Forces act on the same Body.

A Machine for explaining the Laws of Elasticity.

Experiments for illustrating the Nature and Properties of the Center of Gravity

Two Cones joined together by their Bases, and which ascend an inclined Plane. 2d. A Cylindar which ascends an inclined Plane.

Those two Machines serve for proving experimentally, that a Body cannot remain at rest when its Centre of Gravity is not supported. The Plane on which the double Cone moves is formed of two Rulers inclined to each other and to the Horizon, and this double Inclination may be varied at pleasure as the Experiment may require.

A small Carriage with its Appendages.

This Model with the Parts which accompany it, shews the respective Advantages of broad or narrow Wheels, of large or small ones, and what renders Carriages more or less liable to be overturned.

Experiments for illustrating the Theory of simple Machines, the inclined Plane, the Wedge, the Screw, the Lever.

A Machine for shewing the Properties of the inclined Plane.

This Machine is so constructed that the Inclination of the Plane may be varied from the horizontal Line to the vertical, and that the Power may act in any desired Direction.

A Machine for shewing the Nature and Properties of the Wedge.

What forms the Wedge in this Machine are two Planes inclined to each other, the Degree of Inclination can be varied at pleasure, as also the Power, the Weight and the Base of the Wedge.

A Screw which can be taken to Pieces to shew the Principles of its Construction.

A Machine for shewing the Nature of the three Species of Levers.

A large Beam accurately divided, mounted on a Foot, for shewing the Properties of the Lever.

The Power, the Weight, and the Prop or Fulcrum are moveable, and may be easily placed so as to be to each other in any given Proportions.

Two Figures in Eqilibrio on a Pivot, for shewing the Art of Chord or Wire-dancing.

A large Brass Pully, in which the Circumference and the diametral

Lines have only been left, in order to shew that the Pully may be considered as an Assemblage of Levers of the first Species.

At the Back of the Supporter, there is fixed a Lever of the same Species with those which constitute the Diameters of the Pully, to serve as a Proof by the Application of the same Power and Weight.

A Pully whose Axis is moveable in a perpendicular Direction, and which serves to shew the Action of the Power, and of the Weight on this Axis, in different Cases.

A Block with two Pullies. 2d. A Block with four Pullies; another Block whose Pullies are fixed on the same Axis.

All those combined Pullies are of Metal or Ivory, turned on their Axis with great Precision, and all possible Care has been taken to diminish the Friction.

An Assemblage of several Toothed Wheels and Pinions, for shewing that both the one and the other like the Pullies, may be considered as Levers.

Models for shewing the Application of simple Machines in the more compounded ones. The Capstan, the Crane, the Pile-driver, Wind-mills, Water-mills, &c.

At the Back of the Supporter, are fixed an Assemblage of Levers which correspond in the same Manner as the Diameters of the Wheels on the other Side, to serve as a Proof by the Application of the same Power and Weight.

A Model of *Archimedes*'s Screw, whose Effects are rendered sensible by the Motion of several small Balls of Ivory, which are raised successively.

A Model of an Endless Screw, which drives an Axis. 2d. A Model of a Press. 3d. A Model of a Capstan. 4th. A Model of a Crane. 5th. A Model of an Engine for driving Piles.

A Jack, of a particular Construction, for raising great Weights. A common Balance, for shewing the Defects to which this Machine is liable, and how they may be remedied.

A large Roman Balance, contrived for making the Experiments of *Sanctorius*.

This Machine is so constructed, that a Person may weigh himself without the Assistance of another.

A Model of a Screen for winnowing Corn by the Means of an artificial Wind, and several Screens of a particular Construction.

A Model of a Saw for cuting at the same Time several Flints, Agates, Cornelians, &c. and at one Stroke, to form Tables for Snuff-Boxes, and other Works. An horizontal Turning Leath, adapted for grinding Glasses for Telescopes, Microscopes, &c.

A Model of a common Wind-Mill. 2d. A Model of a Polish Wind-Mill. 3d. A Model of a Water-Mill for extracting Oil. 4th. A Model of a Water-Mill for winnowing and grinding Corn, drawing up the Sacks, and boulting the Flour.

As all those Models are intended to shew the Application of simple Machines in the more compounded ones, Care has been taken to leave exposed or to cover with Glass, the Pieces destined for Motion, and the Proportion of the Parts have been carefully observed.

A Machine for shewing the Effects of Friction, in Machines more correct, and of a more extensive Use than any hitherto invented.

Second Class.

Machines for making Experiments on the Motion, Gravity and Equilibrium of Fluids.

II.

A large Cistern lined with Lead, with a Cock to it, which serves for making several hydrostatical Experiments.

Two large cylindrical Glasses mounted on a common Base, between which is erected a Stem which carries a Beam of a Balance.

This Machine is very commodious in several Operations which regard the Weight or Equilibrium of Fluids.

Experiments for shewing the Properties of Fluids.

A small Bottle with a Glass Stopper, and heavier in this State than a Quantity of Water of the same Bulk.

A Glass Tube, a Part of which rises perpendicularly, and the other forms several Flexions for shewing the Height of Fluids in Vessels which have a Communication with each other.

A small Barrel with a Cock to it, and a bent Tube which serves for demonstrating the same Principle, with some curious Applications.

A Glass Vessel, partly filled with a coloured Fluid, to which is adjusted a large Glass Tube, and a small sucking Pump, which serves to shew that Columns of the same Fluid are of the same specific Gravity.

A long Tube of Glass with a Cock at the lower Extremity, and mounted on a graduated Ruler, to which is adjusted a Pendulum which beats Seconds.

This Machine serves to shew how the Parts of a Fluid press each other, and in what Ratio the Effluxes thereof are performed.

A Bladder filled with a coloured Fluid, to which is fitted a Glass Tube, which serves to shew that Fluids exert their Pressure in all Directions.

A Vessel whose Bottom bursts by the Pressure of a small Quantity of a Fluid.

Experiments for shewing the Pressure of Fluids upon the the Bottoms and Sides of the Vessels that contain them.

A large Machine, which serves to shew the Pressure of Fluids on the Bottoms and Sides of Vessels which contain them.

This Machine consists of several fine Vessels of Glass, which are adjusted successively on a common Base, the Piston which serves as a Bottom, is sufficiently moveable as not to cause any sensible Error by Friction, the Columns of the Fluid remain always at the same Height, and the Power acts uniformly.

An Hydrometer with six small cylindrical Vases, which are filled with different Fluids.

Two small Cruets, mounted each on a Pedestal, which serve for the Experiments by which Water is apparently changed into Wine, and Wine into Water.

Two Vases of different Forms, which serve to make a heavier Fluid assume the Place of a lighter in the same Vessel, without mixing.

Experiments for illustrating the Action of Fluids upon Bodies immersed in them.

A Vessel perfectly cylindrical of Copper, with a Solid of the same Metal, and of the same Figure, which fills it exactly, for shewing how much a Body immersed in a Fluid, loses of its Weight.

A Vase of Glass suspended to the Arm of a Balance, for making Experiments of the same Kind.

Two Balls, one of Ivory, and the other of Lead of the same Weight, prepared to be suspended to the Arm of the Balance just mentioned, for shewing, that what a Body loses of its Weight when immersed in a Fluid, is proportional to its Bulk.

A cylindrical Vase of Glass filled with Water, with several human Figures of Enamel, of which some are lighter and the others heavier than a like Portion of the Fluid in which they are immersed.

A Machine for shewing that the relative Gravity of a Body immersed in a Fluid, is changed when the Fluid is condensed or rarified.

This Machine renders palpable by a very quick Operation, the Effects which the different Temperatures of the Air produce in the different Kinds of Thermometers hitherto invented.

A human Figure of Enamel, which is made to move in Water by Compression. 2d. Two large Tubes of Glass mounted in a Frame, in which two Figures move by a Compression which is not perceived by the Spectator.

A Model of the Diving Bell, and the Appurtenances of a Diver.

An hydrostatic Balance, with all its Appendages.

A Model of a curious Machine for raising up Vessels that are sunk.

Experiments for illustrating the Effect produced by the Pressure, &c. of Fluids.

A Water Level. A simple Syphon. 2d. A Fountain Syphon mounted on a Pedestal. 3d. A Syphon with its Vase to be placed in Vacuo. 4th. A double Syphon. 5th. A Syphon whose Branches are moveable by the Means of a Joint. 6th. Tantalus's Cup. 7th. A large Syphon whose Branches are moveable, necessary in Experiments made with the Air-Pump.

All those different Species of Syphons are of Glass, that the Motion of the Fluids may be more easily perceived.

A Model of a Sucking-Pump. 2d. A Model of a Lifting-Pump. 3d. A Model of a Sucking and Lifting Pump. 4th. A Model of the Engine under London-Bridge, that raises Water by Forcing-Pumps.

5th. A Model of a new Pump whose Sucker has no Friction, an intermitting Fountain, *Hiero*'s Fountain.

All those Models of Pumps and Fountains are of Glass, in all those Parts in which the Action passes, and the Motion of the Valves and Suckers, are easily perceived.

Experiments on Ice and artificial Congelations.

Several Cisterns and other Vases for making Experiments on Ice, and artificial Congelations. 2d. An Assortment of different Salts and Fluids for congealing Water with a Vase, in which without Ice, a Cold capable of freezing, may be produced.

THIRD CLASS.

Machines for making Experiments on the Air.

III.

A double barrelled Air-Pump mounted on a very solid Base.

The Pistons are put in Motion by a Handle. Instead of Valves Stop-Cocks are made Use of, which are opened and shut, and that by the same Motion which raises and lowers the Pistons; there is affixed to the Pump a whirling Machine, for the Experiments where it is necessary.

Experiments for shewing the Nature and Properties of the Air.

A single barrelled Air-Pump, mounted on a solid Base.

In the Construction of the whirling Machine, which serves as an Appendage to this Pump, Care has been taken, that the Axis of the great Wheel may move along its Frame, in order to straiten the Chord, and that the horizontal Pulley, which receives the whirling Axis, may be raised or lowered as the Height of the Receiver may require.

A large Receiver fitted for making Experiments on Bodies put in Motion in Vacuo. 2d. A Receiver of less Size fitted for the same Uses. 3d. A long and narrow Receiver fitted also for the same Uses.

Those Vases are fitted for the above Uses, by the Means of a Brass Box, filled with a Sort of prepared Leather, through which passes a Steel Axle-Tree, which communicates the Motion within the Receiver without letting the Air enter.

Experiments on Fire in Vacuo.

An Apparatus necessary for making the Experiments on Fire in Vacuo.

Electrical Experiments in Vacuo.

An Apparatus for making electrical Experiments in Vacuo.

A large Receiver fitted for operating in Vacuo; a tall narrow Receiver fitted for the same Uses.

Those Vases are fitted for the above Uses, by Means of a Brass Box prepared as above, through which passes a Shaft of Metal, whose Extremity is fitted for receiving different Sorts of Pincers, and other Instruments.

Four Cruets mounted on one common Pedestal, and suspended so as to have their Contents poured out in Vacuo, which serve for mixing different Fluids therein. 2d. Two Cruets suspended in the same Manner.

This Machine is so contrived, that the Cruets may be raised or lowered, and brought nearer to each other, as may be required.

An Apparatus for essaying Inflammations in Vacuo.

A Receiver composed of several Pieces, very tall, at the upper End of which, a Machine is adapted with which may be repeated six Times, the Experiment of the descent of Bodies in Vacuo, when the Air is but once exhausted.

Experiments for shewing the Descent of Bodies in Vacuo.

A large Vase of Glass adjusted to a Receiver, and disposed for depriving Fishes in Water of Air.

A large Globe of Glass, joined to a Receiver by a Neck, to which is adapted a Stop-Cock, for making Experiments on the Vapours in the Air. 2d. Two Vases of Comparison having for a common Base a small Receiver, for similar Uses.

Experiments for shewing that the Air is filled with Vapours.

A Receiver, to which are adapted two Barometers, one of Mercury and the other of coloured Water.

Two large Receivers with a hollow Button at Top. 2d. Two Receivers of a middle Size. 3d. Four small Receivers. 4th. A Machine very commodious for sealing up Vases hermetically, *&c.*

Six small truncated Barometers of different Lengths, mounted each on a small Base, to which a Scale is adapted. 2d. Six small gage Tubes, for compressed and rarified Air.

These Gage Instruments are more commodious for Use than any hitherto made, and it is well known of what Importance it is in making Experiments, to be assured of the Degree of Rarefraction, or of the Condensation of the Air.

Experiments for ascertaining the Degree of Compression and Rarefraction of the Air.

A Receiver for making Experiments on burnt, or infected Air.

Two large Copper Hemispheres, to one of which is adapted a Ring, and to the other a Stop-Cock.

Experiments on burnt and infected Air.

A Fountain Bottle, and a Vase to place it in, with several spouting Pipes, which are successively adjusted on it.

A small Receiver for applying the Hand to the Air-Pump.

A Receiver of very thick Glass for bursting a Bladder.

A Supporter, and a small Vase of Glass to place Eggs under a Receiver of the Air-Pump.

A small Receiver with a sharp edged Brim, to cut an Apple, or any like Body.

Experiments for shewing the Spring of the Air and its Applications.

A large Glass Tube, at the Top of which is adjusted, a Wooden Vase for proving the Porosoty of Vegetables.

A Tube of Crystal whose Bottom is of Leather, covered with Mercury, to shew that animal Substances are porous.

A Bladder suspended in a Reciver. 2d. A Bladder in a cylindrical Vase of Metal charged with a great Weight.

A Machine for compressing Air.

fectly dark, with a Tablet and Circles of Metal for opening Passages to the Rays of the Sun, of different Magnitudes and Figures.

A plain Mirror of Metal mounted on a Stem which can be lengthened and shorted, and on which the Mirror can be raised, lowered, inclined, and turned round, for introducing the Rays of the Sun into a darkened Room. 2d. A Mirror of Glass mounted as the former, and for the same Uses.

Four Glasses of different Colours, mounted in Tortoise Shell. 2d. Four Mirrors of Glass mounted in the same Manner.

A large Glass Lens of six Feet Focus Length, mounted on a Pedestal whose Stem can be lengthed or shortened. 2d. A Glass Lens of a shorter Focus mounted, so that it can be raised, lowered or inclined.

Experiments for shewing the different Refrangibility of the Rays of Light.

A Frame, in which is adjusted a Glass Lens between two vertical Planes, for shewing that some Rays of Light unite in a shorter Focus than others.

This Machine is so contrived, that the Experiment may be made upon any Ray separately, and may be adjusted to the Motion of the Sun.

A large concave Glass mounted. 2d. A large multilateral Glass mounted. 3d. Two Polyhedrons of very pure Glass. 4th Two concave Mirrors of Glass.

A very large convex Glass, composed of two curved Glasses mounted on a Pedestal, for making Experiments on the Refraction of Light through different Fluids.

A large vertical Plane for receiving the Image of the Sun when it has passed through the Prism. 2d. A smaller Plane, to which is adapted, an excentric Circle for making the Rays of Light of different Colours, pass successively.

A Cloth six Feet square spread on a Frame, which can be raised and lowered for receiving the Images produced by the Magic Lanthorn, and the Camera Obscura.

Experiments for illustrating the Laws of Vision.

An artificial Eye with Spectacles for different Ages, for shewing how the Defects of Sight are remedied by the Help of Glasses.

A Cornea of an Insect adapted to a small Microscope for shewing that the Eyes of those Animals, for the most Part, are Multipliers.

An Assortment of Fluids for Experiments on the Colours which result from their Mixture.

Invisible Ink, the Writing of which appears and disappears several Times, when heated at the Fire. 2d, Sympathetic Ink.

Experiments for illustrating the Doctrine of the Reflection of Light.

A large Mirror of Metal, concave on one Side, and convex on the other, mounted on a Pedestal. Two convex Mirrors of Paste-board silvered over, with their Appendages, for some catoptrical Experiments.

A cylindrical Mirror of Metal, with thirty Anamorphoses. 2d. A conic Mirror of Metal, with six Anamorphoses. A pyramidal Mirror of Metal, with four Anamorphoses.

To all those Mirrors is adapted a Machine for regulating the Point of View.

A Picture, commonly called the magical one, on account of the Effect of the multilateral Glass, for dioptrical Anamorphoses.

A Magic Lantern, enlightened by the Rays of the Sun. 2d. A Magic Lantern enlightened by a Lamp and a concave Mirror. Experiments for illustrating the Theory of the Construction of optical Instruments, the Magic-Lantern, Camera Obscura, reflecting and refracting Telescopes, Microscopes, &c.

Although this Machine is become very common, it is not however despicable; the most eminent Philosophers of the present Age, have not thought it unworthy of a Place among their Machines, and have given ample Descriptions of it. The above mentioned one, presents a Sight so much the more agreeable, as the Objects appear animated, and are perfectly well designed.

A Camera Obscura of a new Construction, with a Stool and Table, and other Conveniencies for designing.

A kind of Telescope for observing Objects which present themselves at Right-angles to the Tube. 2d. A *Newtonian* Telescope, with which the Objects are viewed sideways, or in a Line which forms an acute Angle with the incident Rays of those Objects. 3d. A catoptrical Telescope two Feet long, which magnifies the Objects 300 Times. 4th. An Achromatic Telescope 12 Feet long.

A portable Microscrope, with the Instruments necessary for observing. 2d. A larger Microscope, with a greater Number of Instruments and Lenses for increasing or lessening its magnifying Power. 3d. A Microscope which has six different Degrees of magnifying Power, with Mirrors of Reflection and Lenses for increasing the Light; it is mounted so that it can be moved in all Directions with great Ease, and has a Machine of a new Contrivance for fixing it at its true Point. The Drawer of its Chest contains every Thing necessary for the different Observations to which it may be applied.

A double Lens mounted in Tortoise Shell for Observations on Insects, and other Operations where the Microscope is not commodious.

An Apparatus for making Experiments on the Transparency and Opacity of Bodies, consisting in Squares of polished Glass, limpid Liquors of different Densities, &c.

SIXTH CLASS.

Machines for making magnetic and electrical Experiments.

VI.

A small Table one Foot long, and eight Inches broad.

A Magnet cut, but not mounted. 2d. A Magnet cut and suspended in a little Boat of Ebony. 3d. A Magnet mounted and adjusted to a whirling Machine. 4th. An artificial Magnet mounted on a Pedestal of Ebony. Experiments on Magnetism.

A Box filled with the Fileings of Iron. 2d. A Bason with little Swans and Frogs of Enamel. 3d. A Box filled with small Ends of Iron and Brass Wire. 4th. A Box filled with small Iron Rings. 5th. A Box containing several Iron Balls, and some Cylanders of the same Metal.

Two large magnetic Needles of poliſhed Iron, placed one at the Top of the other, and mounted on a Pedeſtal. 2d. A Dipping-Needle mounted on a Pedeſtal.

A ſquare Rod of poliſhed Iron two Feet and a half long. 2d. A round Rod of poliſhed Iron two Feet long. 3d. A thin Plate of poliſhed Iron eighteen Inches long. 4th. A Stand of varniſhed Wood.

A Braſs Circle garniſhed with Pivots, for placing twelve ſmall Steel Needles.

A Glaſs Vaſe mounted on a Pedeſtal for placing a magnetic Needle in Water.

A Machine which ſerves for trying the Force of a Magnet.

A Dial Compaſs. 2d. A truncated Compaſs for determining the Meridian of a Place, &c. 3d. A Sea Compaſs, ſeveral Steel Needles of different Sizes adapted for magnetic Experiments.

Experiments on Electricity.

A large Tube of Cryſtal. 2d. Two ſmaller ones and not ſo thick. 3d. A large Glaſs Tube very thick, two Feet long. 4th. A Glaſs Tube three Feet and a half long, with a Stop-Cock, to be applied to the Air-Pump.

A thick ſquare Rod of poliſhed Glaſs, about eighteen Inches long. 2d. A round ſolid Rod of Cryſtal.

A large Globe of Cryſtal adjuſted to a whirling Machine. 2d. A Globe of Cryſtal, the Inſide of which is laid over with Sealing-Wax, to which is adapted a Stop-Cock to be applied to the Air-Pump, and afterwards to a whirling Machine.

A large Stand, whoſe Tablet is made of Sealing-Wax. 2d. A Glaſs Stand fourteen Inches high. 3d. A Stand of Cryſtal of a different Form from the preceding one, for containing Fluids, and Bodies of a round Figure.

A Stick of Sealing-Wax one Inch Diameter, and one Foot long. 2d. A Tube of Sealing-Wax of the ſame Diameter and Length as the Stick.

A Stick of Sulphur one Inch Diameter, and eighteen Inches long. 2d. A Globe of Sulphur three Inches Diameter. 3d. A Cone of Sulphur covered with a Vaſe of Cryſtal of the ſame Figure. 4th. A Cone of Sealing-Wax covered as the former. 5th. A ſmall Globe of Amber and another of Gum.

Six ſmall Cups of Ivory. 2d. A ſmall poliſhed Copper Pyramid for making Experiments on the Communication of Electricity.

A Suſpenſory garniſhed with Ribbands of different Colours. 2d. A Suſpenſory garniſhed with ſilk Twiſt for communicating Electricity to living Bodies. 3d. Thread Twiſt, with a Wooden Ball, for communicating Electricity a great Way off.

A Cake of Roſin and Gum weighing eight Pounds. A Cake of Roſin weighing twelve Pounds.

A Pallet of Paſte-board covered with Gauſe, and garniſhed with Gold Leaf, Balls of Cotton and the Down of Feathers.

A Receiver without a Bottom for the Experiments of Tranſmiſſion.

A Box containing ſix Rackets of Gauſe of different Colours. 2d. A Box containing Plates of different Metals, Wood, Paſte-board and Glaſs.

Experiments on the Tranſmiſſion of Electricity.

A Glaſs garniſhed with a Circle of Metal for containing Water.

A Bar of Iron one Inch ſquare and three Feet long.

A ſmall Globe of Chriſtal mounted ſo that it can be rubed in Vacuo, to which is adapted a Stop-cock to be applied to the Air-pump.

A compleat Aſſortment of every Thing neceſſary for electrical Experiments, either in Air or in Vacuo.

Plates of Braſs, Part of which has been beat cold, the other when tempered in Fire.

A large Paſte-board covered on one Side with Leaf-Gold, and on the other with Leaf-Silver, for ſhewing the Ductility of thoſe Metals.

A Metal compoſed of Iron and Antimony, the Filings of which burſt into Flame by the Friction of the File. 2d. Sounding Lead. 3d. An Amalgama of Tin and Mercury for colouring the Inſide of Glaſs-Veſſels.

Seventh Class.
Machines of Coſmography.
VII.

A large Planetarium five Feet and a Half Diameter, with all its Appendages for ſhewing the different Motions of the Planets, and the Relations of the celeſtial Bodies with the Earth.

Experiments for illuſtrating the Theory of the primary and ſecondary Planets.

A Box containing the Pieces neceſſary for explaining what concerns the Motions and Relations of the Sun, the Earth and the Moon.

This Box only ſuppoſes a Table five Feet Diameter, in the Middle of which it is faſtened.

Two Globes, one celeſtial and the other tereſtrial, one Foot Diameter, conſtructed on the lateſt Obſervations, coloured and varniſhed, mounted on four pillared Pedeſtals, with Meridians and Horizons of a particular Kind of Paſte-board.

Two Armillary Spheres, of the ſame Diameter as the Globes, the one according to the Ptolemaic, the other according to the Copernican Syſtem, coloured and varniſhed, mounted on Pedeſtals of Ebony.

A ſmall tereſtrial Globe, three Inches and a half Diameter, coloured and varniſhed, with a Meridian and Quadrant of Altitude.

Two Globes, one tereſtrial and the other celeſtial, 18 Inches Diameter, coloured and varniſhed, mounted on pillared Pedeſtals, with Meridians, horary Circles, Compaſſes of Braſs, engraved and poliſhed.

The ſame Globes varniſhed and poliſhed, with Meridians, horary Circles, Braſs Compaſſes, mounted on a turning Pedeſtal of a new Conſtruction.

Experiments for illustrating the Doctrine of the Sphere.

The celestial Globe is of an azure blue. The Figures of the Constellations are perceived as Shades, the principal Circles of the Sphere are marked in Silver, as also on the terestrial Globe; the Stars are raised in Gold, each in their proper Size, so that at one View, the natural State of the Heavens is perceived without Confusion.

Two large Planispheres, mounted on a Frame with Gold Stars, and garnished with Meridians and Horizons.

A white Globe one Foot Diameter, mounted on a Stand, with some Instruments belonging to it.

A new Dial, which serves for tracing the Meridian of a Place.

An astronomical Quadrant two Feet Radius, with two Divisions of Nonius; a moveable and immoveable Telescope, and an exterior Micrometer. 2d. An astronomical mural Quadrant four Feet Radius.

A Sextant four Feet Radius. 2d. A Sextant one Foot Radius for taking corresponding Altitudes.

Observations shewing the Use of astronomical Instruments the Quadrant, the Sextant, the Meridian-telescope, the Parallatic-telescope, the Micrometer, &c.

A Quadrant two Feet and a half Radius, with a Transom and double Joint, for measuring Angles on Land.

A meridian Telescope or a passage Instrument, four Feet long, and its Axis two Feet. 2d. A parallatic Telescope with its Axis, which serves for following the Parallel of a Star. 3d. An equatorial Telescope moveable by the Means of several graduated Circles, with its objective Micrometer. 4th. A Telescope moveable on an Axis, with an horizontal and vertical Circle graduated, and an Helioscope.

A Micrometer, to be applied to a moveable Telescope for measuring the Diameters, the Differences of the right Ascensions and Declinations of the celestial Bodies. 2d. A Micrometer to be applied to an astronomical Quadrant. 3d. An achromatic Micrometer.

An Octant 18 Inches Radius, for observing the Altitudes and Distances of the Moon from the Stars on Sea.

A Clock adapted for astronomical Observations, whose Pendulum is so composed as to correct the Dilatation to which Metals are liable. 2d. A Telescope conducted by a Clock for designing the Spots of the Moon, &c.

EIGHTH CLASS.

Machines of Meteorology.

Meteorologic Observations.

A large Thermometer, constructed on the Principles of *Reaumur*. 2d. A Thermometer constructed on the same Principles mounted to accompany a Barometer. 3d. A Thermometer, constructed on the same Principles, to be exposed in open Air.

A portable Thermometer one Foot long, constructed on the same Principles. 2d. A portable Thermometer contrived so as to be plunged into Fluids, in order to determine their Degree of Heat or Cold. 3d. A Thermometer constructed with Mercury, for Experiments where the Heat exceeds that of boiling Water.

The Thermometer of *Florence*. 2. A Thermometer of Air with Mercury. 3d. A Thermometer of Air, with coloured Liquor.

Obfervations fhewing when the Denfity of the Air is diminifhed either by the Expanfion produced by Heat or by Caufes which diminifh its Weight.

A kind of Pyramid, garnifhed with feveral Thermometers of Water, Oil, Spirit of Wine, falt Water, Mercury, for fhewing the Dilatability of each of thofe Fluids.

A large Thermometer filled with coloured Water, for fhewing the Dilatability of Glafs.

A double Barometer. 2d. The Barometer of *Bernoully*. 3d. A Barometer bent in its upper Part.

Thofe three Machines ferve for fhewing the Means employed for rendering the Variation in the Weight or Spring of the Air more fenfible.

The Barometer fhortened, by the Oppofition of the two Columns of Mercury to one Column of Air. 2d. The Barometer fhortened, by a Remainder of Air in the upper Part. 3d. The Barometer of *Amonton*.

Obfervations fhewing when the Denfity of the Air is diminifhed by the Caufes which diminifh its Weight.

Thofe Machines, ferve for fhewing the Methods employed for rendering the Barometer portable.

The fimple and luminous Barometer mounted, to accompany the Thermometer, conftructed on the Principles of *Reaumur*.

This Barometer differs from the common ones by the Manner it is filled, by the Form of the Vafe in which it is plunged, and the Exactitude of its Effects.

The fame Barometer rendered portable in any Direction, or in any kind of Carriage. 2d. The fame Barometer rendered portable in a walking Cane.

This Barometer has this Advantage, that the inferior Surface of the Mercury is feen, which is well known to be of Ufe.

A Dial Hygrometer very fenfible. 2d. An Hygrometer of another Conftruction.

A Pyrometer, or Machine for meafuring the Action of Fire on Bodies, whofe Dilatation is not immediately perceived.

Experiments for fhewing the Dilatation of Metals.

In the Conftruction of this Machine, every Imperfection to which it has been hitherto liable is removed, the Degree of Heat is eafily regulated, and every Precaution neceffary, has been taken to hinder the Duft or the Humidity to fpoil the Polifh or the Motion of the Pieces.

An Anemometer, or Machine for difcovering the Direction and Velocity of the Wind, with the Time during which it continues.

Conclusion.

Such is the Plan of the Collection of Machines which the Nobility and Gentry of the Kingdom of *Ireland* have purchafed, and whofe Conftruction and Application to Experimental Inquiries, they have ordered to be defcribed, and publifhed, for the Ufe of the Mathematical School eftablifhed under their immediate Infpection, purfuant to their Refolution of of the 4th *February*, 1768.

PLAN of the System of the Moral World.

——Servare modum, finemque tueri,
Naturamque sequi, patriæque impendere vitam,
Non sibi sed toti genitum se credere mundo.

LUCAN.

I.

MEN in the State of Nature, being apt to allow no other Rule for determining the Difference which might arise among them, but what is common to the brute Creation, namely, superior Strength. The Establishment of civil Society should be considered as a Compact against Injustice and Violence, a Compact intended to form a Kind of Balance between the different Parts of Mankind; but the moral Equilibrium, like the phisical one, is rarely perfect and durable. Interest, Necessity, and Pleasure, brought Men together, but the same Motives induce them continually to use their Endeavours to enjoy the Advantages of Society, without bearing the Charges necessary to its Support: and in this Sense, Men, as soon as they enter into Society, may be said to be in a State of War; Laws are the Ties, more or less efficacious, intended to suspend their Hostilities, but the prodigious Extent of the Globe, the Differerence in the Nature of the Regions of the Earth and its Inhabitants, not allowing Mankind to live under one and the same Government, it was natural that Men should divide themselves into a certain Number of States, distinguished by the different Systems of Laws which they are bound to obey. Had all Mankind united under one Government, they would have formed a languid Body, extended without Vigour on the Surface of the Earth. The different States are so many strong and active Bodies, which lending each other mutual Assistance, form but one, and whose reciprocal Action supports the Life and Motion of the Whole.

Origin of civil Society.

II.

The different Forms of Government in the World.

All the States with which we are acquainted, partake of three Forms of Government, *viz.* the Republican, Monarchical, and Despotic. In some Places Monarchy inclines to Despotism, in others the Monarchical is combined with the Republican, &c. Those three Species of Government are so entirely distinct, that properly speaking, they have nothing in common: We should therefore form of those three, so many distinct Classes, and endeavour to investigate the Laws peculiar to each; it will be easy afterwards to modify those Laws in their Application to any Government whatsoever, in proportion as they relate more or less to those different Forms.

In the different States, the Laws should be conformable to their Nature, that is, to what constitutes them, and to their Principle, or to that which supports and gives them Vigour. The Law relative to the Nature of Democracy is first explained; it is shewn how the People in some respects are Monarchs, and in other Subjects; how they elect and judge their Magistrates, and how their Magistrates decide in certain Cases, &c. Then the Laws relative to the Nature of Monarchies are unfolded; the Degrees of delegated Power and intermediate Ranks that intervene between the Monarch and the Subject, the Duties of the Body to be appointed, the Guardian of the Laws to mediate between the Prince and the Subject are properly settled: In fine, it is proved, that the Nature of Despotism requires, that the Tyrant should exert his Authority, either in his own Person, or by some other who represents him; afterwards the Principles of the three Forms of Governments is pointed out; it is proved, that the Principle of Democracy is the Love of Equality, whereby is meant, not an absolute, rigorous, and consequently chimerical Equality, but that happy Equilibrium which renders all its Members equally subject to the Laws, and equally interested in their Support: That in Monarchies, where a single Person is the Dispencer of Distinctions and Rewards, the Principle is Honour, to wit Ambition and the Love of Esteem; and in Despotism, Fear. The more vigirously those Principles operate, the greater the Stability of the Government; and the more they are relaxed and corrupted, the more it inclines to Destruction.

The Laws derived from the Nature of Democracies.

The Laws derived from the Nature of Monarchies.

The Laws derived from the Nature of Despotism.

In what consist the Principles of the three Forms of Government.

The System of Education, suitable to each Form of Government, follows: It is proved, that they ought to be conformable to the Principle of each Government: That in Monarchies, the principal Object of Education should be the Art of pleasing; as productive of Refinement of Taste; Urbanity of Manners, an Address that is natural, and yet engaging, whereby Civil Commerce is rendered easy and flowing. In despotic States, the principal Object should be to inspire Terror and implicit Obedience; in Republics all the Powers of Education are required; every noble Sentiment should be carefully instilled; Magnanimity, Equity, Temperance, Humanity, Fortitude, a noble Disinterestedness, from whence arises the Love of our Country.

The Laws of Education relative to the Principle of each Form of government.

The Laws relative to the Principle of each Government next occur; it is shewn, that in Republics, their principal Object should be to support Equality and Oeconomy; in Monarchies to maintain the Dignity of the Nobility, without oppressing the People; in Despotic Governments, to keep all Ranks quiet. Then the Differences which the Principles of the three Forms of Government should produce in the Number and Object of the Laws, in the Form of Judgments and Nature of Punishments is explained; it is proved, that the Constitution of Monarchies being invariable, in order that Justice may be rendered in a Manner more uniform and less arbitrary:

The Laws derived from the Principle of each Form of Government.

More civil Laws and Tribunals are required, which are accurately described; that in temperate Governments, whether Monarchical or Republican, criminal Laws cannot be attended with too many Formalities; that the Punishments should not only be proportioned to the Crime, but as moderate as possible; that the Idea annexed to the Punishment, frequently will operate more powerfully than its Intensity; that in Republicks, the Judgment should be conformable to the Law, because no Individual has a Right to alter it; in Monarchies, the Clemency of the Sovereign may abate its Rigour; but the Crimes should be always judged by Magistrates appointed to take Cognizance of them. In fine, that it is principally in Democracies, that the Laws should be severe against Luxury, Dissoluteness of Manners, and the Seduction of the Sex.

Advantages peculiar to each Form of Government.

The Advantages peculiar to each Government, is, in fine, enumerated; it is proved, that the Republican is better suited to small States, the Monarchical to great Empires; that Republicks are more subject to Excesses, Monarchies to Abuses; that in Republicks the Laws are executed with more Deliberation, in Monarchies with more Expedition. As to despotic Governments, to point out the Means necessary for its Support, is in effect to sap its Foundation; the Perfection of this Government is its Ruin; and the exact System of Despotism is at once the severest Satire, and the most formidable Scourge of Tyrants.

III.

Liberty is the Prerogative of every temperate Government.

Is not to be confounded with Independancy.

Considered with respect to the Constitution.

The general Law of all Governments, at least temperate ones, and consequently just, is political Liberty; the full Enjoyment of which should be secured to each Individual: This Liberty is not the absurd Licence of doing whatever one pleases, but the Privilege of doing whatever is permitted or authorised by Law; it may be considered either as it relates to the Constitution or to the Individual. It is shewn, that in the Constitution of every State, there are two Powers, the Legislative and Executive, and that this latter has two Objects, the internal and external Policy; in the legal Distribution of those different Sorts of Power, consists the greatest Perfection of political Liberty, with respect to the Constitution; in Proof of which are explained the Constitution of the Republic of *Rome*, and that of *Great-Britain*: It is shewn, that the Principle of the latter is founded on the fundamental Law of the ancient *Germans*; namely, that Affairs of small Consequence were determined by the Chiefs, and those of Importance were referred to the General Assembly of the whole Nation, after being previously examined by the Chiefs. Political Liberty considered, with respect to Individuals, consists in the Security which the Law affords them, whereby one Individual is not in Dread of another. It is shewn, that it is principally by the Nature and Proportion of Punishments that this Liberty is established or destroyed: That Crimes against Religion

Exists principally in England.

Considered with respect to Individuals.

ſhould be puniſhed by the Privation of the Advantages which Religion drocures; the Crimes againſt good Morals, by Infamy; Crimes againſt the public Tranquility, by Priſon or Exile; Crimes againſt private Security, by corporal Puniſhments: That Writings are leſs criminal than Deeds; meer Thoughts are not puniſhable; Accuſation without a regular Proceſs, Spies, anonymous Letters; all thoſe Engines of Tyranny, equally infamous with reſpect to the Inſtruments and the Employers, ſhould be proſcribed in every good Government, that no Accuſations ſhould be urged but in Face of the Law, which always puniſhes Guilt or Calumny: In every other Caſe, the Magiſtrate ſhould ſay, *we ſhould abſolve from Suſpicion, the Man who wants an Accuſer, without wanting an Enemy.* That it is an excellent Inſtitution to have public Officers appointed, who in the Name of the State may proſecute Criminals: This will produce all the Advantages of Informers, without their Inconveniencies and Infamy.

Liberty conſidered with reſpect to the levying of Taxes and the public Revenues.

The Nature and Manner of impoſing and collecting Taxes is afterwards explained: It is proved, that they ſhould be proportioned to Liberty; conſequently in Democracies they may be heavier than in other Governments, without being burthenſome; becauſe each Individual conſiders them as a Tribute he pays himſelf, and which ſecures the Tranquility and Fortune of each Member: Beſides, in Democracies, the Miſapplication of the public Revenues is more difficult, becauſe it is more eaſily diſcovered and puniſhed; each Individual having a Right to call the Treaſurer to an Account. That in every Form of Government, thoſe Taxes that are laid on Merchandizes are leaſt burthenſome, becauſe the Conſumer pays without perceiving it: That the exceſſive Number of Troops in Time of Peace, is only a Pretext to overcharge the People with Taxes; a Means of enervating the State, and an Inſtrument of Servitude. In fine, that the collecting of the Duties and Taxes by Officers appointed for this Purpoſe, whereby the whole Product enters the public Treaſury, is by far leſs burthenſome to the People, and conſequently more advantageous than the farming out of the ſame Duties and Taxes, which always leaves in the Hands of a few private Perſons, a Part of the Revenues of the State.

The Augmentation of the Number of Troops enervates the State.

IV.

Particular Circumſtances which ſhould modify the different Forms of Government.

The Circumſtances independant of the Nature of the Form of Government, which ſhould modify the Laws, ariſe principally from the Nature of the different Regions of the Earth, and the different Characters of the People which inhabit them. Thoſe ariſing from the Nature of the Regions of the Earth, are two-fold; ſome regard the Climate, others the Soil. No Body doubts but the Climate has an Influence on the habitual Diſpoſition of Bodies, conſequently on the Characters, the Laws ſhould be therefore conformable to the Nature of the Climate in

The Climate produces the Difference in the Characters and Passions of Men.

indifferent Matters, and on the contrary check its vicious Effects; an exact Enumeration of which is made, and the Laws for correcting them explained, it is shewn, how in Countries where the Heat of the Climate inclines the People to Indolence, the Laws encourage them to Labour; where the Use of Spirituous Liquors is prejudicial, they are discouraged, &c.

Slavery is inconsistent with the Law of Nature and the civil Law.

The Use of Slaves being authorised in the hot Countries of *Asia* and *America*, and prohibited in the temperate Climates of *Europe*, the Lawfulness of civil Slavery is next enquired into; it is proved, that Men having no more Power over the Liberty than over the Lives of one another, Slavery in general is inconsistent with the Law of Nature; that there has never been perhaps but one just Law in Favour of Slavery, *viz.* the *Roman* Law, whereby the Debtor was rendered the Slave of the Creditor; the Limitation of this Servitude, both as to the Degree and as to the Time, is pointed out. That Slavery at the utmost can be tolerated in despotic States, where free Men, too weak against the Government, seek for their own Advantage, to become the Slaves of those who tyrannize over the State; or else in Climates where Heat so enervates the Body, and weakens the Spirits, that Men cannot be brought to undergo painful Duties only by the Fear of Punishment.

Countries where it may be tolerated.

Domestic Slavery dedends on the Climate.

From thence we pass to the Consideration of the domestic Servitude of Women in certain Climates: It is shewn, that it should take Place in those Countries where they are in a State of cohabiting with Men before they are able to make Use of their Reason; marriagable by the Laws of the Climate, Infants by those of Nature. That this Subjection is still more necessary in those Countries where Poligamy is established, a Custom in some Degree founded on the Nature of the Climate and the Ratio of the Nunber of Women to that of Men; then the Nature of Repudiation and Divorce is examined, and it is proved, that if once allowed, it should be allowed in Favour of Women as well as of Men.

Political Slavery.

In fine, political Slavery is treated of; it is proved, that the Climate which has such Influence in producing domestic and civil Servitude, has not less in reducing one People under the Obedience of another; that the Northern People having more Strength and Courage than those of Southern Climates, the former are destined to preserve, the latter to lose their Liberty; in Confirmation of which, the various Revolutions which *Europe*, *Asia*, &c. have undergone, is unfolded; the Causes of the Rise and Fall of Empires is pointed out, particularly those of the *Roman* Empire; it is proved, that its Rise was principally owing to the Love of Liberty, of Industry, and of Country; Principles instilled into the Minds of the People from their earliest Infancy; to those intestine Dissentions, which kept all their Powers in Action, and which

It Reigns principally in hot Countries.

ceased at the Approach of an Enemy; to their intrepid Constancy under Misfortunes, which made them never dispair of the Republick; to that Principle from which they never receded, of never concluding Peace until they were victorious; to the Institution of Triumphs, which animated their Generals with a noble Emulation; to the Protection they granted Rebels against their Sovereigns; to their wise Policy of leaving to the Vanquished their Religion and their Customs; in fine, to their Maxim of never engaging in War with two powerful Enemies at once, submitting to every Insult from one, until they had crushed the other. That its Fall was occasioned by the too great Extent of the Empire, which changed the popular Tumults into civil Wars; by their Wars abroad, which forcing the Citizens to too long an Absence, made them lose insensibly the Republican Spirit; by the Corruption which the Luxury of *Asia* introduced; by the Proscriptions of *Sylla*, which debased the Spirit of the Nation, and prepared it for Slavery; by the Necessity they were in of submitting to a Master, when their Liberty became burthensome to them; by the Necessity they were in of changing their Maxims, in changing their Form of Government; by that Succession of Monsters, who reigned almost without Interruption, from *Tiberius* to *Nerva*, and from *Comodus* to *Constantine*; in fine, by the Translation and Division of the Empire, which was destroyed, first in the *West*, by the Power of the *Barbarians*; and after having languished many Ages in the *East*, under weak or vicious Emperors, insensibly expired.

Enumeration of the Causes of the Rise and Fall of the Roman Empire.

The Laws relative to the Nature of the Soil is next explained; it is shewn, that Democracies are better suited than Monarchies to barren and mountainous Countries, which require all the Industry of their Inhabitants; that a People who till the Soil, require more Laws than a Nation of Shepherds, and those more than a People who live by Hunting; those who know the Use of Coin, than those who are ignorant of it.

The Influence of the Nature of the Soil on the Laws.

The Laws relative to the Genius of the different People of the Earth at length is disclosed, and it is proved, that Vanity which magnifies Objects, is a good Resort of Government; Pride, which depresses them, is a dangerous one; that the Legislator, in some measure, should respect Prejudices, Passions, and Abuses; as the Laws should not be the best, considered in themselves, but with respect the People for which they are made; for Example, a People of a gay Character require easy Laws; those of harsh Characters, more severe ones. The Manners and Customs are not to be changed by Laws, but by Recompences and Examples: In fine, what the different Religions have, conformable or contrary to the Genius and Situation of the People who profess them, is explained.

The Laws considered with respect to the Genius of the Inhabitants of the Earth.

V.

The Relations of which the different Forms of Government are susceptable.

The different States considered with respect to each other, may yield mutual Assistance, or cause mutual Injury. The Assistance they afford is principally derived from Commerce, its Laws are therefore to be unfolded; it is proved, that though the Spirit of Commerce naturally produces a Spirit of Interest, opposed to the Sublimity of moral Virtues, yet it renders a People naturally just, and banishes Idleness and Rapine. That free Nations, who live under moderate Governments, should apply themselves to it more than those who are enslaved; that one Nation should not exclude another from its Commerce without important Reasons; that the Liberty however of Commerce does not consist in allowing Merchants to act as they please; a Faculty which would be very often prejudicial to them, but in laying them under such Restraints only, as are necessary to promote Trade; that in Monarchies, the Nobility should not pursue it, much less the Prince: In fine, that there are Nations to whom Commerce is disadvantageous; it is not those who want for nothing, but those who are in want of every thing; as *Poland*, by whose Commerce the Peasants are deprived of their Subsistence, to satisfy the Luxury of their Lords: The Revolutions which Commerce has undergone, is next displayed, and the Cause of the Impoverishment of *Spain* by the Discovery of *America*, pointed out: In fine, Coin being the principal Instrument of Commerce, the Operations upon it are treated of, such as Exchange, Payment of public Debts, &c. whose Laws and Limits are settled.

Virtues which Commerce introduces.

The Liberty of Trade not to be confounded with the Liberty of the Trader.

Should be interdicted to the Nobility in Monarchies.

Population and the Number of Inhabitants being immediately connected with Commerce, and Marriages having for their Object Population, every Thing relative thereto is accurately explained; it is shewn, that public Continence is what promotes Propagation; that in Marriages, though the Consent of Parents is with Reason required, yet it should be subject to Restrictions, as the Law should be as favourable as possible to Marriage; that the Marriage of Mothers with their Sons, on account of the great Disparity of the Ages of the Contractors, could rarely have Propagation for Object, and considered even in this Light, should be prohibited; that the Marriage of the Father with the Daughter might have Propagation for Object, as the Virtue of engendering ceases a great deal later in Men, and has in consequence been authorised in some Countries, as in *Tartary*; that as Nature of herself inclines to Marriage, the Form of Government must be defective, where it stands in Need of being encouraged; that Liberty, Security, moderate Taxes, the Proscription of Luxury, are the true Principles and Support of Population; that Laws notwithstanding may be made with Success, for encouraging Marriages, when, in spight of Corruption, the People are attached to their Country; what Laws have been made to this Purpose, particularly

Marriage to be encouraged.

Incestuous Marriages to be prescribed.

How Population is promoted.

those of *Augustus*, are unfolded; that the Establishment of Hospitals may either favour or hurt Population, according to the Views in which they have been planned; that there should be Hospitals in a State where the greatest Part of the Citizens have no other Resource than their Industry; but that the Assistance which those Hospitals give should be temporary; unhappy the Country where the Multitude of Hospitals and Monasteries, which are only perpetual Hospitals, sets every Body at their Ease, except those who labour.

Hospitals necessary in rich States.

How they are to be conducted

To prevent the mutual Injuries which States may receive from each other, Defence and Attack are rendered necessary; it is shewn, that Republicks by their Nature being but small States, cannot defend themselves but by Alliances; but that it is with Republicks they should be formed. That the defensive Force of Monarchies consists principally in having their Frontiers fortified. That States as well as Men, have a Right to attack each other for their own Preservation, from whence is derived the Right of Conquest, the general Law of which is to do as little Hurt to the Vanquished as possible. That Republicks can make less considerable Conquests than Monarchies; that immense Conquests introduce and establish Despotism; that the great Principle of the Spirit of Conquest should be to render the Condition of the conquered People better, which is fulfilling at once the natural Law and the Maxim of State, how far the *Spaniards* receded from this Principle, in exterminating the *Americans*, whereby their Conquest was reduced to a vast Desert, and they were forced to depopulate their Country, and weaken themselves for ever, even by their Victory, is explained. That it may become necessary to change the Laws of a vanquished People, but never their Manners and Customs. That the most assured Means of preserving a Conquest, is to put the Vanquished and Victors on a Level if possible, by granting them the same Rights and Privileges; how the *Romans* conducted themselves in this Respect, is related; as also how *Cesar* with respect to the *Gauls*.

The Objects of Conquest is not Slavery but Conservation.

Means of preserving a Conquest.

VI.

After having treated in particular of the different Species of Laws, there remains no more to be done, but to compare them together, and to examine them, with respect to the Objects on which they are enacted. Men are governed by different Kinds of Laws, by the natural Law common to each Individual; by the divine Law, which is that of Religion; by the ecclesiastical Law, which is that of the Policy of Religion; by the civil Law, which is that of the Members of the same Community; by the political Law, which is that of the Government of the Community; by the Law of Nations, which is that of Communities considered with respect to each other; each of those have their distinct Objects, which are not to be confounded, nor

The Laws resulting from the Nature, Circumstances, and Relations, of the different Forms of Government.

what belongs to one be regulated by the other; it is neceſſary that the Principles which preſcribe the Laws, reign alſo in the Manner of compoſing them; the Spirit of Moderation ſhould as much as poſſible direct all the Diſpoſitions: In fine, the Stile of the Laws, ſhould be ſimple and grave, it may diſpenſe with Motives, becauſe the Motive is ſuppoſed to exiſt in the Mind of the Legiſlator; but when they are aſſigned, they ſhould be founded on evident Principles.

VII

Concluſion. Such is the Plan of the Syſtem of the Moral World, where the Inhabitants of this Earth are conſidered in their real State, and under all the Relations of which they are ſuſceptible; the moral Philoſopher without dwelling on mere ſpeculative and abſtract Truths, in pointing out the Duties of Man, and the Means of obliging him to diſcharge them, has leſs in View the metaphiſical Perfection of the Laws, than what human Nature will admit of; the Laws that are exiſting, than thoſe which ſhould be eſtabliſhed; and as a Citizen of the World confined to no Nation or Climate; he makes the Laws of a particular People leſs the Object of his Reſearch, than thoſe of all the People of the Univerſe.

PLAN of the Military Art, including the Inſtructions relative to Engineers, Gentlemen of the Artillery, and in general to all Land-Officers.

Intenti expectant Signum, exultantiaque haurit
Corda pavor pulſans, Laudumque arrecta Cupido.

I.

SINCE the Revolution which the Invention of Gunpowder has produced in *Europe*, but above all, ſince Philoſophy born to conſole Mankind, and to make them happy, has been forced to lend its Light to teach Nations how to deſtroy one another, the Art of War forms a Science as vaſt as it is complicated, compoſed of the Aſſemblage of a great Number of Sciences united and connected together, lending each other mutual Aſſiſtance, and which the Youth of this Country who are intended for the the Military State, could never acquire but in a Military School, eſtabliſhed by public Authority, and conducted by a Man of ſuperior Talents and Abilities.

II.

Mathematicks. There the young Officers are firſt brought acquainted with Algebra and Geometry, elementary, tranſcendental and ſublime, to teach them the general Properties of Magnitude and Extention; how to calculate the Relations of their different Parts; how to apply them for determining acceſſible and inacceſſible Angles and Diſtances, tracing of Camps, furveying of Land, drawing of Charts, cubing the Works of Fortifications, &c. and to infufe into them that Spirit of Combination, which is the Foundation of all Arts, where Imagination does not predominate, as neceffary to the Military Gentleman as to the Aftronomer, which has formed *Turenne* and *Coborn*, as *Archimedes* and *Newton.*

III.

Thefe abftract Notions ferve as an Introduction for attaining the Art which teacheth the Properties of Motion, to meafure the Times and Spaces, to calculate the Velocities, and to determine the Laws of Gravity, to command the Elements by which we fubfift, whofe Forces it teaches to fubdue, and learns how to employ all that is at our Reach in Nature, in the moft advantageous Manner, either to affift us in our Enterprizes, by fupplying our Weaknefs, or to fatisfy our Wants, and procure us all Kind of Conveniencies. Mecanicks and Dynamicks.

IV.

They are taught the Application of this admirable Art, more particalarly for regulating the Dimenfions which fuit the Linings of the Works of Fortification, that they may refift the Preffure of the Earth, which they are to fuftain, by determining the Law according to which this Preffure acts. For eftimating the Refiftance that Counterforts are capable of, according to their Length, Thicknefs, and their Diftances from one another, for calculating how the Efforts of Vaults act, in order to deduce general Rules for determining their Thicknefs, according to the Forms that are to be given them in the different Ufes that are made of them in Fortification, either for Subterraneans, City-Gates, Magazeens of Powder, &c. for affigning the Form of Bridges, relative to the fpreading of the Arches, determining the Strefs and Strength of Timber, the Proportions of the Parts of Works, that they may have an equal relative Strength with refpect to the Models, according to which they are executed in large Dimenfions. Military Architecture.

V.

Then is unfolded the Theory of the Force and Action of Gunpowder, as it ferves to regulate the Proportions of Cannons, Mortars, Guns, &c. that of elaftic Fluids, as it teacheth to determine the actual Degree of the Refiftance of the Air to Shells and Bullets, and to affign the real Tract defcribed by thofe millitary Projectiles. Ballifiic.

VI.

Then the Ufe that can be made of the Dilatation and Condenfation of the Air, as of the Force that its Spring acquires by Heat, to move Machines, is explained, by fhewing the Effects of Pumps, defcribing the Properties of all the Kinds that have hitherto been invented; pointing out their Defects and Advantages; to what Degree of Perfection they can be brought; determining the moft advantageous Proportions and Pnumaticks.

Forms of their Parts, and of all the Machines contrived to make them move, either of those intended for the Use of private Persons, for extinguishing Fires, for supplying public Fountains, &c. unfolding the Construction of all those that have been hitherto executed in the different Parts of *Europe*, which are put in Motion either by Animals, by the Course of Rivers, by the Force of Fire, explaining how this Agent, the most powerful in Nature, has been managed with the greatest Art; afterwards is shewn how to calculate the Force of the Wind, the Advantages that can be drawn from it, for draining an aquatic or maracageous Land, or to water a dry Ground; exemplified by what has been practised in the different Parts of *Europe* in this Way.

VII.

The Art of conducting, raising, and managing Water, is next disclosed; it is shewn how to raise Water above the Level of its Source by Means of its Gravity, without making Use of the Parts which enter into the ordinary Composition of Machines; how to discover by Calculation, if a Water of a given Source, or raised to a given Height, by any Machine, can attain to a given Place, either by Trenches, Aqueducts, or Pipes; how to construct Basons, Water-Houses, and Cisterns to preserve it; how to distribute it through the different Parts of a City, determining the most advantageous Dimensions and Dispositions of the Conduits, and describing the most useful and ingenious hitherto executed.

Hydraulicks

As nothing is more agreeable to the Sight than Water-Works, the Manner of laying them out, and the Construction of the Machines imagined to raise the Water into the Reservoirs, which are the Soul of all those Operations, are unfolded, in order that the Engineer may be able to point out to those who are willing to embellish their Gardens, what suits them as to the Expence they are willing to be at, or the Situation of the Place; and that the Officer may be able to judge of the Beauty of Objects of this Kind.

Water, being of all Agents, that from which the greatest Advantage can be drawn for animating Machines, it is shewn how to apply it to the Wheels of the different Kinds of Mills; what Velocity they should have relative to the Current which moves them, in order that the Machines may be capable of the greatest Effect; entering into the Detail of all their different Species; calculating the Force necessary to put them in Motion; the Effects they are capable of, by Calculations, comprehending the Friction of their Parts, and the other Accidents inseperable from Practice; determining when they act upon inclined Planes, the Angle they should form with the Horizon. In fine, comparing such Machines as are contrived for the same Purpose, in order to discover which are to be preferred, according to the local Circumstances and Conveniencies for their Execution.

VIII.

The Art of rendering Works capable of resisting the violent or slow Action of Water, presents itself next; the various Machines made use of in draining, and of sinking Piles, is described; then all that concerns the Construction of Sluices, as also the Manner of employing them, according to the different Uses to which they are applied, either in levelling the Canals of Navigation; draining of Marshes; rendering Rivers navigable; forming artificial Inundations; making of Harbours, &c. Hydraulick Architecture.

IX.

In order to render those Researches of real Use to the young Officers, they are initiated in the Art of delineating Objects, as it teacheth how to represent all the Parts of Works already constructed, or that are intended to be constructed by Plans of them taken parallel to the Horizon, which shew the Distribution of all their Parts, their Dimensions, &c. by Profiles or Cuts of them taken perpendicular to the Horizon, which shew the Heights, Situations, &c. of all the Parts, by Plans of Elevation, or Cuts of the exterior Parts of the Work; in fine, by perspective Plans or Cuts, which represent the Object as seen at a certain Distance, which will enable them to judge of the Effect that all the Parts together produce. Draughting.

X.

These Studies prepare the young Officers for attaining to a Proficiency in the Art of defending and attacking, which comprehend the Method of fortifying regular Poligons, according to the different Systems, shewing their Advantages with regard to the local Circumstances, and how far they have been followed with Success in the Fortifications of the most celebrated Towns in *Europe*; the Construction and Disposition of Batteries, the Management of Artillery, the pointing of Mortars and Cannon, the conducting of Trenches, the Manner of distributing the different Stages of Mines, the Form of their Excavation, the Rangement of the Chambers, the best contrived for the husbanding the Ground and the Annoyance of the Enemy, the Construction of Lines and the Mensuration of their Parts, the tracing of Camps, entrenched or not entrenched, in even or uneven Ground, the tracing of the Camps of Armies which besiege, included in Lines of Circumvallation and Contravallation, the Attack of a regular or irregular fortified Place, situated in an equal or an unequal Ground, exemplified by the Plans of the most celebrated Sieges, joining Theory to Practice, neglecting not one Detail that may be of Importance. All these Operations being made in large Dimensions, and a Front of Fortification being raised accompanied with the other detached Works to be attacked and defended as in a real Action. Attack and Defence.

XI.

Geography. Geography, as an Introduction to Hiſtory, is uſeful to all Perſons, but the Profeſſion for which Youth is intended ſhould decide of the Manner more or leſs extenſive, it is to be taught; the young Officers ſhould have an exact Knowledge of the Countries which are commonly the Theatre of War, they are therefore inſtructed in Topography in the greateſt Detail, employing the Method of refering to the different Places, the Paſſages in Hiſtory which may render it remarkable, prefering military Facts to all others; by this Means their Notions are rendered more fixed, and their Memories though more burthened, will become ſtronger.

XII.

Hiſtory. The Life of Man is inſufficient to ſtudy Hiſtory in Detail, the Manner of teaching it ſhould therefore be adapted to the State of Life for which Youth is intended: Thoſe who are deſtined for the Law, ſhould be taught it, as it ſerves to diſcover the Spirit and Syſtem of the Laws of which they will one Day be the Diſpenſers; thoſe who are intended for the Church, as it relates to Religion and the eccleſiaſtical Diſcipline; the young Officers are taught it, as they may draw Inſtruction from the military Details, as it furniſhes Examples of Virtue, Courage, Prudence, Greatneſs of Soul, Attachment to their Country and Sovereign; they are made to remark in Antient Hiſtory that admirable Diſcipline, that Subordination which rendered a ſmall Number of Men the Maſters of the World; they are taught how to gather from the Hiſtory of their own Country, ſo neceſſary and ſo neglected, the preſent State of Affairs, the Rights of their King and Country, the Intereſt of other Countries and Sovereigns, &c.

XIII.

Tacticks. The Theory and Practice of the different Parts of the Military Service being neceſſary to all Officers, they are inſtructed in what regards the Service of Camps, the Service of Towns, Reviews, Armaments, Equipments, &c. As to military Exerciſes, and Evolutions, all who are acquainted with the actual State of military Affairs, know how neceſſary it is to have a great Number of Officers ſufficiently inſtructed in the Art of exerciſing Troops; it is manifeſt that a continual Practice is the ſureſt Means to attain to a Proficiency in this Art; the young Officers therefore are taught the Management of Arms, and trained up to the different Evolutions, which one Day they will make others execute.

XIV.

Order of the Studies. The Order that is followed in the Employ of the Day is ſuch, that the Variety and Succeſſion of Objects may ſerve as a Recreation, which is the moſt infallible Means to haſten Inſtruction. The Leſſons of Algebra, Geometry, Mechanicks, Hidroſtaticks, Hydraulicks, Geography, Hiſtory, &c. are firſt given, and thoſe on the various Branches of Drawing ſucceed.

XV.

As Youth is liable to take a Diſguſt againſt abſtract Knowledge, when its Application is not rendered ſenſible, the Teachers of Mathematicks and Drawing frequently put in Practice in the Field, the Mathematical, Mechanical, &c. Operations which are ſuſceptible, and which have been already delineated on Paper, Deſign at ſight, Views, Landſcapes, &c. this Method has the Advantage of procuring the Pupils an Amuſement which inſtructs them, and rendering palpable the Truths that have been preſented them, it inſpires them at the ſame Time with a Deſire of learning new ones, and making them execute after Nature agreeable Operations, it is a ſure Means of forming their Taſte. Practical Operations.

XVI.

As the Inequality of Ages and Genius, and even of the good and bad Diſpoſitions of the Pupils, cauſe a great Difference, the State of the Examination is divided into three Claſſes. In the firſt are thoſe who diſtinguiſh themſelves the moſt by their Application; in the ſecond are compriſed thoſe who do their beſt; the third comprehends thoſe from whom little is expected. This State is laid before the Society, in order that it may have an exact Knowledge of the Progreſs of each. Public Examinations.

XVII.

Such are the Means, my brave Countrymen, which the DUBLIN SOCIETY have purſuant to their Reſolution of the 4th of February, 1768, procured you, to enable you to ſtudy with Succeſs, how to eſtabliſh a Concert and an Harmony of Motion amongſt thoſe vaſt Bodies ſtiled Armies; how to combine all the Springs which ought to concur together; how to calculate the Activity of Forces, and the Time of Execution; how to take away from Fortune her Aſſendant, and to enchain her by Prudence; how to ſeize on Poſts, and to defend them; how to profit of the Ground, and take away from the Enemy the Advantage of theirs; not to be dejected by Dangers, nor elated by Succeſs; how to retire, change the Plan of Operation; how in the Glance of an Eye to Form the moſt deciſſive Reſolutions; how to ſeize with Tranquility the rapid Inſtants which decide Victories, draw Advantages from the Faults of the Enemy; commit none, or what is greater, repair them, in which conſiſts the ART OF WAR. Concluſion.

PLAN of the Mercantile Arts, including the Instructions relative to those who are intended for Trade.

Docuit quæ maximus Atlas.

I.

Dignity of the Trader.

WISE Regulations and well concerted Encouragements will contribute very little to promote Trade, unless they be rendered practicable, operative, and useful, by the Skill and Address of the judicious and industrious Trader; it is he who employs the Poor, rewards the Ingenious, encourages the Industrious, interchanges the Produce and Manufactures of one Country for those of another, binds and links together in one Chane of Interest, the Universality of the human Species and thus becomes a Blessing to Mankind, a Credit to his Country, a Source of Affluence to all around him, his Family, and himself. The Extent of Knowledge and Abilities notwithstanding, requisite to fit Youth for so great and valuable Purposes, have not been attended to in this Country, and those of the commercial Profession have laboured under the same Disadvantages in Point of Education, as the different Classes of Men we have already spoke of.

II.

The Disadvantages in Point of Education those of the commercial Profession labour under.

A Number of Years are spent and frequently lost in drudging through the common Forms of a Grammer School, where Youth are obliged to learn what is dark and difficult, and what must afterwards cost them much Pains to unlearn, and if long pursued must in the End retard the quickest Parts, and go near to eclipse the brightest Genius: whilst on the contrary, if the Grammar School Studies were properly directed and carefully pursued, they would learn to pass a proper Judgment on what they read, with regard to Language, Thoughts, Reflections, Principles, and Facts, to admire and imitate the Solid more than the Bright, the True more than the Marvellous, the personal Merit and good Sense more than the external and adventitious; their Taste for Writing and Living might be in some Measure formed, their Judgment rectified, the first Principles of Honour and Equity instilled, the Love of Virtue and Abhorrence of Vice excited in their Minds: *quare ergo liberalibus Studiis Filios erudimus? non quia Virtutem dare possunt, sed quia Animum ad accipiendam Virtutem præparant, quemadmodum prima illa ut Antiqui vocabant, Literatura, per quam Pueris Elementa traduntur, non docet liberales Artes, sed mox percipiendis Locum parat, sic liberales Artes non perducunt Animum ad Virtutem, sed expediunt.*

III.

At a certain Age, not after certain Acquisitions, a Master of Mathematicks is looked out for, and in this Case great Pretentions, attested by his own Word, and low Prices, are sufficient Credentials to recommend him, although neither the Teacher nor the Student reap much Advantage from it. When the Round of this Teacher's Form is once finished, the Student is then turned over to the Compting-House, where he is employed during the Time of his Apprenticeship, in copying Letters, going of Messages, and waiting on the Post-Office. The Master, though he hath Talents for communicating, hath not Time for attending to the Instruction of an Apprentice, who, on the other Hand, hath been so little accustomed to think, that this Improvement by Self-Application will be very inconsiderable, besides his Time of Life, and constant Habit of Indulgence, render him more susceptible of pleasurable Impressions, than of Improvement in Business, the more especially when he was not previously prepared to understand it; wherefore it is not at all surprising, if many, who having no Foundation in Knowledge to qualify them for the Compting-House, profit little from the Expence and Time of an Apprenticeship, and from seeing Business conducted with all the Skill and Address of the most accomplished Merchant: The Consequence must no Doubt, be fatal to Numbers, and the public Interest, as well as private, must suffer greatly by every Instance of this Nature. It is true, that there have been, and still are, Gentlemen, who, destitute of all previous mercantile Instruction, without Money, and without Friends, by the uncommon Strength of natural Abilities, supported only by their own indefatigable Industry and Application, and perhaps favoured with an extraordinary Series of fortunate Events, have acquired great Estates; but such Instances are rare, and rather to be admired than imitated; for we see many set out with large Capitals, who have shone in the commercial World while their Capitals lasted, as Meteors do in the natural, but like them, soon destroyed themselves, and involved in their Ruin all such who were so unhappy as to be within the Sphere of their Influence. *Novimus Novitios, qui cum se Mercaturæ vix dederunt, in magnis Mercimoniis se implicantes, Rem suam male gessisse; et profecto imperitos Mercatores, multis Captionibus suppositos, multisque insidiis expositos experientia videmus.*

IV.

Commerce is not a Game of Chance, but a Science, in which he who is most skilled bids fairest for Success, whereas the Man who shoots at Random, and leaves the Direction to Fortune, may go miserably wide of the Mark; of which the People of this Country at length made sensible, have come to the Resolution of no longer trusting the future Prospects of their Children in the World to a Foundation so weak and uncertain: but setting a proper Value on Education, are determined to be as careful in having the Minds of their Children adorned with Virtue and good Sense, as they are in setting off what relates to their Bodies. A School is erected in this Kingdom for training up Youth to Business, where every Master has a Salary proportioned to the Difficulty of his Department:

Establishment of a mercantile School.

The moſt intelligent Traders being appointed the Superintendants of this School, who take Care that none be admitted whoſe Parts are not previouſly enquired into, and whoſe Genius is not, in ſome Meaſure, turned to act with Dignity in the mercantile Profeſſion; who enquire often into the Morals and Proficiency of the Students, converſe frequently with the Maſters on the Subject of Trade, and admit the Students according to their Seniority in Letters to ſuch Converſations; and in ſhort, take every other Method of encouraging Maſters and Students to Induſtry and Attention, that they may go through the tedious and difficult Taſk with Alacrity and Spirit. That by theſe Means, our Youth may be long acquainted with the Arts of gaining, before they learn to ſpend Money; they may not be grown old in Debauchery and Riot before they are initiated into Buſineſs, and we may ſoon ſee a Spirit of Induſtry, Knowledge, Humanity, and good Senſe, diffuſe itſelf among all Ranks and Denominations, whilſt Idleneſs and Folly, with all their miſchiveous Train, may be baniſhed our Streets.

V.

Order of the Studies.

Mathematicks and Drawing.

In this School the young Merchant is firſt brought acquainted with Arithmetick, numeral and ſpecious, which of all other Sciences is the moſt neceſſary to the mercantile Profeſſion, the Teaching of which requires much Skill and Knowledge. Before it is applied to Computations in Buſineſs, every Rule is ſtated, exemplified, and illuſtrated, in an eaſy intelligible Manner, and the Examples ſo multiplied and diverſified, that the Learner may be thoroughly grounded, and have Reaſon always ready for what he does: All the various Compendiums which ſerve to abreviate Operations are diſtinctly ſhewn and demonſtrated: That Facility and Diſpatch may be equally familiar, Theory then is reduced to Practice, in all the Caſes which can occur to the Merchant, the Banker, the Cuſtom-Houſe, and Inſurance-Office, to which every Obſervation relative to Inſurance, Factorage, Exchange, and ſuch other Branches of Buſineſs are joined, which ſerve to illuſtrate the Uſe of the different Examples. This not only forms the Mind of the young Merchant to Buſineſs, but when he comes to act for himſelf, will prevent many tedious and expenſive Suits, which an Ignorance in the practical Arts of negociating them is frequently apt to create.

He is then initiated in Geometry, elementary, tranſcendental, and ſublime, which of all other Studies, contribute moſt to invigorate the Mind, to free it from Prejudices, Credulity, Superſtition, and to accuſtom it to Attention and demonſtrative Reaſoning: The Theory is reduced to Practice in the Menſuration of Surfaces and Solids, Heights and Diſtances, and in conſtructing the Inſtruments he hath Occaſion to uſe in laying down Plans and Maps of Countries, ſelling Land by Meaſure, aſcertaining the Price of Labour, and determining the Quantity of Liquors, for regulating of their Price and Duty, &c. ſome Proficiency in the different Branches of Drawing enables him to carry into Execution theſe practical Operations.

VI.

Geography and Hiſtory

The young Merchant is inſtructed in the Uſe of the Globes, Maps, &c. and brought acquainted with the Situation, Extent, Produce, Manufactures, Commerce, Ports, Policies, and Regulations with reſpect to Trade of all the Nations in the World; how the ſeveral Parts of the World are connected together in their mutual Intercourſe of Commerce, how the Redundancies of one Country ſupplies the Wants of another, in what Articles the Markets are ſcarce, and in what they are overſtocked, to enable him at all Times to foreſee when any Branch of Trade in which he is concerned is likely to be ſtagnated, and to take his Meaſures accordingly for preventing the bad Conſequences; the national Commerce in general, the Trade of the Place where he lives, the Laws, Cuſtoms, and Uſuages relative to the Buſineſs of the Merchant, the Penalties to which he is liable, and the Privileges to which he is intitled; the Duties, Impoſts, and other Charges, laid upon the Produce of thoſe Iſlands in other Countries, with all the known Maxims that relate to the Proſperity of Trade, open alſo a wide Field for Improvement in Matters of real Uſe.

VII.

Navigation.

Naval Architecture teaches the young Merchant the principal Parts of the Veſſel and their Proportions, the Dimenſions of her Bottom, the Curvature of her Flanks, the Sally of her Stem, &c. how to form the Plan of a Ship, how to diſcover by Calculation whether the Veſſel conſtructed according to the Device, will have the Qualities requiſite to her Deſtination, whether the Weight is proportioned to the Solidity of her Bottom, her Stability to the Quantity of Sail ſhe is to carry, the Rapidity of her Motion to her Capacity: the Method of gauging Veſſels, how to regulate the Duties on Anchorage, and other Duties of the ſame Species, &c.

The young Merchant is alſo brought acquainted with the Uſe of the Sea Compaſs, the Conſtruction and Uſe of Sea Charts, the Principles of Aſtronomy applied for finding the Latitude, the Variation of the Compaſs and Longitude at Sea; in fine the Manner of ſolving the different Problems relative to Navigation by graphical Operations and Calculation.

VIII.

Moral Philoſophy.

It will be of little Conſequence to have the Underſtanding improved, if the Heart be totally neglected; Man was made by Nature for Society, but the Merchant both by Nature and Practice: The young Merchant is therefore taught the Nature and Eſſence of Good, its Principles, Powers, and Effects, how to blend Self-Love with Benevolence, to moderate his Passions, to subject all his Actions to the Test of Reason, and that it is his Duty and Interest to found all his Dealing on Inregrity and Honour, as he that accustoms himself to unfair Dealing will, by Degrees, be reconcilled to every Species of Fraud, till Ruin and Infamy become the Consequence.

The Principle of Law and Government likewise constitute a Part of the mercantile Plan of Instruction, by which they learn to whom Obedience is due, for what it is paid, and in what Degree it may justly be required; and to give proper Instructions to their Representatives in the great Council of the Nation when they are deliberating on any Act which may be detrimental to the Interest of the Community with respect to Commerce, or any other Privilege whatsoever.

IX.

Composition, The Study of Composition not only teaches but accustoms the young Merchant to range his Thoughts, Arguments, and Proofs, in a proper Order, and to cloath them in that Dress, which Circumstances render most natural; by this Means he is not only enabled to read the Works of the best Authors witn Taste and Propriety, to observe the Elegance, Justness, Force, and Delicacy of the Turns and Expressions, and still more the Truth and Solidity of the Thoughts; hereby will the Connection, Disposition, Force, and Gradation of the different Proofs of a Discourse be obvious and familiar to him, while at the same Time he is led by Degrees to speak and write with Freedom and Elegance, which will infalliably raise the Opinion of the young Merchant in the Eye of his Correspondents, and of the Public.

X.

Book-Keeping, A Merchant ought to know upon all Occasions what is in his Power to do without embarrasing himself, and have such an Idea of his Dealings, and those with whom he deals, that his Speculations may be always within his Sphere, to effect which the Method of arranging and adjusting Merchants Transactions is, like other Sciences, communicated in a rational and demonstrative Manner, and not mechanically by Rules depending on the Memory alone. The Principles upon which the Science is founded is likewise reduced to Practice by proper Examples in foreign and domestic Transactions, such as Buying and Selling, Importing, Exporting, for proper Company, and Commission, Account, Drawing, and Remitting too, freighting and hiring Vessels for different Parts of the World, making Insurances and Under-writing, and the various other Articles that may be supposed to diversify the Business of the practical Compting-House. The Nature of all those Transactions, and the Manner of negociating them, are particularly explained as they occur, the Forms of Invoices and Bills of Sales, together with the Nature of all intermediate Accounts, which may be made uſe of to anſwer particular Purpoſes, are laid open; and the Form of all ſuch Writs as may be ſuppoſed to have been connected with the Tranſactions in the Waſtebook, are rendered ſo familiar, that the young Merchant may be able to make them out at once without the Aſſiſtance of Copies.

XI.

In order to accuſtom the young Merchants to think, write, and act like Men, before they come upon the real Stage of Action, an epiſtolary Correſpondence is eſtabliſhed among them, in order to accuſtom them to digeſt well whatever they read, and improve their Stile under the Correction of an accurate Maſter, to that clear, pointed, and conciſe Manner of Writing which ought, particularly, to diſtinguiſh a Merchant. Fictitious Differences among Merchants are likewiſe ſubmitted to their Judgement, ſometimes to two by the Way of Arbitration, and again to a Jury, whilſt one aſſumes the Character of the Plaintiff, and another that of the Defendant, and each gives in ſuch Memorials or Repreſentations, according to the Nature of the Facts diſcuſſed, as he thinks moſt proper to ſupport the Cauſe, the Patronage of which was aſſigned him. Practical Negotiations.

XII.

Thus the Education of the young Merchant is conducted, that his Knowledge may be ſo particular, and his Morals ſo ſecured, that he may be Proof againſt the Arts of the Deceitful, the Snares of the Diſingenuous, and the Temptations of the Wicked; that he may in a ſhort Time be ſo expert in every Part of the Buſineſs of the practical Compting-Houſe, that when he comes to act for himſelf, every Advantage in Trade will lie open to him, that his Knowledge, Skill, and Addreſs, may carry him through all Obſtacles to his Advancement, his Talents ſupply the Place of a large Capital, and when the beaten Track of Buſineſs becomes leſs advantageous, by being in too many Hands, he may ſtrike out knew Paths for himſelf, and thus bring a Balance of Wealth, not only to himſelf, but to the Community with which he is connected, by Branches of Trade unknown before. Concluſion.

PLAN of the Naval Art, Including the Inſtructions relative to Ship-Builders, Sea-Officers, and in general to all theſe who are any Way concerned in the Buſineſs of the Sea.

Qui dubiis auſus committere fluctibus Alnum,
Quas Natura negat, præbuit Arte Vias. CLAUD.

I.

AS nothing is executed in the Military Way, but by the Direction of Geometry and Mechanicks, no leſs indiſpenſible is the Uſe of theſe Sciences in Naval Operations, *viz.* Ship-building, ſtowing, working, and conducting Veſſels through the Sea. A Ship is ſo complicated a Machine, its various Parts have ſo cloſe and ſo hidden a Depandance on one another, and the Qualities it ought to be endued with, are ſo many in Number, and ſo difficult to be reconciled, the Mechaniſm of its Motions depends upon ſo many Inſtruments, which have an eſſential Relation to each other, &c. that it is only by Experience, aided by the ſublimeſt Geometry, it has been diſcovered, that all its Actions are ſubjected to invariable Laws, and that we can attain to certain Rules, which could enable the Maſter Ship-builders to give their Veſſels the moſt advantageous Forms, relative to the Services for which they are deſtined, and inſtruct the Navigator how to draw from the Wind the greateſt Force, to diſpoſe of it at Pleaſure, and to traverſe the vaſteſt Seas without Danger and without Fear.

Notwthſtanding which, Mathematicks reduced by the Teachers of them in this Kingdom, to a few groſs practical Rules, their Application to Sea Affairs, and to all other uſeful Enterpriſes, has not as yet been introduced; this Neglect has not only retarded the Progreſs that the Study of the Mathematicks otherwiſe would have made, by hindering it from being known that they are the Means the moſt proper to ſupply the Limitation of our natural Faculties, and that it is from them that all uſeful Arts are to receive their Perfection. But in the preſent Caſe, cannot but be attended with the moſt fatal Conſequences, and the Diſaſters that happen but too often at Sea, are undoubtedly, in a great Meaſure, owing to it.

II.

Naval Architecture.

The conſtructing and repairing of Veſſels is entirely abandoned to the Direction of Ship-Carpenters, whoſe Knowledge is confined to a few groſs obſcure Rules, which leave the Diſpoſition of almoſt all the Work to Chance, or to the Caprice of Workmen; they rely in the moſt important Circumſtances, on the blindeſt Practice, on that which is the moſt liable to Error; they change the upper Part of the Ship, they add a new Deck, or take one away, they alter totally the Form of her Bottom, &c. Making all thoſe Changes, without knowing what Effects will enſue, even thoſe that would manifeſt themſelves in the Harbour, though they could determine them after the moſt infallible and preciſe Manner, in employing the leaſt Knowledge of Geometry, and the ſimpleſt Operations of Arithmetick.

It was therefore neceſſary that a Marine School ſhould be eſtabliſhed, where the Youth who are intended for the Buſineſs of the Sea, ſhould be taught the Nature of Fluids, and the Mecaniſm of floating Bodies, how to conſider the Ship as a phyſical heterogeneous Body in all its different Situations, and relative to its different Uſes; repreſenting it to themſelves not only when it is loaden, and at Anchor, but alſo when it ſails, when it goes well, doubles a Cape, gets difficultly clear of a Coaſt,

&c. ſo that Geometry and Mechanicks taking the Place that Chance and blind Practice had uſurped, Maſter Shipbuilders may exerciſe their Employments with Diſcernment; ſubſtituting luminous and precise Rules in the Place of their imperfect practical ones; they may be no more expoſed to the Trouble and Shame of attempting any thing raſhly, but may be enabled to aſſign and foreſee the Succeſs of their Enterpriſes, and producing no Plans but what are ſupported by juſtifiable Calculations, in which each Quality the Ship ought to have, are diſcuſſed and eſtimated with Exactneſs; we can ſee, in verifying their Calculations, what Streſs can be laid upon their Promiſes; we may have infallible Means of deciding in Favour of the different Plans propoſed for the ſame Ship, and the Multitude of their Opinions, far from being hurtful, may on the contrary be profitable, ſince it will often furniſh an Occaſion of making a better Choice.

III.

The Ship being built, it is the Buſineſs of the Navigator to diſtribute the Loading in ſuch a Manner, that ſhe may ſail without Danger, and at the ſame Time receive with the greateſt Facility whatever Motions are to be given her, that is, he is to diſcover her moſt eligible Poſition in the Water, he is to diſpoſe her Sails after a ſuitable Manner to oblige the Veſſel to take the Route he intends to follow upon all Occaſions, and to make her go well in ſpight of the Agitation of the Sea, and the Violence of the Wind, which often oppoſes; for this Effect, in a Glance of an Eye, he muſt be capable of rendering preſent to his Mind all the moveable Parts of the Ship, which he muſt look upon as a Body which he animates as he does his own, and that it is as it were an Extention of it; ſeize the actual State of Things in their continual Change, and form the moſt deciſive Reſolutions, which he muſt draw from no other Fund but his own Breaſt. This is without doubt, the moſt difficult Part of the Navigator's Art, but at the ſame Time, the moſt important for him to poſſeſs, as it furniſhes him with the ſureſt Reſources in immergent Occaſions, and renders him ſuperior in Battle. It is ſurpriſing with what Readineſs, the Ship well diſpoſed, obeys, as it were, the Orders of the ſkilful Seaman; but on the contrary, if he does not know all the Nicety of this Part of his Art, his Ship, though excellent, is no more than a heavy Maſs, which receives all its Motions from the Caprice of Winds and Weaves, which in ſpight of his Courage and deſperate Efforts, becomes but too ſurely a Prey to the Enemy, or ends very ſoon its Deſtiny by Shipwreck.

Mechanical Navigation.

Notwithſtanding which, no Attempt had been made in this Kingdom to leſſen the Difficulties of attaining to a Proficiency in this Branch of the Naval Art, by inſtructing Sea-Officers in it after a methodical Manner. It was entirely abandoned to blind Practice, as if it could not be ſubjected to exact

Rules in the Employment of the phyfical Means which it makes ufe of to move the Veffel. When a Maneuvre is executed in the Prefence of a young Sea-Officer, he does not know very often for what it is done, or how the Inftruments that are made ufe of act; he is furrounded with Perfons too bufy to give him the leaft Eclaircifement; we may judge from thence how much Time he muft lofe to learn thefe grofs Notions, which are to ferve him inftead of Theory: The imperfect Knowledge which the young Sea-Officer will attain to, will be (to the Difgrace of human Reafon,) the Fruit of many Years unwearied Labour; and neverthelefs, as it will favour of its defective Origin, it will not give him fufficient Infight, and will leave him without exact Rules, which he can abfolutely rely upon; he will give, for Example, a certain Obliquity to the Sails; he will receive the Wind with a determined Incidence, but will he know whether there is nothing to be changed in one Senfe or the other, in one or the other Difpofition, his only Rule is fervily to copy what he has feen practifed perhaps erroneoufly by others on like Occafions; it was therefore neceffary that the Youth intended for the Sea, fhould be methodically inftructed in the ufeful Maxims of the Doctrine of the moveable Forces, applied to the Bufinefs of the Sea, fo that rendering them familiar to themfelves in taking Share in all the Maneuvres they will fee executed, in order to apply them mechanically, without the painful Help of Reflection; they might fee nothing for which they were not prepared beforehand, and of which they could give an Explication to themfelves; and as they would not be obliged to execute any Maneuvre blindly, they might be fenfible of the happy Effects that a reflected Exercife can produce, and the Quality of a good Practitioner would be lefs difficult to acquire.

IV.

The Art of Piloting.

The Navigator not only ought to know how to produce the different Motions of his Ship, but he is to obferve all the Particularities of its Route, efteem its daily Pofition, and the Courfe he is to fteer, to arrive at the Harbour where he is to go: This is the only Branch of the Naval Art that is taught by Rule; but it is a general Complaint among Seamen, that very little of what is learned in Schools, is of real Ufe; which contributes very much to confirm them in the dangerous Error, that Theory is of little or no Service; this proceeds from the Generality of Teachers having not fufficient Skill to conform their Plans of Teaching to the Exigencies of Seamen, in fhewing them how to modify their Rules of Navigation, according to the different Cafes of Sailing; how to reduce to the fmalleft Compafs, the Errors to which the Meafures made ufe of for determining the Courfe and Diftance, are liable to, and how to make proper Allowances for them, which would enable them, as often as the Reckoning would not agree with the Obfervation, to judge on which

Side lay the Error, and consequently how to correct them; all which supposes in the Teacher a profound Knowledge of the Theory of the Art, and a perfect Knowledge of all the Circumstances of the Ship's Motion, in all Cases of Wind and Weather.

Their not being sufficiently exercised in Astronomy, and astronomical Observations, make them neglect instructing Sea-Officers how to chuse the most favourable Circumstances for observing either by Night or Day. The only Observations practised by Sea-Officers, are the Sun's meridional Height, and its setting; they are entirely unacquainted with the Stars, though their Observations could be of great Use, particularly when the Sun does not serve, being observeable at all Hours of the Night, and the Incertitude to which the Reckoning is liable demands that the Sea-Officers should let no Occasion slip of taking Observations every Day; moreover the most reasonable Hopes of determining the Longitude at Sea, is founded on the Observation of the Distance of the Moon from a Star, or from the Sun; this Method gives actually the Longitude to half a Degree, and has the Advantage of being as easy put in Practice as that for determining the Latitude. If they had a little Skill in astronomical Observations, they could determine the Positions of so many Places, even of this Kingdom, which are placed in Charts after an uncertain Estimation; but on the contrary, they do not know even how to verify the Instruments that are in use at Sea, particularly their Compasses and Quadrants; for want of such a Knowledge, they are obliged to take them upon the bare Word of the Workman, who is interested to get them off his Hands at any Rate; and though they ought to be verified every Voyage, on Account of the Accidents that might arise to them, it is not done. This Particular, however minute, nevertheless is worthy of Attention, since nothing should be neglected in the present Case, seeing, in spight of all the Care that can be taken, the Errors that are committed being but too sensible, and as great ones may be occasioned in the Reckoning by the Imperfection of the Instruments, as in Deductions deduced from Calculation.

We may conclude from these Considerations, that the Ship-builders and Navigators of this Kingdom were no way apprised of the important Resources they could draw from Geometry and Mechanicks, though in no Profession so eminent as in theirs, and that they could never be sufficiently skilled in their respective Arts, until a Marine School was established, conducted by a Person exercised sufficiently in the sublime Mathematicks, as to be able to understand the different mathematical Tracts that have been published in great Number of late Years, upon the different Branches of the Naval Art, such as Ship-building, Stowing, working Vessels at Sea, &c. by the most eminent Mathematicians of *Europe*, who should make it his Business to communicate to them after

Establishment of a Marine School.

a methodical Manner, all the Improvements their respective Arts have received, and receive daily from Mathematicks.

V.

Draughting. He is aided in this important Employment by Drawing-masters, as the Ship-builders cannot finish properly their Plans, without a Tincture of this Art, and some Proficiency in it, may enable the Navigator to take Views of Lands, draw such Coasts, and plan such Harbours, as the Ship should touch at, which will contribute very much to render the Geography of our Globe more correct, and lessen the Dangers of Navigation; but what is perhaps of more Consequence, it will make them acquire the Habit of observing Objects with Distinctness, and recollect exactly every Part of them, and recall all the Circumstances of their Appearances. In one Word, as the Science, which is entirely occupied in weighing, measuring and comparing Magnitudes, is necessary in all Stations and Occurrences of Life, so the Art which teaches how to represent them to the Eye is indispensible.

AN EXTRACT from the Plan of the School of Mechanic Arts, where Architects, Painters, Sculptors, and in general all Artists and Manufacturers receive the Instructions in Geometry, Perspective, Staticks, Dynamicks, Physicks, &c. which suit their respective Professions, and may contribute to improve their Taste and their Talents.*

Rem quam ago, non opinionem sed opus esse, eamque non Sectæ alicujus aut placiti, sed utilitatis esse et amplitudinis immensæ fundamenta.

BACON.

I.

In the mechanic Arts is to be considered the Theory and Practice. HOWEVER vigorous, indefatigable, or supple is the naked Hand of Man, it is capable of producing but a small Number of Effects. He can perform great Matters but by the Help of Instruments and Rules, which are as Muscles superadded to his Arms. The different Systems of Instruments and Rules conspiring to the same End, hitherto invented to impress certain Forms on the Productions of Nature, either to supply our Wants, our Pleasures, our Amusements, our Curiosity, &c. constitute the mechanic Arts.

Every Art has its Theory and Practice; its Theory is grounded on Geometry, Perspective, Staticks, Dynamicks, whose Precepts corrected by those of Physicks, as it procures the Knowledge of the Materials, their Qualities, Elasticity, Inflexibility, Friction, the Effects of the Air, Water, Cold, Heat, Aridity, &c. produce the Rules and Instruments of the Art. Practice is the habitual Use of those Instruments and Rules.

* This Plan being too extensive is omitted for the present

It is ſcarce poſſible to improve the Practice without Theory, and reciprocally to be Maſter of the Theory without Practice, as there is in every Art a great Number of Circumſtances relative to the Materials, to the Inſtruments, and to the Operation which can be learned only by Uſe. It is the Buſineſs of Practice to point out the Difficulties, and to furniſh the Phenomena. It is the Buſineſs of the Theory to explain the Phenomena, to remove Difficulties and to open the Road to further Improvement; from whence it follows, that only ſuch Artiſts who have a competent Knowledge of the Theory, can become eminent in their Profeſſion.

The Knowledge of the Theory abſolutely neceſſary to every Artiſt.

But unfortunately ſuch is the Influence of Prejudice in this Country, that Artiſts, Mechanicks, &c. are conſidered as incapable of acquiring any Knowledge in the Principles of their reſpective Profeſſions, and our Youth deſtined to receive a liberal Education, are taught to think it beneath them to give a conſtant Application to Experiments and particular ſenſible Objects, for to practice or even to ſtudy the mechanic Arts, is to ſtoop to Things whoſe Reſearch is laborious, the Meditation ignoble, the Expoſition difficult, the Exerciſe diſhonourable, the Number endleſs, and the Value inconſiderable. Prejudice which has debaſed an uſeful and eſtimable Claſs of Men, and peopled our Towns with arrogant Reaſoners, uſeleſs Comtemplators, and the Country with idle and haughty Landlords.

The Judicious, ſenſible of the Injuſtice and of the fatal Conſequences attending this Contempt for the mechanio Arts, the Induſtry of the People and Eſtabliſhment of Manufactures being the moſt aſſured Riches of this Country, have come to the Reſolution that the Juſtice which is due to the Arts and Manufactures, ſhall be rendered them; that the mechanick Arts ſhall be raiſed from that State of Meaneſs, which Prejudice has hitherto kept them; that the Protection of the Noblemen and Gentlemen of Fortune ſhall ſecure the Artiſts and Mechanicks from that Indigence in which they languiſh, who have thought themſelves contemptible becauſe they have been deſpiſed; that they ſhall be taught to have a better Opinion of themſelves, as being the only Means of obtaining from them more perfect Productions.

A School of mechanic Arts is eſtabliſhed, where all the Phenomena of the Arts are collected, to determine the Artiſts to ſtudy, teach the Men of Genius to think uſefully, and the Opulent to make a proper Uſe of their Authority and their Rewards. There the Artiſts receive the Inſtructions they ſtand in need of, they are delivered from a Number of Prejudices, particularly that from which ſcarce any are free, of imagining that their Art has acquired the laſt Degrees of Perfection; their narrow Views expoſing them often to attribute, to the Nature of Things, Defects which ariſe wholly from themſelves; Difficulties appearing to

The Eſtabliſhment of a School of mechanic Arts.

them unsurmountable, when they are ignorant of the Means of removing them. They are rendered capable of reflecting and combining, and of discovering, in short, the only Means of excelling; the Means of saving the Matter, and the Time, of aiding Industry, eitherby a new Machine, or by a more commodious Method of Working. There Experiments are made, to advance whose Success, every one contributes, the Ingenious direct, the Artist executes, and the Man of Fortune defrays the Expence of the Materials, Labour and Time. There Inspectors are appointed who take Care that good Stuff is employed in our Manufactures, and that they are properly supplied with Hands; that each Operation employs a different Man, and that each Workman shall do, during his Life, but one Thing only; from whence it will result, that each will be well and expeditiously executed, and the best Work will be also the cheapest. Thus, in a short Time, our Arts and Manufactures will be brought to as great a Degree of Perfection, as in any other Part of *Europe*.

GENERAL CONCLUSION.

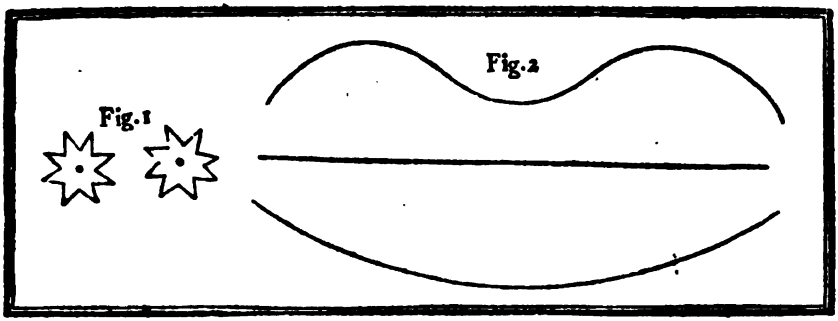
Such is the Plan of the new Scene of useful and agreeable Knowledge calculated for all Stations in Life, which the Nobility and Gentry of the Kingdom of *Ireland*, pursuant to their Resolution of the 4th of February 1768, have opened to Youth, in the Drawing-School established under their immediate Inspection. Encouraging Men of Genius and Education, from all Parts, to appear as Teachers, inviting the Artists and Connoisseurs to devote their Attention to excite the Emulation of the Pupils by adjudging and distributing the Premiums granted to engage them to advance more and more their Studies to the Point of Perfection, and taking under their Patronage such young Citizens favoured by Nature more than by Fortune, who discover happy Dispositions and superior Talents for the Service of their Country.

ERRATA.

Page LXIII Line 15, for the Centrifugal Force diminishes the Centrifugal Force, read the Centrifugal Force diminishes the Centripetal Force.

Page LXXI Line 14, for $\frac{400}{49\frac{1}{2}}$ read $\frac{400}{94\frac{1}{2}}$

Page LXXXV Line 41, for this Expression 69 for (a), 70 for (b), read, this Expression, for (a) 70, for (b) 69.



DEFINITIONS.

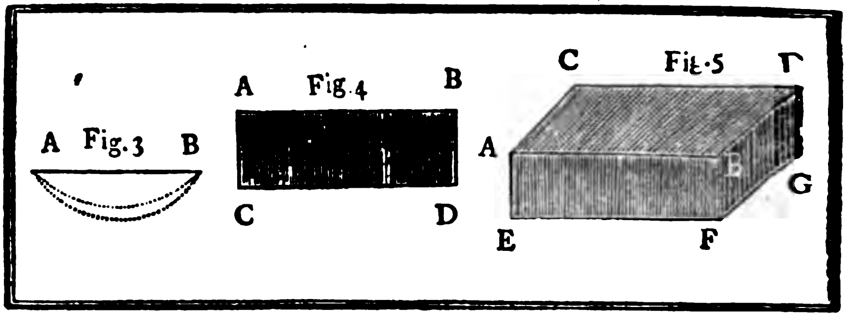
I.

A *Point*, is that which has no parts, or which hath no magnitude. *Fig.* 1.

IN this definition, as well as in the second and fifth, Euclid *simply explains the manner of conceiving the first objects of* Geometry, a Point, a Line, *and* a Superficies; *he does not demonstrate that there are such objects in the class of real beings. These notions, though very useful in geometry, are only abstractions which are not to be realised, by being represented as existing independent of the mind, where they took their rise. There are no mathematical points in nature, (at least what* Euclid *says does not prove it); but there exist things which have extension, which may be treated as simple marks without magnitude, as often as they are considered not as composed of parts, but merely as the limits of some other magnitude. Thus, when it is required to measure the distance of two stars, the Astronomer proceeds, as if those stars were indivisible points: and he is in the right; since he does not propose to determine their magnitude, but the distance that separates them, of which they are looked upon as the terms. The same is to be understood with respect of the other notions of this kind. We represent under the form of a line, or of a length without breadth, every magnitude whose length alone is the object of our consideration, whatever may be its breadth, its depth, or its other qualities. The imagination, always disposed to transform into realities what has none, forms of those abstractions a class of beings which seem to exist independent of the mind. The Geometer has a right to adopt those beings, as they may serve to render his speculations on magnitude, considered in different points of view, more intelligible; but it is by no means allowed to him, to form wrong notions as to their origin and their real use.*

II.

A *Line* is Length without breadth. Fig. 2.



DEFINITIONS.

III.

THE *Extremities of a Line*, are points (A, B,). *Fig.* 3.

IV.

A *straight Line*, is that which lies evenly between its extreme points (A, P,). *Fig.* 3.

This definition is imperfect, since it presents no essential character of a straight line; for which reason, Euclid *could make no use of it: it is no more quoted in the body of the work. He is obliged to have recourse to other principles (for example, to the 12th axiom) as often as he has occasion of employing truths, which depend on a perfect definition of a straight line.*

V.

A *Superficies*, is that which hath only length and breadth. *Fig.* 4.

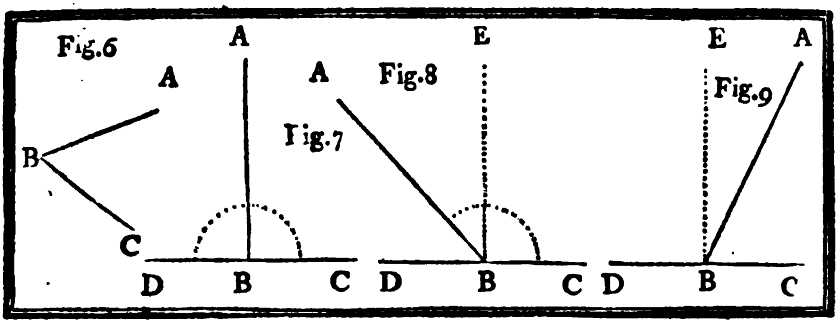
VI.

The *Extremities of a Superficies*, are lines (AB, CD, AC, BD,). *Fig.* 4.

VII.

A *Plane Superficies*, or simply a *Plane*, (AD) is that which lies evenly between its extremities (AB, CD, AC, BD,). *Fig.* 5.

This definition is liable to the same exceptions as the fourth.



DEFINITIONS.

VIII.

A *Plane Angle*, is the inclination of two lines (AB, BC,) to one another, which meet together, and which are ſituated in the ſame plane. Fig. 6.

IX.

A *Plane Rectilineal Angle*, is the inclination of two ſtraight lines to one another. *Fig.* 6.

N. B. *When ſeveral angles are at one point* B, *any one of them is expreſſed by three letters, of which the letter that is at the vertex of the angle, that is at the point in which the ſtraight lines that contain the angle meet one another, is put between the other two letters, and one of theſe two is ſomewhere upon one of thoſe ſtraight lines, and the other upon the other line.*

X.

When a ſtraight line (AB) ſtanding on another ſtright line (CD) makes the adjacent angles (ABD, ABC,) equal to one another, each of the angles is called a *right angle*; and the ſtraight line (AB) which ſtands on the other (CD) is called a *perpendicular*. Fig. 7.

XI.

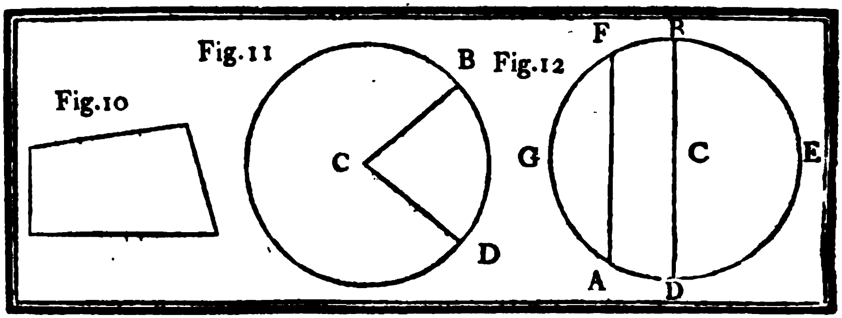
An *Obtuſe Angle*, (ABC) is that which is greater than a right angle (EBC). *Fig.* 8.

XII.

An *Acute Angle*, (ABC) is that which is leſs than a right angle (EBC). *Fig.* 9.

XIII.

A *Term* or *Boundary*, is the extremity of any magnitude.



DEFINITIONS.

XIV.

A *Figure*, is that which is inclosed by one or more boundaries. *Fig.* 10.

XV.

A *Circle*, is a plane figure contained by one line, which is called the *circumference*, and is ſuch that all ſtraight lines (CB, CD,) drawn from a certain point (C) within the figure to the *circumference*, are equal to one another. *Fig.* 11.

XVI.

This point (C) is called the *center* of the circle, and the ſtraight lines (CB, CD,) drawn from the center to the circumference, are called the *Rays*. *Fig.* 11.

XVII.

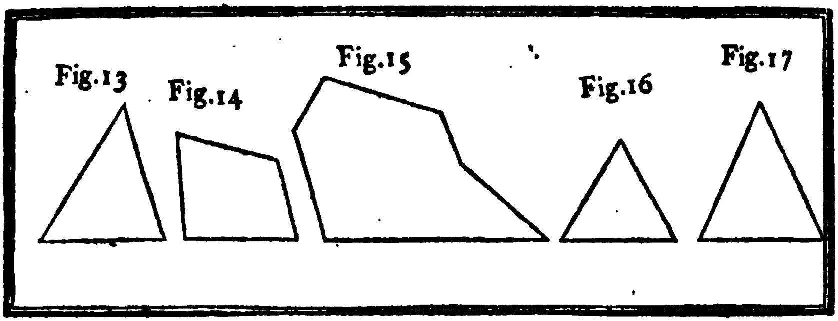
A *Diameter of a Circle*, is a ſtraight line (DB) drawn thro' the center, and terminated both ways by the circumference. *Fig.* 12.

XVIII.

A *Semicircle*, is the plane figure (DEB) contained by a diameter (BD) and the part of the circumference (DEB) cut off by the diameter (DB). *Fig.* 12.

XIX.

A *Segment of a Circle*, is a figure contained by a ſtraight line (AF) called a *Chord*, and the part of the circumference it cuts off (AGF, or AEF) called an *Arc*, *Fig.* 12.



DEFINITIONS.

XX.

Rectilineal Figures, are thoſe which are contained by ſtraight lines. *Fig.* 13, 14, 15, 16, 17.

XXI.

Trilateral Figures, or triangles, are thoſe which are contained by three ſtraight lines. *Fig.* 13, 16, 17.

XXII.

Quadrilateral Figures, are thoſe which are contained by four ſtraight lines. *Fig.* 14.

XXIII.

Multilateral Figures, or polygons, are thoſe which are contained by more than four ſtraight lines. *Fig.* 15.

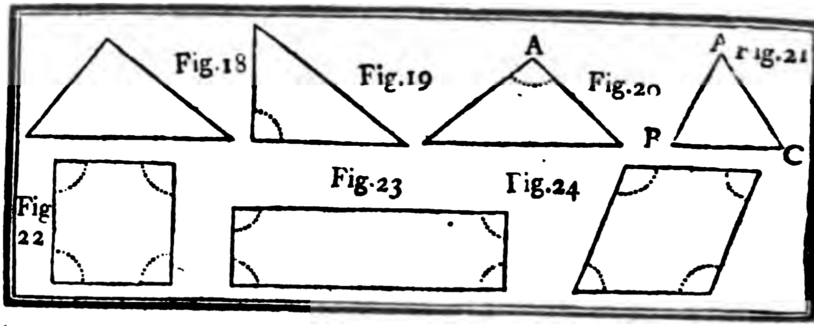
XIV.

As to three ſided figures in particular:

An *Equilateral Triangle*, is that which has three equal ſides. *Fig.* 16.

XXV.

An *Iſoſceles Triangle*, is that which has only two ſides equal. *Fig.* 17.



DEFINITIONS.

XXVI.

A *Scalene Triangle*, is that which has three unequal ſides. *Fig.* 18.

XXVII.

Likewiſe, amorg thoſe ſame trilateral figures:

A *Right angled Triangle*, is that which has a right angle. *Fig.* 19.

XXVIII.

An *Obtuſe angled Triangle*, is that which has an obtuſe angle, (A). *Fig.* 20.

XXIX.

An *Acute angled Triangle*, is that which has three acute angles, (A, B, C,). *Fig.* 21.

XXX.

After the ſame manner in the ſpecies of four ſided figures:

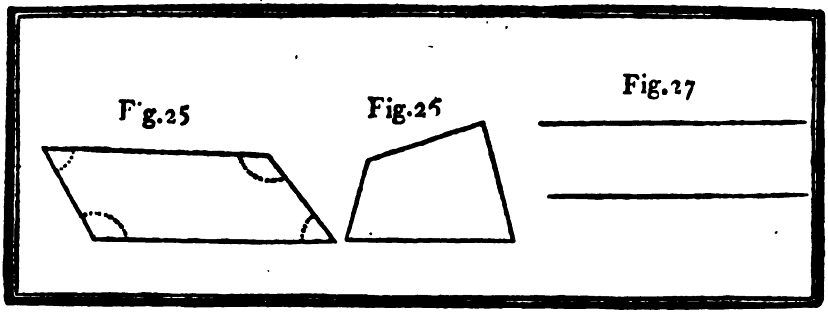
A *Square*, is that which has all its ſides equal, and all its angles right angles. *Fig.* 22.

XXXI.

An *Oblong*, is that which has all its angles right angles, but has not all its ſides equal. *Fig.* 23.

XXXII.

A *Rhombus*, is that which has all its ſides equal, but its angles are not right angles. *Fig.* 24.



DEFINITIONS.

XXXIII.

A *Rhomboid*, is that which has its opposite sides equal to one another, but all its sides are not equal, nor its angles right angles. *Fig.* 25.

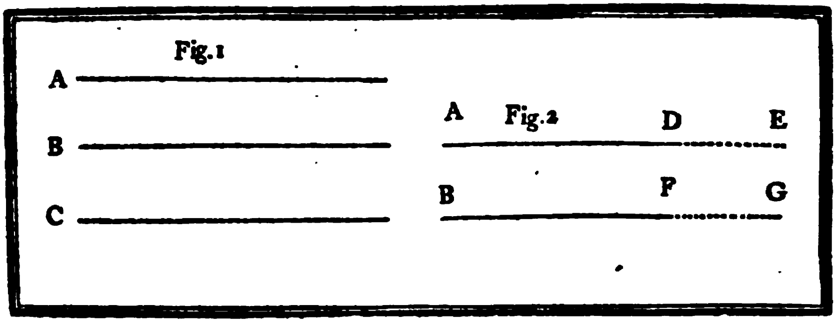
XXXIV.

All other four sided figures besides these, are called *Trapesiums*. Fig. 26.

XXXV.

Parallel straight Lines, are such as are in the same plane, and which being produced ever so far both ways, do not meet. *Fig.* 27.

It is for this reason that every quadrilateral figure whose opposite sides are parallel, is called a Parallelogram. Fig. 25.



POSTULATES.

I.

LET it be granted, that a ſtraight line may be drawn from any one point to any other point.

II.

That a terminated ſtraight line may be produced to any length in a ſtraight line.

III.

And that a circle may be deſcribed from any center, at any diſtance from that center.

AXIOMS; OR, COMMON NOTIONS.

I.

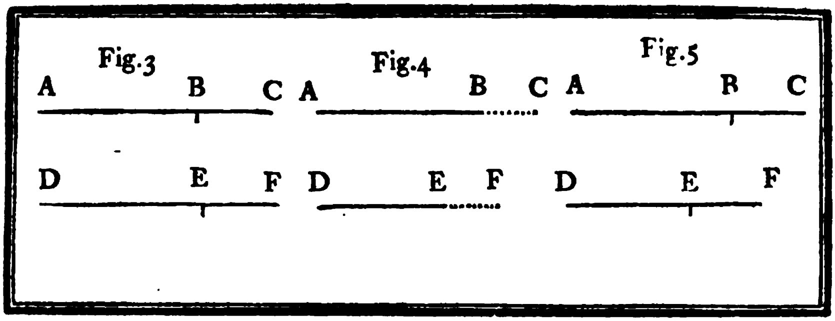
TWO magnitudes, which are equal to the ſame third, are equal to one another.

If the line A *is equal to the line* B, *and the line* C *equal to the ſame line* B, *the line* A *will be equal to the line* C. *Fig.* 1.

II.

If to equal magnitudes be added equal magnitudes, the wholes will be equal.

If to the line AD *be added the part* DE, *and to the line* BF, *which is equal to the line* AD, *be added the part* FG, *equal to the part* DE, *the wholes* AE, BG, *will be equal to one another.*



AXIOMS.

III.

IF equals be taken from equals, the remainders are equal.

If *from the whole line* AC, *be taken the part* BC, *and from the whole line* DF, *equal to* AC, *be taken the part* EF, *equal to* BC; *the remainders* AB, DE, *will be equal.* Fig. 3.

IV.

If equals be added to unequals, the wholes are unequal.

If to the line AB, *be added the part* BC, *and to the line* DE, *less than* AB, *be added the part* EF, *equal to the part* BC; *the wholes* AC, DF, *will be unequal.* Fig. 4.

V.

If equals be taken from unequals, the remainders are unequal.

If from the line AC, *be taken the part* BC, *and from the line* DF, *less than* AC, *be taken the part* EF *equal to* BC; *the remainders* AB, DE, *are unequal.* Fig. 5.

VI.

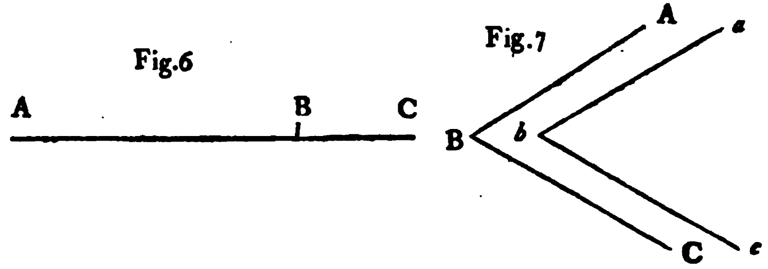
Magnitudes which are double, or *equimultiples of the same magnitude*, are equal to one another.

VII.

Magnitudes which are halves, or *equisubmultiples of the same magnitude*, are equal to one another.

Fig. 6

Fig. 7



AXIOMS.

VIII.

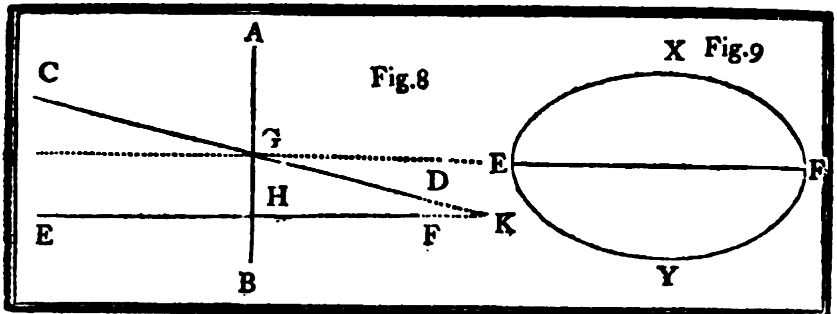
THE whole is greater than its part.

The whole line AC, *is greater than its part* BC. Fig 6.

IX

Magnitudes, which coincide with one another, are equal.

This axiom is called the principle of congruency; the notion of congruency, includes the notion of terms, *and the notion of the* possibility of their coincidence. *Two magnitudes coincide, when their terms perfectly agree; or when they may be contained within the same bounds.* Euclid *regards the principle of congruency as a common notion: he is authorised from the universal practice of determining the equality of magnitudes, by applying one to the other, as in the mensuration of magnitudes by the foot, cubit, pearch,* &c. *or by including them within the same bounds, as in the measure of liquids, of grain, and the like, by pints, gallons, pecks, bushels,* &c. *So that we judge by the eye, or hand, how one agrees with the other, and accordingly determine their equality. It would be wrong to suppose, that such a principle could only conduct to a practice purely mechanical, incompatible with geometrical precission.* Euclid *has found the means of converting this maxim, into a very scientifical principle. On congruency he lays down but a few obvious truths, from which he rigourously demonstrates the more complex ones which depend on this principle. Those obvious truths are as follow.*



AXIOMS.

1. ALL points coincide.
2. Straight lines, which are equal to one another coincide; and reciprocally, ſtraight lines whoſe extremities coincide are equal.
3. If in two equal angles (ABC, *abc*,) the vertexs (B & *b*) coincide, and one of the ſides (BA) with one of the ſides (*ba*) the other ſide (BC) will coincide alſo with the other ſide (*bc*). Likewiſe, all angles whoſe ſides coincide are equal. *Fig.* 7.

Euclid *has not ſeparately enounced, thoſe particular axioms ſubordinate to the general one; he nevertheleſs makes uſe of them, as will eaſily appear in analyzing ſeveral of his demonſtrations.*

X.

All right angles are equal to one another.

XI.

If a ſtraight line (AB) cuts two other ſtraight lines (CD, EF,) ſituated in the ſame plane, ſo as to make the two interior angles (DGH, FHG,) on the ſame ſide of it, taken together, leſs than two right angles; theſe two lines (CD, EF,) continually produced, will at length meet upon the ſide (K) on which are the angles which are leſs than two right angles. *Fig.* 8.

This truth is not ſimple enough, to be placed among the axioms; it is a conſequence of the XXVII *propoſition of the firſt book; it is only there, that it can be properly eſtabliſhed.*

XII.

Two ſtraight lines cannot incloſe a ſpace.

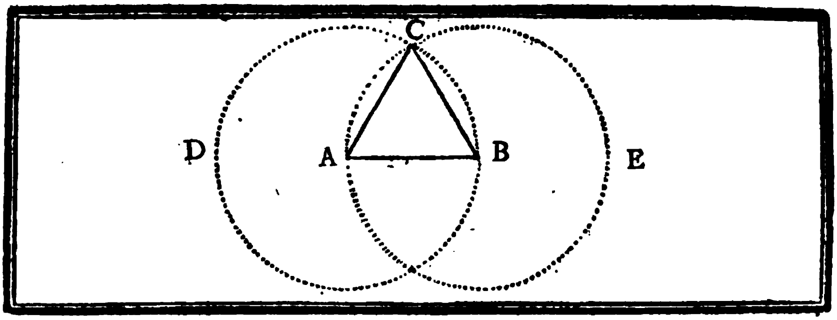
If the two ſtraight lines EF *and* EXF *incloſe a ſpace; thoſe two lines cannot be both ſtraight lines; one of them at leaſt as* EXF *muſt be a curve line.* Fig. 9.

EXPLICATION of the SIGNS.

$\perp$ - - -	Perpendicular.	$\llcorner$ - - -	Right Angle.
$<$ - - -	Greater than	$\triangle$ - - -	Triangle.
$>$ - - -	Leſs than	$=$ - - -	Equal.
$+$ - - -	More.	$\square$ - - -	Square.
$-$ - - -	Leſs.	$\odot$ - - -	Circle.
$\forall$ - - -	Angle.	$\bigcirc$ - - -	Circumference.

ABREVIATIONS.

Plle. - - - Parallel.
Pgr. - - - Parallelogram.
Rgle. - - - Rectangle.



PROPOSITION I. *PROBLEM I.*

UPON a given finite ſtraight line (AB); to conſtruct an equilateral triangle (ABC).

Given *the ſtraight line* AB.

Sought *the conſtruction of an equilateral* △ *upon the finite ſtraight line* AB.

Reſolution.

1. From the center A, at the diſtance AB, deſcribe ⊙ BCD. *Poſ.* 3.
2. From the center B, at the diſtance BA, deſcribe ⊙ ACE. *Poſ.* 3.
3. Mark the point of interſection C.
4. From the point A to the point C, draw the ſtraight line AC. *Poſ.* 1.
5. From the point B to the point C, draw the ſtraight line BC. *Poſ.* 1.

DEMONSTRATION.

BECAUSE the point A is the center of ⊙ BCD *(Ref.* 1.), and the lines AB, AC, are drawn from the center A to the ○ BCD *(Ref.* 4.).

1. Thoſe two lines AB, AC, are rays of the ſame ⊙. *D.* 16. *B.* 1.
2. Conſequently, the line AC is = to the line AB. *D.* 15. *B.* 1.

Likewiſe, becauſe the point B is the center of ⊙ ACE *(Ref.* 2.), and the lines BA, BC, are drawn from the center B to the ○ ACE *(Ref.* 5.).

3. Thoſe two lines are rays of the ſame circle ACE. *D.* 16. *B.* 1.
4. Conſequently, the line BC is alſo = to the ſame line AB. *D.* 15. *B.* 1.
5. Therefore, AC, BC, are each of them = to AB *(Arg.* 2. and 4.).

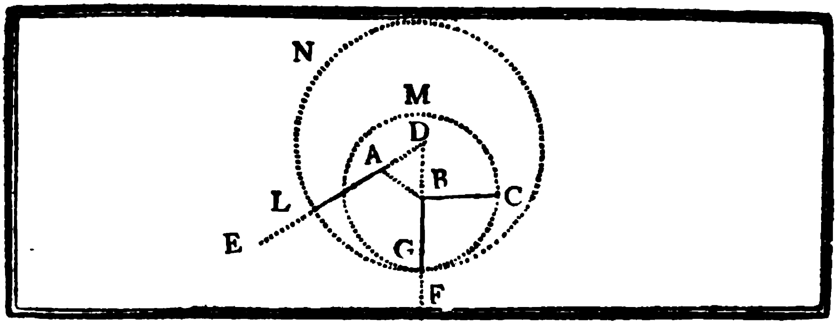
But *if two magnitudes are equal to a ſame third, they are equal to one another.* *Ax.* 1.

6. The line AC is therefore = to the line BC.

But each of thoſe two lines = to one another *(Arg.* 6.), is alſo = to the line AB *(Arg.* 5.).

7. Wherefore, the three lines AB, BC, AC, which form the three ſides of △ ABC, are = to one another.
8. Conſequently, the △ ABC conſtructed upon the given finite ſtraight line AB, is an equilateral triangle. *D.* 24. *B.* 1.

Which was required to be done.



PROPOSITION II. *PROBLEM II.*

FROM a given point (A), to draw a ſtraight line (AL), equal to a given ſtraight line (BC).

Given	Sought
1. *The point* A.	AL = BC.
2. *The ſtraight line* BC.	

Reſolution.

1. From the point A to the point B, draw the ſtraight line AB. *Poſ.* 1.
2. Upon this ſtraight line AB conſtruct the equilateral △ ADB. *P.* 1. *B.* 1.
3. Produce indefinitely the ſides DA and DB of this △. *Poſ.* 2.
4. From the center B, at the diſtance BC, deſcribe ⊙ CGM. *Poſ.* 3.
5. And from the center D, at the diſtance DG, deſcribe ⊙ GLN; which cuts the ſtraight line DA produced, ſomewhere in L. *Poſ.* 3.

DEMONSTRATION.

BECAUSE the lines BC and BG, are drawn from the center B to the ○ CGM *(Reſ.* 4.).

1. Thoſe two lines are rays of the ſame ⊙ CGM. *D.* 16. *B.* 1.
2. Conſequently, BC = BG. *D.* 15. *B.* 1.

And becauſe the lines DG and DL, are drawn from the center D to the ○ GLN *(Reſ.* 5.).

3. Thoſe lines, are alſo rays of the ſame ⊙ GLN. *D.* 16. *B.* 1.
4. Conſequently, DG = DL. *D.* 15. *B.* 1.

But the lines DA & DB, being the ſides of an equilateral △ ADB *(Reſ.* 2.).

5. The line DA, is = to the line DB. *D.* 24. *B.* 1.

Cutting off therefore from the equal lines DG, DL, *(Arg.* 4.); their equal parts DB, DA, *(Arg.* 5.).

6. The remainder AL is = to the remainder BG. *Ax.* 3.

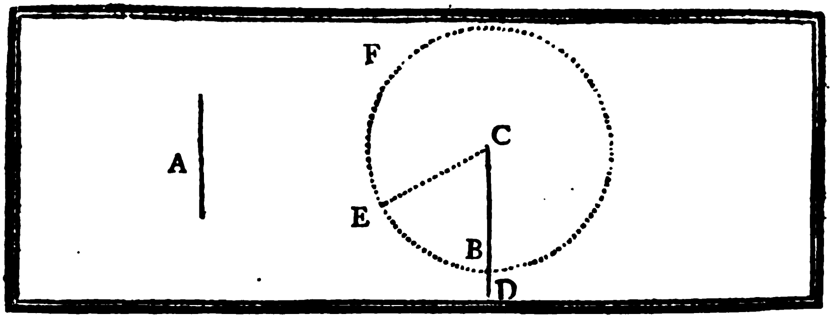
Since therefore the line AL is = to the line BG *(Arg.* 6.), and the line BC is alſo = to the ſame line BG *(Arg.* 2.).

7. The line AL is = to the line BC. *Ax.* 1.

But it is manifeſt that this line AL, is a line drawn from the given point A *(Reſ.* 3.).

8. Wherefore from the given point A, a ſtraight line AL, equal to the given ſtraight line BC, has been drawn.

Which was to be done.



PROPOSITION III. *PROBLEM III.*

TWO unequal ſtraight lines (A & CD) being given; to cut off from the greater (CD) a part (CB) equal to the leſs A.

Given	Sought
the line CD $>$ line A.	*from* CD *to cut off* CB $=$ A.

Reſolution.

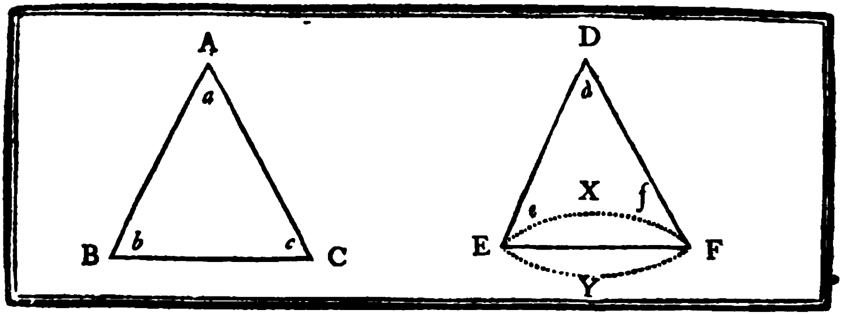
1. From the point C draw the ſtraight line CE $=$ to the given one A. *P.* 2. *B.* 1.
2. From the center C and at the diſtance CE, deſcribe ⊙ CEB; which cuts the greater CD in B. *Poſ.* 1.

DEMONSTRATION.

THE ſtraight lines CB, CE, being drawn from the center C to the ○ BEF (*Reſ.* 2.).

1. They are rays of the ſame ⊙ BEF. *D.* 16. *B.* 1.
2. Conſequently, CB $=$ CE. *D.* 15. *B.* 1.
   But the ſtraight line A being $=$ to the ſtraight line CE (*Reſ.* 1.); and the ſtraight line CB being likewiſe $=$ to CE (*Arg.* 2.).
3. The ſtraight line A is $=$ to the ſtraight line CB. *Ax.* 1.
   And ſince CB is a part of CD.
4. From CD the greater of two ſtraight lines, a part CB has been cut off $=$ to A the leſs.

Which was to be done.



PROPOSITION IV. *THEOREM I.*

IF two triangles (BAC, EDF,), have two sides of the one, equal to two sides of the other, (*i. e.* AB = DE, & AC = DF), & have likewise the angle contained *(a)* equal to the angle contained *(d)*: they will also have the base (BC), equal to the base (EF); & the two other angles *(b & c)* equal to the two other angles *(e & f)* each to each, *viz.* those to which the equal sides are opposite; and the whole triangle (BAC) will be equal to the whole triangle (EDF).

Hypothesis.	Thesis.
I. AB = DE.	*I.* BC = EF.
II. AC = DF.	*II.* $\angle b = \angle e$ & $\angle c = \angle f$.
III. $\angle a = \angle d$.	*III.* △ BAC = △ EDF.

Preparation.

Suppose the △ BAC to be laid upon the △ EDF, in such a manner that

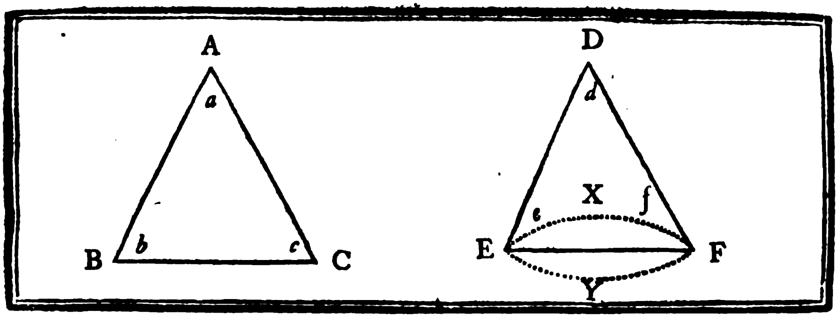
1. The point A falls upon the point D.
2. And the side AB falls upon the side DE.

DEMONSTRATION.

SINCE the line AB is = to the line DE (*Hyp.* 1.), & the point A falls upon the point D (*Prep.* 1.), & the line AB upon the line DE (*Prep.* 2.).

1. The point B will fall necessarily upon the point E. *Ax.* 9.
   Because the $\angle a = \angle d$ (*Hyp.* 3.), & the point A falls upon the point D (*Prep.* 1.), & the side AB upon the side DE (*Prep.* 2.).
2. The side AC will fall necessarily upon the side DF. *Ax.* 9.
   Moreover, since this side AC is = to the side DF.
3. The point C must fall also upon the point F. *Ax.* 9.
4. Wherefore, the extremities B and C of the base BC, coincide with the extremities E and F of the base EF.
5. And consequently, the whole base BC coincides with the whole base EF; for if the base BC did not coincide with the base EF, though the extremities B and C of the base BC, coincide with the extremities E and F of the base EF; two straight lines would inclose a space (EXF or EYF); which is impossible. *Ax.* 12.

Since therefore, the base BC coincides with the base EF (*Arg.* 5.).



6. This base BC will be = to the base EF. *Ax.* 9.

Which was to be demonstrated. I.

Again, the base BC coinciding with the base EF (*Arg.* 5.), & the two other sides AB, AC, of △ BAC coinciding with the the two other sides DE, DF, of △ EDF (*Prep.* 2, *Arg.* 2.).

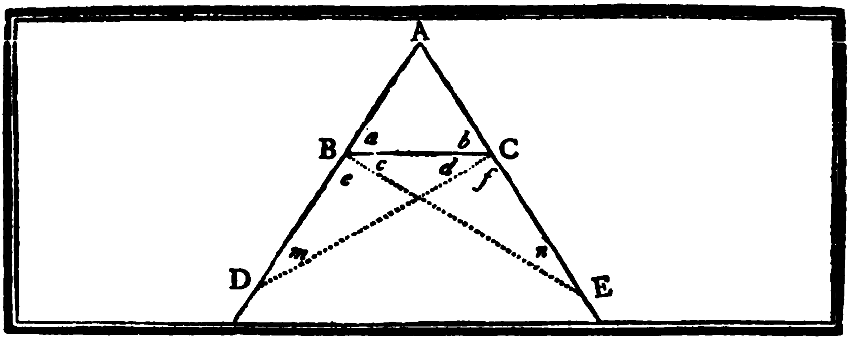
7. Those two △ BAC, EDF, are necessarily equal to one another. *Ax.* 9.

Which was to be demonstrated. III.

In fine, since the ∀*b* & ∀*e* to which the equal sides AC, DF are opposite (*Hyp.* 2.); likewise, the ∀*c* & *f* to which the equal sides AB, DE, are opposite (*Hyp.* 1.), coincide both as to their vertices and their sides (*Arg.* 1, 2, 3, 5.).

8. It follows, that the ∀*b* & ∀*e*, as also the ∀*c* & ∀*f*, to which the equal sides are opposite, are equal to one another. *Ax* 9.

Which was to be demonstrated. II.



PROPOSITION V. *THEOREM II.*

IN every isosceles triangle (BAC): the angles *(a & b)* at the base (BC) are equal, & if the equal sides (AB, AC,) be produced: the angles *(c + e & d + f)* under the base (BC) will be also equal.

Hypothesis.	Thesis.
I. The △ BAC *is an isosceles* △.	*I.* ∀*a* & ∀*b are equal.*
II. AB & AC *are produced indefinitely.*	*II.* ∀*c* + *e* & ∀*d* + *f are also equal.*

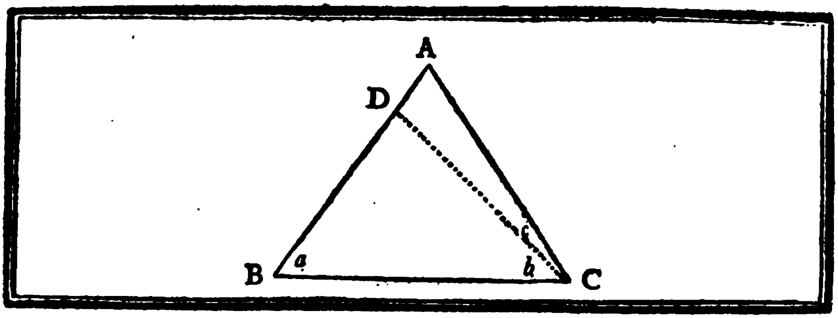
Preparation.

1. In the side AB produced take any point D.
2. Make AE = AD. *P.* 3. *B.* 1.
3. Through the points B & E, as also C & D, draw BE, CD. *Pos.* 1.

Demonstration.

BECAUSE in the △ DAC the two sides AD, AC, are equal to the two sides AE, AB of △ EAB, each to each *(Prep.* 2. *Hyp.* 1.); and the ∀ A contained by those equal sides is common to the two △.

1. The base DC is = to the base BE; & the two remaining ∀*m* & *b* + *d* of △ DAC, are equal to the two remaining ∀*n* & *a* + *c* of △ EAB, each to each of those to which the equal sides are opposite. *P.* 4. *B.* 1.
 And because the whole line AD is = to the whole line AE *(Prep.* 2.), and the part AB = to the part AC *(Hyp.* 1.); *cutting off.* &c.
2. The remainder BD will be = to the remainder CE. *Ax.* 3.
 Again, since in the △ DBC the sides DB, DC, are equal to the sides CE, EB, of △ ECB, each to each *(Arg.* 2. and 1.), & likewise ∀ contained *m* is equal to ∀ contained *n* *(Arg.* 1.).
3. The two remaining ∀ of the one, are = to the two remaining ∀ of the other, each to each, *viz.* ∀*c* + *e* = ∀*d* + *f* & ∀*d* = ∀*c*. *P.* 4. *B.* 1.
 The whole ∀*a* + *c* & *b* + *d* being therefore = to one another, as also their parts ∀*c* & ∀*d* *(Arg.* 1. & 3.); *cutting off* &c.
4. The remaining ∀*a* & *b* are likewise = to one another. *Ax.* 3.
 But those ∀ are the two ∀ at the base BC.
5. Therefore ∀*a* & ∀*b* at the base BC are = to one another.
 Which was to be demonstrated. I.
 Moreover, since ∀*e* + *c* = ∀*d* + *f* *(Arg.* 3.) are the ∀ under the base.
6. It is evident that the ∀*e* + *c* & ∀*d* + *f* under the base, are also = to one another. Which was to be demonstrated. II.



PROPOSITION VI. *THEOREM III.*

If a triangle (ACB) has two angles *(a & b + c)* equal to one another; the sides which are opposite to those equal angles, will be also equal to one another.

Hypothesis.	Thesis.
In the △ ACB, ∀*a* = ∀*b* + *c*.	*The side* CA = *to the side* BA.

Demonstration.

If not,

1. The sides CA, BA, will be necessarily unequal. *C. N.*
2. Consequently one of them, as BA, will be > the other CA. *C. N.*

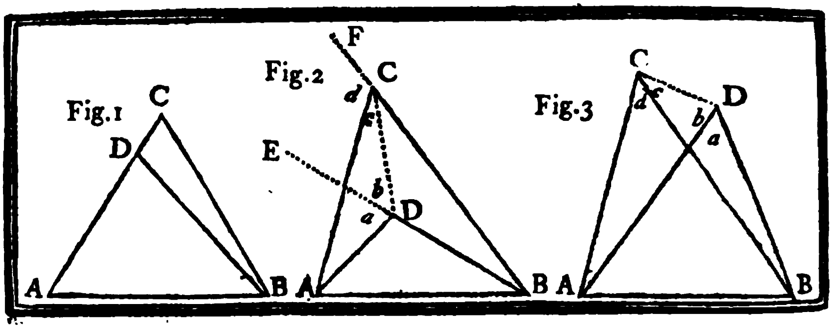
Preparation.

1. Cut off therefore from the > side BA, a part = to the < side CA. *P.* 3. *B.* 1.
2. Draw from the point C to the point D, the straight line CD. *Pos.* 1.

In the △ ACB, DBC, the side BD = to the side CA *(Prep.* 1.), the side BC is common to the two △, & ∀ contained *a* = ∀ contained *b* + *c* *(Hyp.* 1.).

1. Consequently, the two △ [illegible] DBC, have two sides of the one equal to two sides of the other, [illegible]to each, & ∀ ontained *a* = ∀ contained *b* + *c*.
2. Wherefore the △ ACB is = to △ DBC. *P.* 4. *B.* 1.
   But the △ ACB being the whole, & the △ DBC its part.
3. It follows, that the whole would be = to its part.
4. Which is impossible. *Ax.* 8.
   Therefore as the sides CA, BA, which are opposite to the equal ∀ *a* & *b* + *c*, cannot be unequal.
5. Those sides are equal to one another, or CA = BA. *C. N.*

Which was to be demonstrated.



PROPOSITION VII. *THEOREM IV.*

FROM the extremities (A & B) of a ſtraight line (AB), from which have been drawn to the ſame point (C), two ſtraight lines (AC, BC,): there cannot be drawn to any other point (D) ſituated on the ſame ſide of this line, two other ſtraight lines (AD, BD,), equal to the two firſt each to each.

Hypotheſis.

1. AC, BC, *alſo* AD, BD, *are ſtraight lines;*
2. *Drawn from the ſame points* A & B;
3. *To two different points* D & C, *ſituated on the ſame ſide of the line* AB.

Theſis.

It is impoſſible that AC = AD, & BC = BD.

DEMONSTRATION.

If not,

There is on the ſame ſide of the line AB a point D ſo ſituated, that AC = AD, & BC = BD. Conſequently this point will be placed,

CASE 1. Either in the ſide AC, or BC. *Fig.* 1.
CASE 2. Or within the △ ACB. *Fig.* 2.
CASE 3. Or laſtly without the △ ACB. *Fig.* 3.

CASE I. If the point D be ſuppoſed to be in one of the ſides, as in AC. *Fig.* 1.

BECAUSE the point D is ſuppoſed to be a point in AC different from the point C.

1. The line AD is either > or < the line AC. } *C. N.*
2. Conſequently it is impoſſible that AD = AC. }

Which was to be demonſtrated.

CASE II. If the point D be ſuppoſed to be ſituated within the △ ACB. *Fig.* 2.

Preparation.

1. From the point D to the point C, draw the ſtraight line DC. *Poſ.* 1.
2. Produce at will BD to E & BC to F. *Poſ.* 2.

BECAUSE AC is ſuppoſed = AD.

1. The △ CAD will be an iſoſceles △. *D.* 25. *B.* 1.
2. Conſequently the ∀ at the baſe $a + b$ & c will be equal to one another. *P.* 5. *B.* 1.
   And becauſe BC is ſuppoſed = BD.
3. The △ CBD will be likewiſe an iſoſceles △. *D.* 25. *B.* 1.
4. Hence the ∀ under the baſe b & $c + d$, will be alſo equal to one another. *P.* 5. *B.* 1.
   Wherefore, if from ∀$c + d$ be taken its part ∀d.
5. ∀b will be > ∀c. *C. N.*
   And if to the ſame ∀b be afterwards added ∀a.
6. Much more then will the whole ∀$a + b$ be > ∀c. *C. N.*
7. Conſequently ∀$a + b$ & ∀c are not equal. *C. N.*
   But it has been demonſtrated that in conſequence of the ſuppoſition of this caſe, ∀$a + b$ & ∀c ſhould be equal *(Arg.* 2.).
8. From whence it follows that this ſuppoſition cannot ſubſiſt, unleſs thoſe angles at the ſame time be equal and unequal.
9. Which is impoſſible. *C. N.*
10. Therefore the ſuppoſition which makes AC = AD & BC = BD, is in itſelf impoſſible.

Which was to be demonſtrated.

CASE III. If the point D be ſuppoſed to be without the △ ACB. *Fig.* 3.

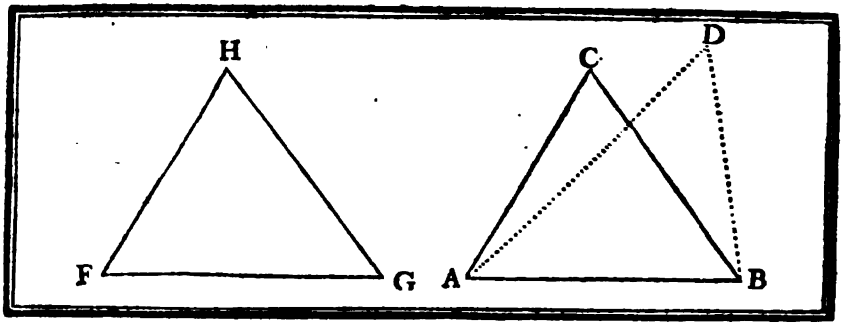
Preparation.

From the point D to the point C let there be drawn the ſtraight line DC. *Poſ.* 1.

BECAUSE AC is ſuppoſed = AD.

1. The △ CAD will be an iſoſceles △. *D.* 25. *B.* 1.
2. Conſequently ∀b & $d + c$ at the baſe are equal to one another. *P.* 5. *B.* 1.
   Again, becauſe BC is likewiſe ſuppoſed = BD;
3. The △ CBD will be an iſoſceles △. *D.* 25. *B.* 1.
4. Hence ∀c & ∀$b + a$ at the baſe will be equal to one another. *P.* 5. *B.* 1.
   If therefore we take from ∀$b + a$ its part ∀a.
5. The ∀c will be > the remaining ∀b. *C. N.*
   And if to this ſame ∀c be added ∀d.
6. Much more then will the whole ∀$c + d$ be > ∀b. *C. N.*
7. Wherefore ∀$c + d$ & ∀b are not equal to one another. *C. N.*
   But it has been proved that in conſequence of the ſuppoſition of this caſe, ∀$d + c$ & ∀b are equal to one another. *(Arg. 2.).*
8. From whence it follows that this ſuppoſition cannot ſubſiſt, unleſs thoſe angles be at the ſame time equal and unequal.
9. Which is impoſſible. *C. N.*
10. Therefore, the ſuppoſition which makes AC = AD & BC = BD is impoſſible.

Which was to be demonſtrated.



PROPOSITION VIII. *THEOREM V.*

IF two triangles (FHG, ACB,), have the three ſides (FH, HG, GF,) of the one equal to the three ſides (AC, CB, BA,) of the other, each to each, they are equal to one another, & the angles contained by the equal ſides are likewiſe equal, each to each.

Hypotheſis.
I. FH = AC.
II. HG = CB.
III. GF = BA.

Theſis.
△ FHG = △ ACB, *and* { ∠F = ∠A. ∠G = ∠B. ∠H = ∠C.

Preparation.

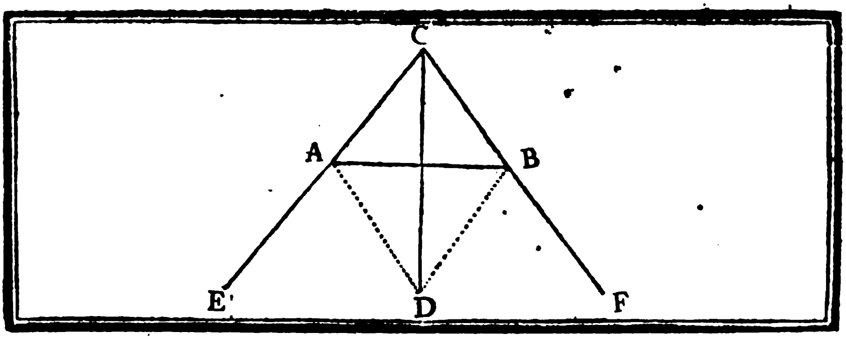
Let the △ FHG be applied to the △ ACB, ſo that,
1. The point F may coincide with the point A.
2. And the baſe FG with the baſe AB.

Demonstration.

BECAUSE the point F coincides with the point A (*Prep.* 1.), & the line FG with the line AB (*Prep.* 2.), & thoſe lines are equal (*Hyp.* 3.).

1. The point G muſt coincide with the point B. *Ax.* 9.
The extreme points F & G of the ſide FG, coinciding therefore with the extreme points A & B of the ſide AB (*Prep.* 1. *Arg.* 1.); & the ſtraight lines FH, GH, being equal to the ſtraight line AC, BC, each to each.
2. The ſtraight lines FH, GH, will neceſſarily coincide with the ſtraight lines AC, BC, each with each.
If not; then from the extremities A & B of a line AB, there may be drawn to two different points C & D, on the ſame ſide of AB, two ſtraight lines AC, BC, equal to two other ſtraight lines AD, BD, each to each. Which is impoſſible. *P.* 7. *B.* 1.
3. Thoſe ſides therefore coincide.
4. But the baſe FG coinciding with the baſe AB (*Prep.* 2.), the ſide FH with the ſide AC, & the ſide GH with the ſide BC, (*Arg.* 2.).
5. It follows, that the △ ACB, FGH, are equal to one another; as likewiſe their ∠ contained by the equal ſides, each to each. *Ax.* 9.

Which was to be demonſtrated.



PROPOSITION IX. *PROBLEM. IV.*

TO divide a given rectilineal angle (ECF), into two equal angles (ECD, FCD,).

Given	Sought
A rectilineal ∀ ECF.	∀ ECD = ∀ FCD.

Resolution.

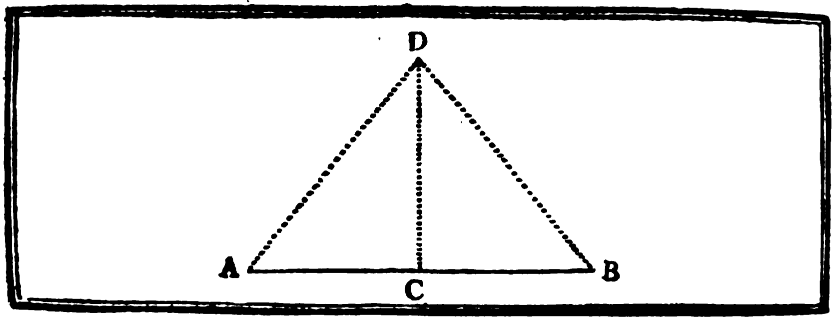
1. Take CA of any length.
2. Make CB = CA. *P.* 3. *B.* 1.
3. From the point A to the point B, draw the straight line AB. *Pos.* 1.
4. Upon the straight line AB, construct the equilateral △ ADB. *P.* 1. *B.* 1.
5. From the point C to the point D, draw the straight line CD. *Pos.* 1.

DEMONSTRATION.

BECAUSE AC = BC (*Ref.* 2.), DA = DB (*Ref.* 4.), and the side DC common to the two △ CAD, CBD.

1. Those two △ have the three sides of the one equal to the three sides of the other, each to each.
2. Consequently the ∀ FCD, ECD, contained by the equal sides CA, CD; & CB, CD, are equal to one another. *P.* 8. *B.* 1.

Which was to be done.



PROPOSITION X. *PROBLEM V.*

TO divide a given finite ſtraight line (AB) into two equal parts (AC, BC,).

Given	Sought
A finite ſtraight line AB.	AC = BC.

Reſolution.

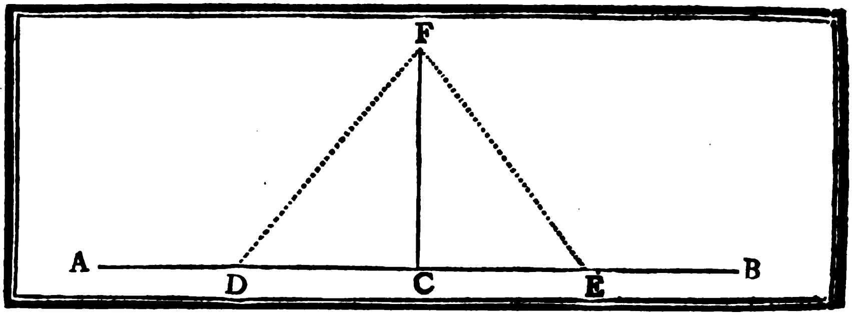
1. Upon the ſtraight line AB conſtruct the equilateral △ ADB. *P.* 1. *B.* 1.
2. Divide into two equal parts ∀ ADB by the ſtraight line DC. *P.* 9. *B.* 1.

DEMONSTRATION.

BECAUSE AD = BD *(Ref.* 1.), & the ſide DC is common to the two △ ADC, BDC, & ∀ contained ADC = ∀ contained BDC *(Ref.* 2.).

1. Thoſe two △ ADC, BDC, have two ſides in the one equal to two ſides in the other, each to each, & ∀ contained ADC = ∀ contained BDC *(Ref.* 2.).
2. Conſequently, the baſe AC = to the baſe BC. *P.* 4. *B.* 1.

Which was to be done.



PROPOSITION XI. *PROBLEM VI.*

FROM a given point (C), in an indefinite ſtraight line (AB), to raiſe a perpendicular (CF) to this line.

Given	Sought
The indefinite ſtraight line AB, & *the point* C *in this ſtraight line.*	*The ſtraight line* CF *raiſed from the point* C ⊥ *upon* AB.

Reſolution.

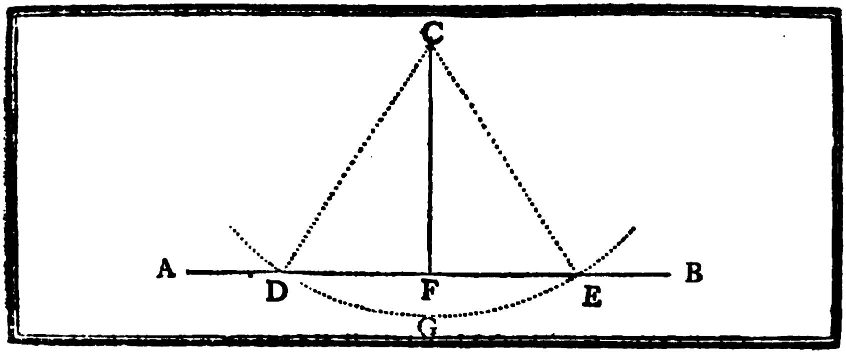
1. On both ſides of the point C take CD, CE, equal to one another. *P.* 3. *B.* 1.
2. Upon the ſtraight line DE, conſtruct the equilateral △ DFE. *P.* 1. *B.* 1.
3. From the point F to the point C, draw the ſtraight line FC. *Poſ.* 1.

DEMONSTRATION.

BECAUSE CD is = to CE (*Reſ.* 1.), FD = FE (*Reſ.* 2.), & the ſide CF is common to the two △ DFC, EFC.

1. It is evident that thoſe two △ have the three ſides of the one, equal to the three ſides of the other, each to each.
2. Conſequently, the adjacent ∀ FCD, FCE, (contained by the equal ſides FC, CD, and FC, CE,) are equal to one another. *P.* 8. *B.* 1.
   But it is the ſtraight line FC, which falling upon AB, forms thoſe adjacent ∀ = to one another.
3. Wherefore, the ſtraight line FC is ⊥ upon AB. *D.* 10. *B.* 1.

Which was to be done.



PROPOSITION XII. *PROBLEM VII.*

FROM a given point (C), without a given indefinite ſtraight line (AB); to let fall a perpendicular (CF) to this line.

Given	Sought
The indefinite ſtraight line AB, & *the point* C *without this line.*	*The ſtraight line* CF, *let fall from the point* C ⊥ *upon* AB.

Reſolution.

1. Take any point G, upon the other ſide of the ſtraight line AB, with reſpect to the point C.
2. From the center C, at the diſtance CG, deſcribe an arc of ⊙ DGE cutting the indefinite line AB in two points D & E. *Poſ.* 3.
3. Divide the line DE into two equal parts in the point F. *P.* 10. *B.* 1.
4. From the point C to the point F, draw the ſtraight line CF. *Poſ.* 1.

Preparation.

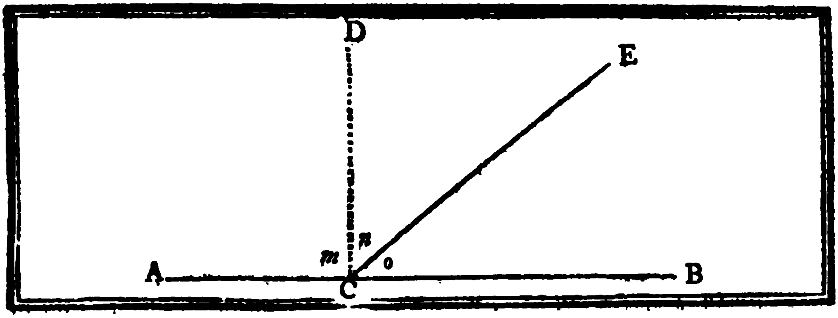
From the point C to the points D & E, draw the ſtraight lines CD & CE. *Poſ.* 1.

DEMONSTRATION.

BECAUSE the lines CD, CE, are drawn from the center C to the ○ DGE (*Reſ.* 2. and *Prep.*).

1. Thoſe lines are rays of the ſame ⊙. *D.* 16. *B.* 1.
2. Conſequently, the line CD is = to the line CE. *D.* 15. *B.* 1.
   Since therefore CD is = to CE (*Arg.* 2.), DF = FE (*Reſ.* 3.), & the ſide CF is common to the two △ DCF, ECF.
3. Thoſe two △ have the three ſides of the one equal to the three ſides of the other, each to each.
4. Wherefore the ∀ CFD, CFE, contained by the equal ſides FC, FD, and FC, FE, are = to one another. *P.* 8. *B.* 1.
   But thoſe two ∀ CFD, CFE, = to one another (*Arg.* 4.), are the adjacent angles formed by the line CF which falls upon the line AB.
5. Therefore, each of thoſe two ∀ CFD, CFE, is a ∟, and the line CF is ⊥ upon the line AB. *D.* 10. *B.* 1.

Which was to be demonſtrated.



PROPOSITION XIII. *THEOREM VI.*

THE angles which one ſtraight line EC makes with another AB upon one ſide of it, are either two right angles, or are together equal to two right angles.

Hypotheſis.	Theſis.
EC *is a ſtraight line meeting* AB *in the point* C.	*I. Either each of* ∀ ACE, ECB, *is a* ∟. *II. Or their ſum is* = *to two* ∟.

SUP. I. If ∀ ACE is = to ∀ ECB.

DEMONSTRATION.

BECAUSE the adjacent angles ACE, ECB, formed by the ſtraight lines CE & AB, are equal to one another *(Sup.)*.

1. It follows, that each of them is a ∟. *D. 10. B. 1.*

Which was to be demonſtrated.

SUP. II. If ∀ ACE is not = to ∀ ECB.

Preparation.

From the point of concurſe C, raiſe upon AB the ⊥ CD. *P. 11. B. 1.*

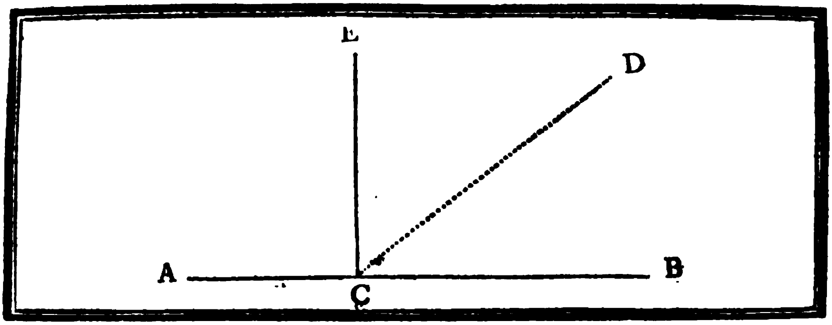
DEMONSTRATION.

BECAUSE DC is ⊥ upon AB *(Prep.)*.

1. The two ∀ DCA & DCB are ∟. *D. 10. B. 1.*
   But as ∀ DCB is = to the two ∀ *n* + *o*; if the ∀ DCA or ∀ *m*, be added to each.
2. The two ∀ DCA + DCB, are = to the three ∀ *m* + *n* + *o*. *Ax. 2.*
   Again, becauſe ∀ ECA is = to the two ∀ *m* + *n*; if the ∀ ECB or ∀ *o* be added to each.
3. The two ∀ ECA, ECB, are alſo = to thoſe ſame three ∀ *m* + *n* + *o*. *Ax. 2.*
4. Conſequently, the two ∀ ECA & ECB are = to the two ∀ DCA & DCB. *Ax. 1.*
   But the two ∀ DCA & DCB, being two ∟ *(Arg. 1.)*.
5. It is evident that the ſum of the two ∀ ECA & ECB, is alſo = to two ∟. *Ax. 1.*

Which was to be demonſtrated.

D 2



PROPOSITION XIV. *THEOREM VII.*

IF two ſtraight lines (AC, BC,), meet at the oppoſite ſides of a ſtraight line (EC), in a point C, making with this ſtraight line (EC) the ſum of the two adjacent angles (ACE, ECB,) equal to two right angles; thoſe two ſtraight lines (AC, BC, will be in one and the ſame ſtraight line.

Hypotheſis.	Theſis.
I. The two lines AC, BC, *meet in the point* C.	*The lines* AC, BC, *are in one & the ſame ſtraight line* AB.
II. The adjacent ∀ ACE + ECB *are* = *to two* ∟.	

DEMONSTRATION.

IF not,

AC may be produced from C to D, ſo that DC & AC may make but one and the ſame ſtraight line ACD. *Poſ.* 2.

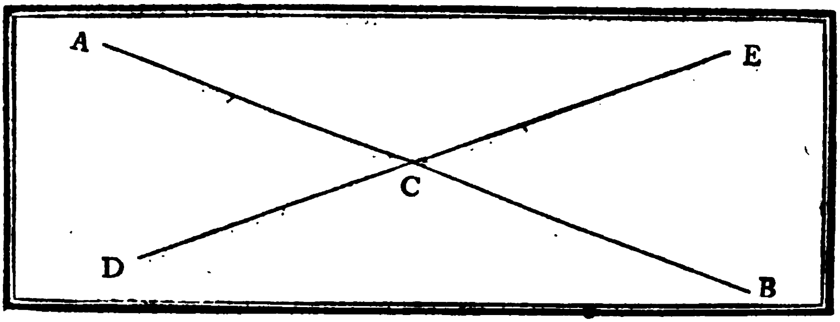
Preparation.

Produce then AC from C to D. *Poſ.* 2.

BECAUSE ACD is a ſtraight line upon which falls the line EC.

1. It follows, that the ſum of the adjacent ∀ ACE + ECD is = to two ∟. *P.* 13. *B.* 1.
   But the ∀ ACE + ECB being alſo = to two ∟ *(Hyp.* 2.).
2. The two ∀ ACE + ECB are therefore = to the two ∀ ACE + ECD. *Ax.* 1.
   Taking away therefore from each the common ∀ ACE.
3. The remaining ∀ ECB, ECD, will be equal to one another. *Ax.* 3.
   But ∀ ECB being the whole & ∀ ECD its part.
4. It follows, that the whole is equal to its part. *Ax.* 1.
5. Which is impoſſible. *Ax.* 8.
6. Conſequently, the lines AC & BC, are in one & the ſame ſtraight line.

Which was to be demonſtrated.



PROPOSITION XV. THEOREM VIII.

IF two straight lines (AB, DE,) cut one another in (C), the vertical or opposite angles (ECA, DCB, & ACD, BCE,) are equal.

Hypothesis.	Thesis.
AB, DE, *are straight lines which cut one another in the point* C.	*I.* ∀ ECA = ∀ DCB. *II.* ∀ ACD = ∀ BCE.

Demonstration.

BECAUSE the straight line AC falls upon the straight line DE (*Hyp.*).

1. The sum of the two adjacent ∀ ECA + ACD is = to two ∟. *P.* 13. *B.* 1.

Again, since the straight line DC falls upon the straight line AB (*Hyp.*).

2. The sum of the adjacent ∀ ACD + DCB is also = to two ∟. *P.* 13. *B.* 1.
3. Consequently, the ∀ ECA + ACD are = to ∀ ACD + DCB. *Ax.* 1.

Taking away therefore from those equal sums (*Arg.* 3.) the common ∀ ACD.

4. The remaining ∀ ECA, DCB, which are vertically opposite, are equal. *Ax.* 3.

Which was to be demonstrated. I.

In the same manner it will be proved:

5. That ∀ ACD is = to ∀ BCE, which is vertically opposite to it.

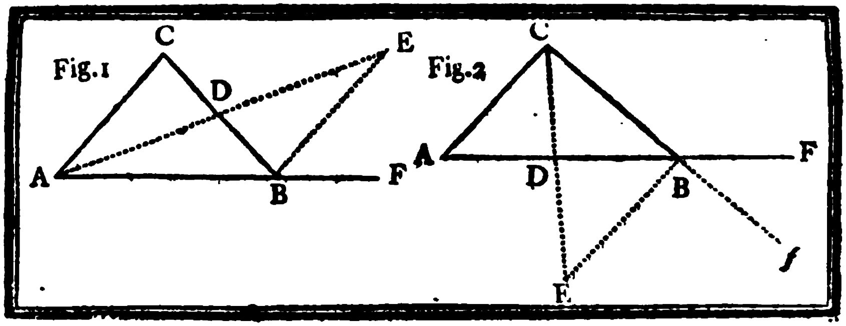
Which was to be demonstrated. II.

COROLLARY I.

FROM this it is manifest, that if two straight lines cut one another, the angles they make at the point where they cut, are together equal to four right angles.

COROLLARY II.

AND consequently, that all the angles made by any number of lines meeting in one point, are together equal to four right angles.



PROPOSITION XVI. *THEOREM IX.*

IF one ſide as (AB) of a triangle (ACB) be produced, the exterior angle (CBF) is greater than either of the interior oppoſite angles (ACB, CAB,).

Hypotheſis.

I. ACB *is a* △.
II. ∀ CBF *is an exterior* ∀ *& formed by the ſide* AB *produced.*
III. ∀ ACB *&* CAB *are the interior oppoſite ones.*

Theſis.

The exterior ∀ CBF > *the interior oppoſite* ∀ ACB *or* CAB.

Preparation.

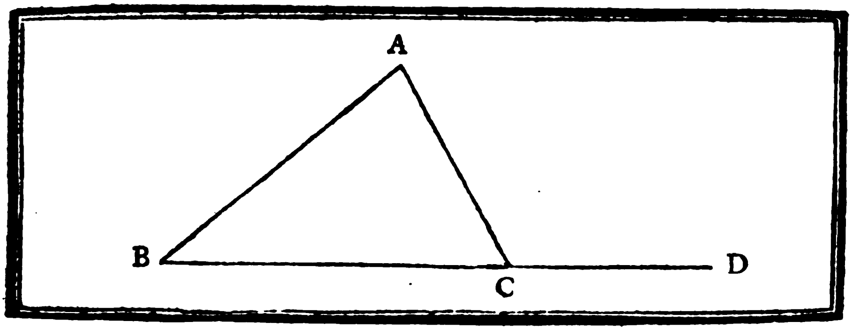
1. Divide CB into two equal parts at the point D. (*Fig.* 1.) *P.* 10. *B.* 1.
2. From the point A to the point D, draw the line AD, & produce it indefinitely to E. *Poſ.* 1.
3. Make DE = DA. *P.* 3. *B.* 1.
4. From the point B to the point E, draw the ſtraight line BE. *Poſ.* 1.

DEMONSTRATION.

THE ſtraight lines AE, BC, (*Fig.* 1.) interſect each other at the point D. (*Prep.* 2.).

1. Conſequently, the oppoſite vertical ∀ CDA, BDE, are = to one another. *P.* 15. *B.* 1.
Wherefore ſince in the △ ACD, DEB, the ſide CD is = to the ſide DB (*Prep.* 1.), AD = DE (*Prep.* 3.), & ∀ contained CDA is = to ∀ contained BDE (*Arg.* 1.).
2. It follows, that the remaining ∀ of the one are equal to the remaining ∀ of the other, each to each of thoſe to which the equal ſides are oppoſite. *P.* 4. *B.* 1.
But the ∀ ACD, DBE, are oppoſite to the equal ſides AD, DE, (*Prep.* 3.).
3. Therefore ∀ ACD is = to ∀ DBE.
But ∀ CBF being the whole, & ∀ DBE its part.
4. It follows, that ∀ CBF > ∀ DBE. *Ax.* 8.
5. Wherefore the exterior ∀ CBF is alſo > the interior ∀ ACB. *C. N.*
In the ſame manner, dividing the ſide AB into two equal parts in the point D (*Fig.* 2.) it will be proved.
6. That the exterior ∀ AB*f* is > the interior ∀ CAB.
But this ∀ AB*f* is vertically oppoſite to ∀ CBF.
7. Wherefore ∀ AB*f* = ∀ CBF. *P.* 15. *B.* 1.
8. Conſequently, the exterior ∀ CBF is > the interior ∀ CAB. *C. N.*

Which was to be demonſtrated.



PROPOSITION XVII. *THEOREM X.*

ANY two angles as (ABC, ACB,) of a triangle (BAC), are less than two right angles.

Hypothesis.	Thesis.
ABC *is a* △.	*The* ∠ ABC + ACB *are* < *two* ∟.

Preparation.

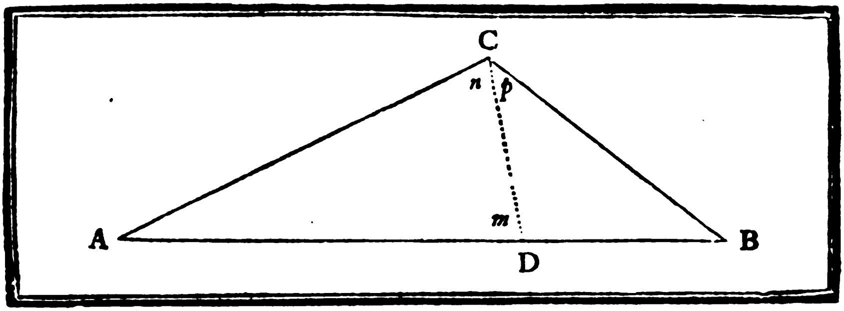
Produce the side BC (upon which the two ∠ ABC, ACB, are placed) to D. *Pos.* 2.

DEMONSTRATION.

BECAUSE ∠ ACD is an exterior ∠ of the △ BAC.

1. It is > its interior opposite one ABC.
   Since therefore ∠ ACD is > ∠ ABC; if the ∠ ACB be added to each. *P.* 16. *B.* 1.
2. The ∠ ACD + ACB will be > the ∠ ABC + ACB. *Ax.* 4.
   But the ∠ ACD + ACB are the adjacent ∠, formed by the straight line AC, which falls upon BD *(Prep.)*.
3. Consequently, those ∠ ACD + ACB are = to two ∟. *P.* 13. *B.* 1.
   But the ∠ ACD + ACB being = to two ∟ *(Arg.* 3.) & those same ∠ being > the ∠ ABC + ACB *(Arg.* 2.).
4. It follows, that the ∠ ABC + ACB are < two ∟. *C. N.*

Which was to be demonstrated.



PROPOSITION XVIII. *THEOREM. XI.*

IN every triangle (ACB); the greater side is oppoſite to the greater angle.

Hypotheſis.	Theſis.
ACB *is a* △, *whoſe ſide* AB *is* > AC.	∠ ACB, *oppoſite to* > *ſide* AB, *is greater than* ∠ ABC *oppoſite to the leſſer ſide* AC.

Preparation.

Becauſe the ſide AB is > AC *(Hyp.)*.

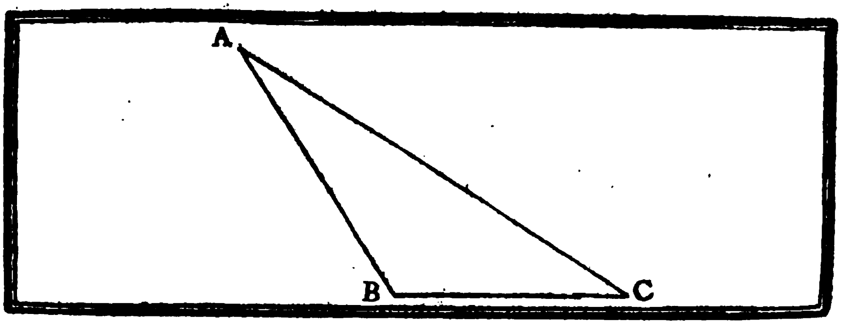
1. Make AD = AC. *P.* 3. *B.* 1.
2. From the point C to the point D, draw the ſtraight line CD. *Poſ.* 1.

DEMONSTRATION.

BECAUSE the ſide AD is = to the ſide AC *(Prep.* 1.*)*.

1. The △ ACD is an iſoſceles △. *D.* 25. *B.* 1.
2. Conſequently, the ∠ *m* & *n* at the baſe CD are = to one another. *P.* 5. *B.* 1.
   But ∠ *m* being an exterior ∠ of △ DCB.
3. It follows, that it is > the interior oppoſite ∠ DBC. *P.* 16. *B.* 1.
   But ∠ *m* is = to ∠ *n* *(Arg.* 2.)
4. Therefore ∠ *n* is alſo > ∠ DBC. *C. N.*
   And if to ∠ *n* be added ∠ *p*.
5. Much more will ∠ *n* + *p* or ∠ ACB, oppoſite to the greater ſide AB, be > ∠ DBC, or ABC, oppoſite to the leſſer ſide AC. *C. N.*

Which was to be demonſtrated.



PROPOSITION XIX. *THEOREM XII.*

IN every triangle (BAC), the greater angle, has the greater side opposite to it.

Hypothesis.	Thesis.
In the △ BAC, ∀ C *is* > ∀ A.	*The side* AB *opposite to* ∀ C *is* > *the side* CB *opposite to* ∀ A.

DEMONSTRATION.

IF not,

The side AB is either equal, or less than the side CB. C. *N.*

CASE I. Suppose AB to be = to CB.

BECAUSE the side AB is = to the side CB *(Sup. 1.)*.

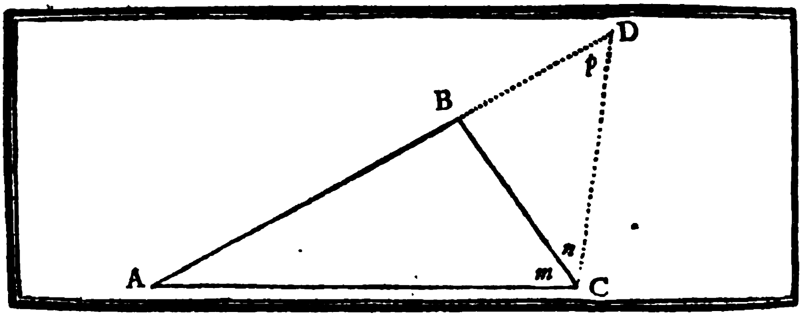
1. The △ BAC is an isosceles △. *D. 25. B. 1.*
2. Consequently, the ∀ C & A at the base, are = to one another. *P. 5. B. 1.*
   But those ∀ C & A are not = to one another *(Hyp.)*.
3. Therefore neither are the sides AB, CB = to one another.

CASE II. Suppose AB to be < CB.

BECAUSE the side AB is < the side CB *(Sup. 2.)*.

1. It follows, that ∀ C opposite to the lesser side AB, is < ∀ A opposite to the greater side CB. *P. 18. B. 1.*
   But ∀ C is not < ∀ A *(Hyp.)*.
2. Consequently, the side AB cannot be < the side CB.
   The side AB being therefore neither = to the side CB *(Case 1.)*; nor < the side CB *(Case 2.)*.
3. It follows, that this side AB is > the side CB. *C. N.*

Which was to be demonstrated.



PROPOSITION XX. THEOREM XIII.

ANY two ſides (AB, BC,) of a triangle (ABC) are together greater than the third ſide (AC).

Hypotheſis.	Theſis.
ABC *is a* △.	*Any two ſides, as* AB + BC, *are* > *the third* AC.

Preparation.

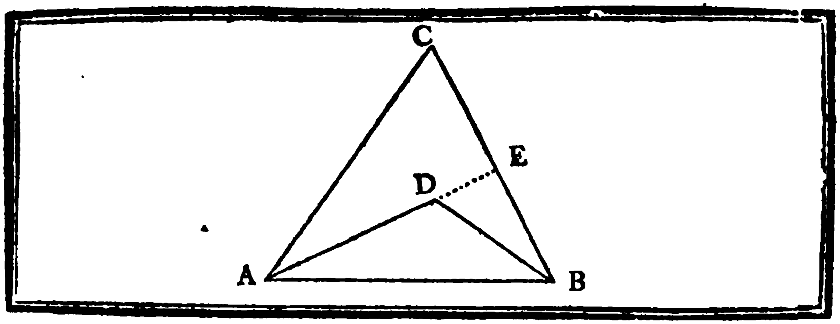
1. Produce one of the two ſides, as AB, towards D indefinitely. *Poſ.* 2.
2. Make BD = to BC. *P.* 3. *B.* 1.
3. From the point C to the point D, draw the ſtraight line CD. *Poſ.* 1.

DEMONSTRATION.

BECAUSE in the △ BDC the ſide BD is = to the ſide BC (*Prep.* 2.).

1. This △ is an iſoſceles △. *D.* 25. *B.* 1.
2. Conſequently, the ∀ at the baſe *n* & *p* are = to one another. *P.* 5. *B.* 1.
But ∀ *m* + *n* being the whole, & ∀ *n* its part.
3. It follows, that ∀ *m* + *n* is > ∀ *n*. *Ax.* 8.
But ∀ *m* + *n* being > ∀ *n* (*Arg.* 3.), & this ∀ *n* being = to ∀ *p*, (*Arg.* 2.).
4. It is evident that ∀ *m* + *n* is > ∀ *p*. *C. N.*
Since therefore in the △ ADC, ∀ *m* + *n* is > ∀ *p* (*Arg.* 4.).
5. The ſide AD oppoſite to the greater ∀ *m* + *n* is alſo > the ſide AC oppoſite to the leſſer ∀ *p*. *P.* 19. *B.* 1.
But becauſe the ſtraight line BD is = to the ſtraight line BC (*Prep.* 2.), if the ſide AB be added to both.
6. It follows, that AB + BD or AD is = to the ſum of the two ſides AB + BC. *Ax.* 2.
But AD is > the ſide AC (*Arg.* 5.).
7. Wherefore, the ſum of the two ſides AB + BC is alſo > the third ſide AC. *C. N.*

Which was to be demonſtrated.



PROPOSITION XXI. *THEOREM XIV.*

IF from the ends (A & B) of the side (AB) of any triangle (ACB) there be drawn to a point (D) within the triangle, two straight lines (DA, DB,); these straight lines will be less than the other two sides (CA, CB,) of the triangle; but will contain a greater angle (ADB).

Hypothesis.	Thesis.
DA, DB, *are two straight lines drawn from the points* A & B *to the point* D, *within the* △ ACB.	*I.* DA + DB < CA + CB. *II.* ∀ ADB > ∀ C.

Preparation.

Produce the straight line DA, until it meets the side CB in E. *Pos.* 2.

DEMONSTRATION.

BECAUSE the figure ACE is a △ (*D.* 21. *B.* 1.).

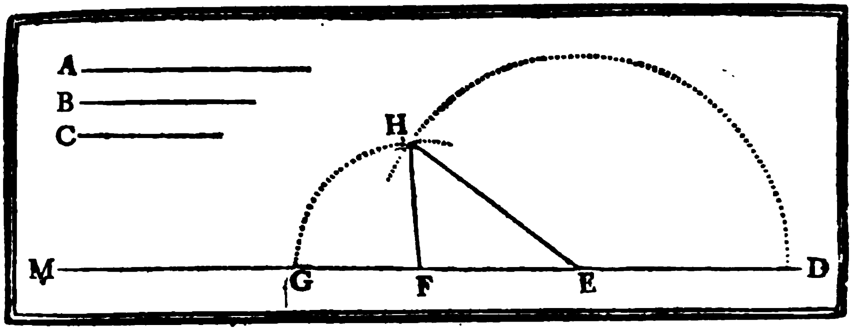
1. The two sides CA + CE are > the third AE. *P.* 20. *B.* 1.
If the line EB be added to each of these.
2. The sides CA + CB (that is CA + CE + EB) are > the lines AE + EB. *Ax.* 4.
Again, the figure DEB being also a △ (*D.* 21. *B.* 1.).
3. The two sides EB + ED are > the third DB. *P.* 20. *B.* 1.
If we add to each of these the line DA.
4. The lines AE + EB (that is DA + ED + EB) are > the lines DA + DB. *Ax.* 4.
But it has been proved that the sides CA + CB are > the lines AE + EB (*Arg.* 2.).
5. Much more then will the sides CA + CB be > the lines DA + DB. *C. N.*

Which was to be demonstrated. I.

AGAIN, because ∀ ADB is an exterior ∀ of △ DEB (*Prep.*), & the ∀ DEB is its interior opposite one.

1. It follows, that ∀ ADB is > ∀ DEB. *P.* 16. *B.* 1.
2. For the same reason; ∀ DEB is > ∀ C.
But since ∀ ADB > ∀ DEB (*Arg.* 1.), & ∀ DEB > ∀ C (*Arg.* 2.).
3. It is evident, that ∀ ADB is much > ∀ C. *C. N.*

Which was to be demonstrated. II.



PROPOSITION XXII. *PROBLEM VIII.*

TO make a triangle (FHE) of which the ſides ſhall be equal to three given ſtraight lines (A, B, C,); ſuppoſing any two whatever of theſe given ſtraight lines to be greater than the third.

Given	Sought
The ſtraight lines A, B, C, *ſuch that* $A+B>C$, $A+C>B$, $C+B>A$.	*The conſtruction of a* $\triangle$ FHE *ſuch, that* EH *may be* $=$ A, FE $=$ B, *&* FH $=$ C.

Reſolution.

1. Draw the indefinite ſtraight line DM. *Poſ.* 1.
2. Make ED $=$ to the given A, FE $=$ to the given B, & FG $=$ to the given C. *P.* 3. *B.* 1.
3. From the center E at the diſtance ED, deſcribe the $\odot$ DH. } *Poſ.* 3.
4. From the center F at the diſtance FG, deſcribe the $\odot$ GH. }
5. From the points E & F, to the point of interſection H, draw the ſtraight lines EH, FH. *Poſ.* 2.

DEMONSTRATION.

THE ſtraight lines ED, EH, being drawn from the center E to the $\bigcirc$ DH (*Reſ.* 3 & 5.).

1. Thoſe two ſtraight lines ED, EH, are rays of the ſame $\odot$ DH. *D.* 16. *B.* 1.
2. Conſequently, the ſtraight line ED is $=$ to the ſtraight line EH. *D.* 15. *B.* 1.

 Since therefore ED is $=$ to EH (*Arg.* 2.), & the given ſtraight line A is alſo $=$ to the ſame line ED (*Reſ.* 2.).

3. It follows, that EH is $=$ to the given A. *Ax.* 1.

 After the ſame manner it will be proved, that

4. The line FH is $=$ to the given C.

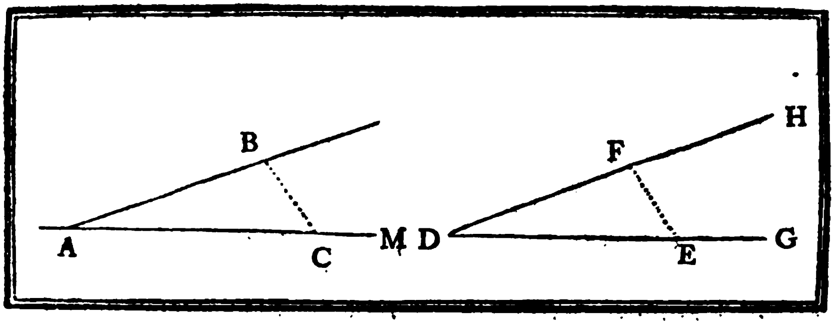
 But the ſide EH being $=$ to the given A (*Arg.* 3.), the ſide FH $=$ to the given C (*Arg.* 4.), & in fine the ſide FE $=$ to the given B. (*Reſ.* 2.).

5. It is evident, that the three ſides EH, FE, FH, of $\triangle$ FHE, are $=$ to the three given ſtraight lines A, B, C.

Which was to be done.

REMARK.

THE condition added, that any two of the given lines ſhould be greater than the third, is eſſential, in conſequence of the XX prop. of the I. Book; *without this reſtriction the circles deſcribed from the centers* E & F *would not cut one another; a defect which would render the conſtruction impoſſible.*



PROPOSITION XXIII. *PROBLEM IX.*

AT a given point (A) in a given ſtraight line (AM) to make a rectilineal angle (BAC) equal to another given rectilineal angle (HDG).

Given	Sought
I. *An indefinite ſtraight line* AM.	*An angle* BAC *made on* AM,
II. *The point* A *in the ſtraight line* AM.	*at the point* A = *to* ∀ HDG.
III. *The rectilineal angle* HDG.	

Reſolution.

1. In the ſides DG, DH, of the given ∀ HDG, take any two points E & F.
2. From the point E to the point F, draw the ſtraight line EF. *Poſ.* 1.
3. Upon the indefinite ſtraight line AM & at the point A, conſtruct a △ ABC whoſe three ſides ſhall be = to the three ſides of △ DFE. *P.* 22. *B.* 1.

DEMONSTRATION.

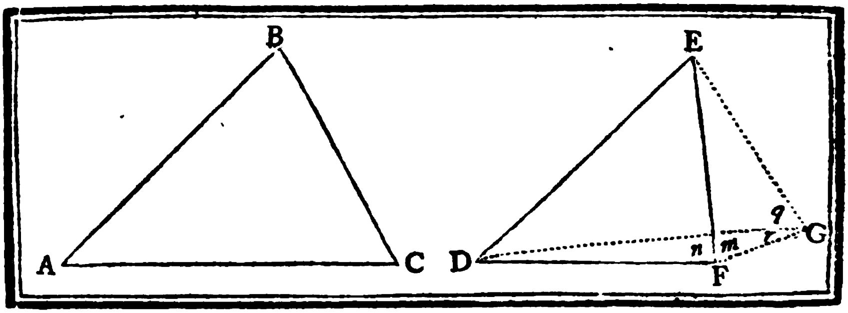
BECAUSE the three ſides AB, AC, BC, of △ ABC are = to the three ſides DF, DE, FE, of △ DFE, each to each (*Ref.* 3.).

1. It follows, that the ∀ BAC & HDG, oppoſite to the equal ſides BC, FE, are = to one another. *P.* 8. *B.* 1.

But ∀ BAC being = to the given ∀ HDG; as alſo made on the the given ſtraight line AM, at the given point A (*Ref.* 3.).

2. It follows, that at the given point A, in the given ſtraight line AM, the rectilineal ∀ BAC is made = to the given rectilineal ∀ HDG.

Which was to be done.



PROPOSITION XXIV *THEOREM. XV.*

IF two triangles (ABC, DEF,) have two sides (BA, BC,) of the one equal to two sides (ED, EF,) of the other, each to each; but the angle contained (B) greater than the angle contained (DEF); the base (AC) opposite to the greater angle, will be also greater than the base (DF) opposite to the lesser angle.

Hypothesis.	Thesis
I. BA = ED.	*The base* AC is > *the base* DF.
II. BC = EF.	
III. ∀ B > ∀ DEF.	

Preparation.

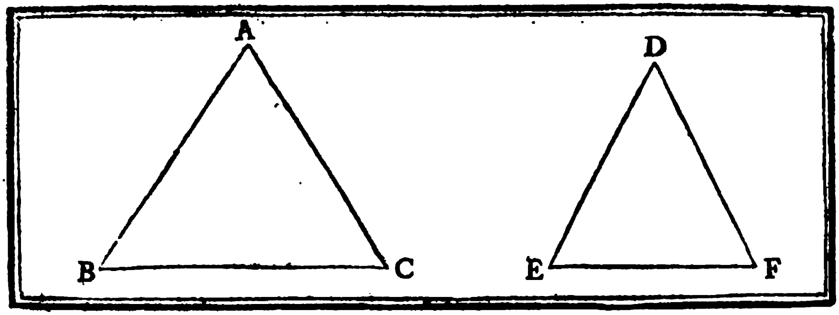
1. At the point E, in the line DE, make ∀ DEG = to the given ∀ B. *P.* 23. *B.* 1.
2. Make EG = to BC or to EF. *P.* 3. *B.* 1.
4. From the points D & F to the point G, draw the straight lines DG, FG. *Pos.* 1.

DEMONSTRATION.

BECAUSE in the △ ABC the sides BA, BC, are = to the sides ED, EG, of △ DEG (*Hyp.* 1, *Prep.* 2.), & ∀ contained B = to ∀ contained DEG (*Prep.* 1.).

1. It follows, that the base AC is = to the base DG. *P.* 4. *B.* 1.
   Again, because EG is = to the side EF (*Prep.* 2, *Hyp.* 2.).
2. The △ FEG is an isosceles △. *D.* 25. *B.* 1.
3. Consequently, ∀ *m* = ∀ *r* + *q*. *P.* 5. *B.* 1.
   Since therefore ∀ *m* = ∀ *r* + *q* (*Arg.* 3.); if from the last be taken its part *q*.
4. The ∀ *m* will be > ∀ *r*. *C. N.*
   And if to ∀ *m* be added ∀ *n*.
5. Much more will the whole ∀ *m* + *n* be > ∀ *r*. *C. N.*
6. Consequently, the side DG opposite to the greater ∀ *m* + *n*, is > the side DF opposite to the lesser ∀ *r*. *P.* 19. *B.* 1.
   But the straight line DG being > DF (*Arg.* 6.), & this same straight line DG being = to the base AC (*Arg.* 1.).
7. It is evident that the base AC is also > the base DF. *C. N.*

Which was to be demonstrated.



PROPOSITION XXV. *THEOREM XVI.*

IF two triangles (BAC, EDF,) have two ſides of the one equal to two ſides of the other, each to each, but the baſe (BC) of the one greater than the baſe (EF) of the other; the angle (BAC) oppoſite to the greater baſe (BC), will be alſo greater than the angle (D) oppoſite to the leſſer baſe (EF).

Hypotheſis.	Theſis.
I. AB = DE.	*The angle* A *oppoſite to the greater*
II. AC = DF.	*baſe* BC, *is* > ∀ D *oppoſite to the leſſer*
III. BC > EF.	*baſe* EF.

DEMONSTRATION.

IF not,
The angle A is either equal or leſs than the angle D. *C. N.*

CASE I. Suppoſe ∀ A to be = to ∀ D.

BECAUSE ∀ A is = to ∀ D (*Sup.* 1.), & the ſides AB, AC, & DE, DF, which contain thoſe ∀, are equal each to each, (*Hyp.* 1 & 2.).

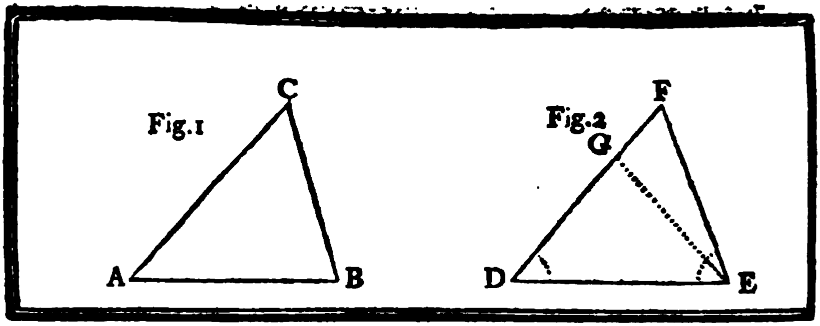
1. The baſe BC is = to the baſe EF. *P.* 4. *B.* 1.
   But the baſe BC is not = to the baſe EF (*Hyp.* 3.).
2. Therefore ∀ A cannot be = to ∀ D.

CASE II. Suppoſe ∀ A to be < ∀ D.

BECAUSE ∀ A is < ∀ D (*Sup.* 2.), & the ſides AB, AC, & DE, DF, which contain thoſe ∀ are equal, each to each, (*Hyp.* 1 & 2.).

1. The baſe BC is < the baſe EF. *P.* 24. *B.* 1.
   But the baſe BC is not < the baſe EF (*Hyp.* 3.).
2. Therefore ∀ A is not < ∀ D.
   But it has been ſhewn that neither is it equal to it (*Caſe.* 1.).
3. Conſequently, ∀ A, which is oppoſite to the greater baſe BC, is > ∀ D, which is oppoſite to the leſſer baſe EF.

Which was to be demonſtrated.



PROPOSITION XXVI. *THEOREM XVII.*

IF two triangles (ACB, DFE,) have two angles (A & B) of one, equal to two angles (D & FED) of the other, each to each, & one ſide equal to one ſide, *viz.* either the ſides, as (AB & DE) adjacent to the equal angles; or the ſides, as (AC & DF) oppoſite to equal angles in each: then ſhall the two other ſides (AC, BC, or AB, BC,) be equal to the two other ſides (DF, EF, or DE, EF,) each to each, & the third angle (C) equal to the third angle (F).

Hypotheſis.	CASE I.	Theſis.
I. ∀ A = ∀ D.	When the equal ſides AB, DE, are	*I.* AC = DF.
II. ∀ B = ∀ FED.	adjacent to the equal angles A & D,	*II.* BC = EF.
III. AB = DE.	B & FED (*Fig.* 1 & 2.).	*III.* ∀ C = ∀ F.

Demonstration.

If not,

The ſides are unequal, & one, as DF will be > the other AC.

Preparation.

1. Cut off from the greater ſide DF a part DG = to AC. *P.* 3. *B.* 1.
2. From the point G to the point E, draw the ſtraight line GE. *Poſ.* 1.

BECAUSE in the △ ACB, DGE, the ſide AC is = to the ſide DG, (*Prep.* 1.), AB = DE (*Hyp.* 3.), & ∀ A is = to ∀ D. (*Hyp.* 1.).

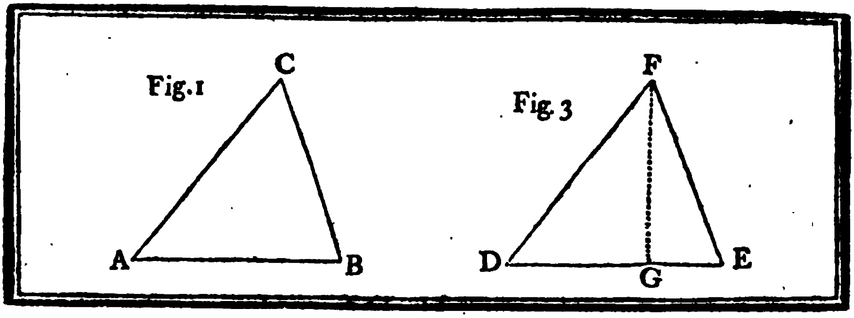
1. The ∀ B & GED oppoſite to the equal ſides AC & DG are equal. *P.* 4. *B.* 1.
   But ∀ B being = to ∀ GED (*Arg.* 1,), & this ſame ∀ B being alſo = to ∀ FED (*Hyp.* 2.).
2. It follows, that ∀ GED is = to ∀ FED. *Ax.* 1.
   But ∀ FED being the whole & ∀ GED its part:
3. The whole would be = to its part.
4. Which is impoſſibe. *Ax.* 8.
5. The ſides AC, DF, are therefore not unequal.
6. Conſequently, they are equal, or AC = DF. *C. N.*

Which was to be demonſtrated. I.

Since then in the △ ACB, DFE, the ſide AC is = to the ſide DF, (*Arg.* 6.), AB = DE (*Hyp.* 3.), & ∀ A is = to ∀ D (*Hyp.* 1).

7. The third ſide BC is alſo = to the third ſide EF, & the ∀ C & F, oppoſite to the equal ſides AB, DE, are alſo = to one another. *P.* 4. *B.* 1.

Which was to be demonſtrated. II & III.



CASE II.

Hypothesis.		Thesis.
I. ∀ A = ∀ D.	When the equal sides AC, DF,	*I.* AB = DE.
II. ∀ B = ∀ E.	are opposite to the equal angles	*II.* BC = EF.
III. AC = DF.	B & E. (*Fig.* 1. & 3.)	*III.* ∀C = ∀F.

DEMONSTRATION.

If not,

The sides AB, DE, are unequal; and one, as DE, will be > the other AB.

Preparation.

1. Cut off from the greater side DE, a part DG = to AB. *P.* 3. *B.* 1.
2. From the point G to the point F, draw the straight line GF. *Pos.* 1.

BECAUSE then in the △ ACB, DFG, the side AC is = to the side DF (*Hyp.* 3.), AB = DG (*Prep.* 1.), & ∀ A is = to ∀ D, (*Hyp.* 1.).

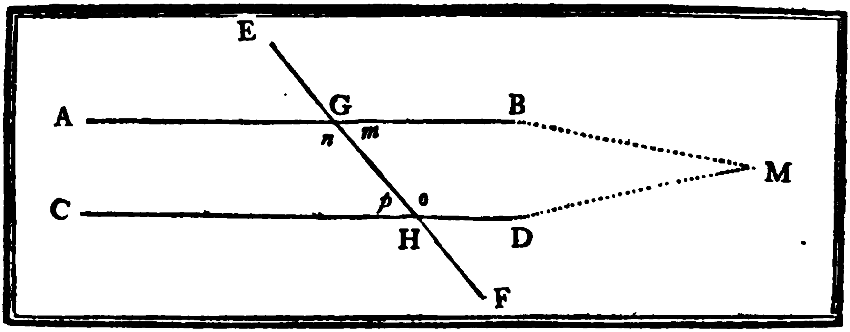
1. The other ∀ B & DGF, to which the equal sides AC, DF are opposite, are = to one another. *P.* 4. *B.* 1.
   The angle B being therefore = ∀ DGF (*Arg.* 1.), & this same ∀ B being also = to ∀ E (*Hyp.* 2.).
2. It follows, that ∀ E is = to ∀ DGF. *Ax.* 1.
   But ∀ DGF is an exterior ∀ of △ GFE, & ∀ E, is its interior opposite one.
3. Therefore the exterior ∀ will be equal to its interior opposite one.
4. Which is impossible. *P.* 16. *B.* 1.
5. Consequently, the sides AB, DE, are not unequal.
6. They are therefore equal, or AB = DE. *C. N.*

Which was to be demonstrated. I.

Since then in the △ ACB, DFE, the side AC is = to the side DF, (*Hyp.* 3.), AB = DE (*Arg.* 6.), & ∀ A is = to ∀ D (*Hyp.* 1.).

7. It is evident, that the third side BC is = to the third side EF, & the ∀ C & F, to which the equal sides AB, DE, are opposite, are equal to one another. *P.* 4. *B.* 1.

Which was to be demonstrated. II. & III.



PROPOSITION XXVII. *THEOREM XVIII.*

IF a ſtraight line (EF), falling upon two other ſtraight lines (AB, CD,) ſituated in the ſame plane, makes the alternate angles *(m & p, or n & o,)* equal to one another: theſe two ſtraight lines (AB, CD,) ſhall be parallel.

Hypotheſis.	Theſis.
I. AB, CD, *are two ſtraight lines in the ſame plane.*	*The lines* AB, CD,
II. The line EF *cuts them ſo that* ∀ *m* = ∀ *p*, *or* ∀ *n* = ∀ *o*.	*are* plle.

DEMONSTRATION.

IF not,
The ſtraight lines AB, CD, produced will meet either towards BD or towards AC. *D.* 35. *B.* 1.

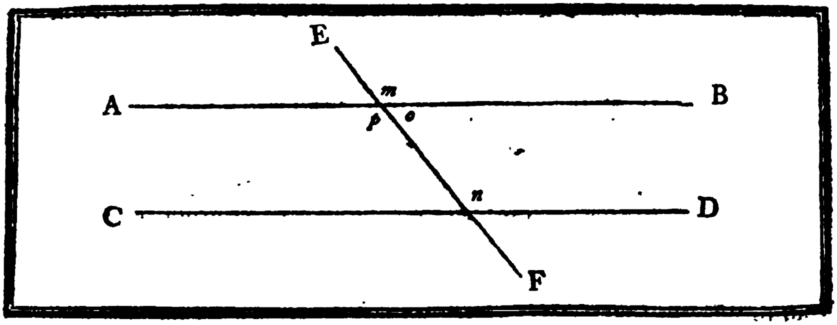
Preparation.

Let them be produced & meet towards BD in the point M. *Poſ.* 2.

BECAUSE the ∀ *n* is an exterior angle of △ GMH, & ∀ *o* its interior oppoſite one.

1. The ∀ *n* is > ∀ *o*. *P.* 16. *B.* 1.
   But ∀ *n* is = to ∀ *o* *(Hyp.* 2.).
2. This ∀ *n* is therefore not > ∀ *o*. *C. N.*
3. Conſequently, it is impoſſible that the ſtraight lines AB, CD, ſhould meet in a point as M.
4. From whence it follows that they are plle ſtraight lines. *D.* 35. *B.* 1.

Which was to be demonſtrated.



PROPOSITION XXVIII. *THEOREM XIX.*

IF a ſtraight line (EF) falling upon two other ſtraight lines (AB, CD,) ſituated in the ſame plane, makes the exterior angle *(m)* equal to the interior & oppoſite *(n)* upon the ſame ſide, or makes the interior angles *(o + n)* upon the ſame ſide equal to two right angles; thoſe two ſtraight lines AB, CD, ſhall be parallel to one another.

CASE I.

Hypotheſis.	Theſis.
∀ *m* = ∀ *n*.	AB, CD, *are* plle *lines.*

DEMONSTRATION.

BECAUSE the ∀ *m* & *p* are vertical or oppſite ∀.

1. They are = to one another. *P.* 15. *B.* 1.
   The ∀ *p* being therefore = to ∀ *m (Arg.* 1.), & ∀ *n* being = to the ſame ∀ *m (Hyp.).*
2. It is evident that ∀ *p* is alſo = to ∀ *n*. *Ax.* 1.
   But the equal ∀ *p* & *n (Arg.* 2.), are alſo alternate ∀.
3. Conſequently, the ſtraight lines AB, CD, are plle. *P.* 27. *B.* 1.

CASE II.

Hypotheſis.	Theſis.
The ∀ *o* + *n are = to* 2 ∟.	AB, CD, *are* plle. *lines.*

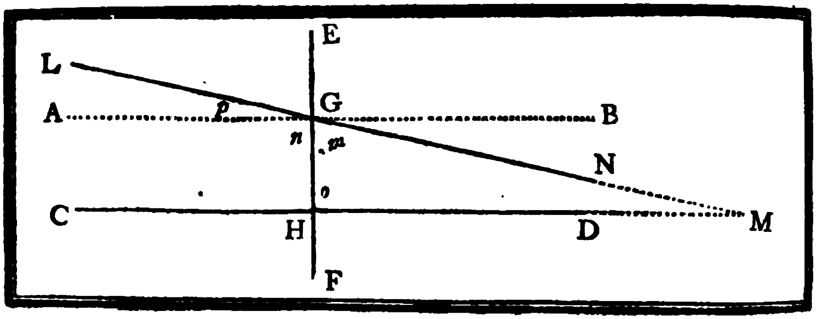
DEMONSTRATION.

BECAUSE the ſtraight line EF falling upon the ſtraight line AB, forms with it the adjacent ∀ *o* & *p*.

1. Thoſe ∀ *o* + *p* are = to two ∟. *P.* 13. *B.* 1.
   The ∀ *o* + *p* being therefore = to two ∟. *(Arg.* 1.), & the ∀ *o* + *n* being alſo = to two ∟ *(Hyp.).*
2. It follows, that the ∀ *o* + *p* are = to ∀ *o* + *n*. *Ax.* 1.
   And if the common angle *o* be taken away from both ſides.
3. The remaining ∀ *p* & *n* will be equal to one another. *Ax.* 3.
   But thoſe equal ∀ *p* & *n (Arg.* 3.), are at the ſame time alternate ∀.
4. Conſequently, the ſtraight lines AB, CD, are plle. *P.* 27. *B.* 1.

Which was to be demonſtrated.

F 2



LEMMA.

IF a ſtraight line (EF), meeting two ſtraight lines (LN, CD,) ſituated in the ſame plane, makes the alternate angles *(p + n & o)* unequal; thoſe two ſtraight lines (LN & CD,) being continually produced, will at length meet in (M), upon that ſide on which is the leſſer of the alternate angles *(o)*.

Preparation.

For ſince ∀ *p* + *n* is ſupoſed > ∀ *o*.

1. There may be made in the greater ∀ *p* + *n*, on the ſtraight line EF, at the point G, an angle *n* = ∀ *o*. *P. 23. B. 1.*
2. And AG may be produced at will to B. *Poſ. 2.*

DEMONSTRATION.

BECAUSE the two lines AB, CD, are cut by a third EF, ſo that the alternate ∀ *n* & *o* are = to one another *(Prep. 1.)*.

1. Thoſe two lines AB, CD, are plle. *P. 27. B. 1.*

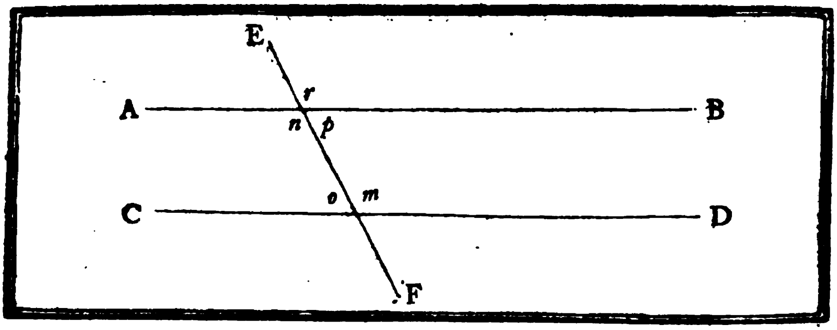
   But the line LN cuts one of the two plles, *viz.* AB in G.
2. Therefore, if produced ſufficiently, it will cut alſo the other CD ſomewhere in M, upon that ſide on which is the leſſer of the alternate ∀ *o**. *C. N.*

Which was to be demonſtrated.

COROLLARY.

WHEN ∀ *o* < ∀ *p* + *n*, the two interior angles *o* + *m* are neceſſarily < two ∟; ſince the two angles *p* + *n* & *m* are equal to two ∟. *P. 13. B. 1.* Conſequently, when the two interior ∀, are < two ∟; the lines LN, CD, which form thoſe angles with EF, will meet ſomewhere on the ſide of the line EF, where thoſe angles are ſituated, provided they are produced ſufficiently.

* *Euclid regards as a ſelf evident principle that,* a ſtraight line (EF), which cuts one of two parallels as (AB) will neceſſarily cut the other (CD), provided this cutting line (EF) be ſufficiently produced. *See the prep. of propoſitions XXX, XXXVII, and ſeveral others.*



PROPOSITION XXIX. *THEOREM XX.*

IF a ſtraight line (EF), falls upon two parallel ſtraight lines (AB, CD), it makes the alternate angles *(n & m)* equal to one another; and the exterior angle *(r)* equal to the interior & oppoſite upon the ſame ſide *(m)*; and likewiſe the two interior angles upon the ſame ſides *(p+ m)* equal to two right angles.

Hypotheſis.	Theſis.
AB, CD, *are two* plle *lines, cut by the ſame ſtraight line* EF.	*I.* ∀ *n* = ∀ *m*.
	II. ∀ *r* = ∀ *m*.
	III. ∀ *p* + *m* = *to* 2∟.

DEMONSTRATION.

IF not,

The ∀ *m* & *n* are unequal, — *C. N.*

And one of them as ∀ *m* will be < the other ∀ *n*.

BECAUSE the ∀ *m* is < ∀ *n*; if the ∀ *p* be added to both.

1. The ∀ *m* + *p* will be < the ∀ *n* + *p*. — *Ax.* 4.

But ſince the ∀ *n* & ∀ *p* are adjacent ∀, formed by the ſtraight line EF which falls upon AB.

2. Theſe ∀ *n* + *p* are = to two ∟. — *P.* 13. *B.* 1.
3. Conſequently, the ∀ *m* + *p* (leſs than the ∀ *n* + *p*) are alſo < two ∟. — *C. N.*
4. From whence it follows, that the lines AB, CD, are not plle. — *Cor. of lem.*

But the ſtraight lines AB, CD, are plle. *(Hyp.)*.

5. Conſequently, the ∀ *m* & *n* are not unequal. — *P.* 27. *B.* 1.
6. They are therefore equal, or ∀ *n* = ∀ *m*. — *C. N.*

Which was to be demonſtrated. I.

Moreover, ∀ *r* & ∀ *n* being vertically oppoſite.

7. Theſe angles are = to one another. — *P.* 15. *B.* 1.

But ∀ *m* being = to ∀ *n* *(Arg.* 6.), & ∀ *r* being = to the ſame ∀ *n*, *(Arg.* 7.).

8. It follows, that ∀ *r* is = to ∀ *m*. — *Ax.* 1.

Which was to be demonſtrated. II.

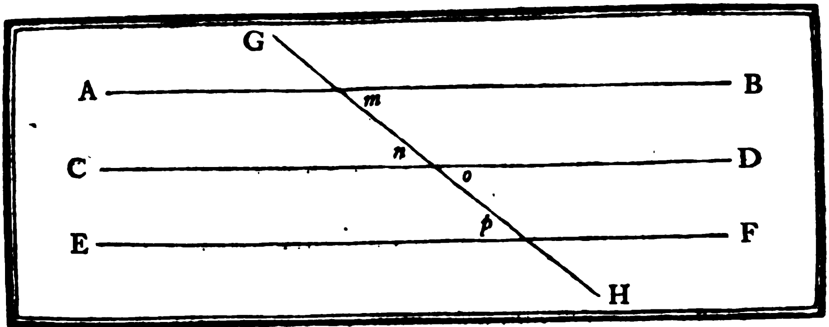
Likewiſe, ∀ *n* being = to ∀ *m* *(Arg.* 6.); if ∀ *p* be added to both.

9. The ∀ *n* + *p* will be = to ∀ *m* + *p*. — *Ax.* 2.

But the ∀ *n* + *p* are = to two ∟ *(Arg.* 2.).

10. From whence it follows that the ∀ *m* + *p* are alſo = to two ∟. — *Ax.* 1.

Which was to be demonſtrated. III.



PROPOSITION XXX. *THEOREM XXI.*

THE ſtraight lines (AB, EF), which are parallel to the ſame ſtraight line (CD), are parallel to one another.

Hypotheſis.	Theſis.
AB, EF, *are ſtraight lines*, plle *to* CD.	*The ſtraight lines* AB, EF *are* plle *to one another*.

Preparation.

Draw the ſtraight line GH, cutting the three lines AB, CD, EF.

DEMONSTRATION.

BECAUSE the ſtraight lines AB, CD, are two plles, *(Hyp.)* cut by the ſame ſtraight line GH. *(Prep)*.

1. The alternate ∀ *m* & *n* are = to one another. *P.* 29. *B.* 1.

   Likewiſe ſince the ſtraight lines CD, EF are two plles. (*Hyp.*) cut by the ſame ſtraight line GH. *(Prep)*.

2. The exterior angle *n* is = to its interior oppoſite one on the ſame ſide *p*. *P.* 29. *B.* 1.

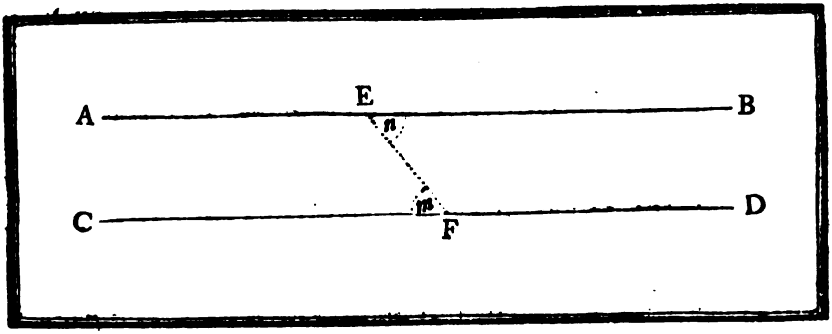
   But ∀ *n* being = to ∀ *m* (*Arg.* 1). & the ſame ∀ *n* being alſo = to ∀ *p* (*Arg.* 2).

3. The ∀ *m* & *p* will be = to one another. *Ax.* 1.

   But theſe ∀ *m* & *p* (*Arg.* 3.) are alternate ∀, formed by the two ſtraight lines AB, EF, which are cut by the ſtraight line GH.

4. Conſequently, theſe ſtraight lines AB, EF are plle. *P.* 27. *B* 1.

Which was to be demonſtrated



PROPOSITION XXXI. *PROBLEM X.*

TO draw a ſtraight line (AB), thro' a given point (E), parallel to a given ſtraight line (CD).

Given	Sought
The ſtraight line CD *and the point* E.	*The ſtraight line* AB, plle *to* CD, *& paſſing thro' the point* E.

Reſolution.

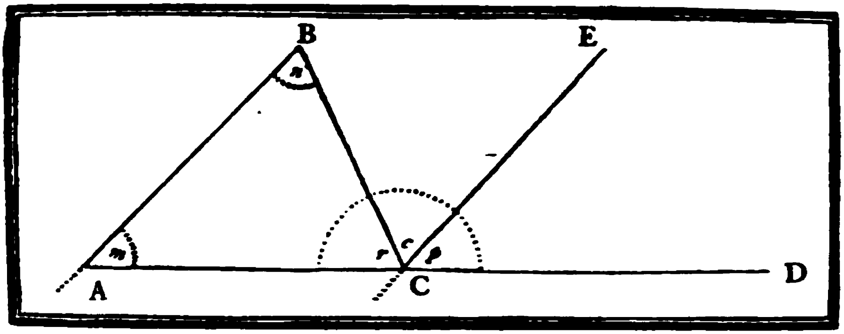
1. In the given ſtraight line CD take any point F.
2. From the point F to the point E, draw the ſtraight line FE. *Poſ.* 1.
3. At the point E in the ſtraight line FE, make ∀ *n* = to ∀ *m*. *P.* 23. *B.* 1.
4. And produce the ſide EB to A. *Poſ.* 2.

DEMONSTRATION.

BECAUSE the alternate ∀ *m* & *n*, formed by the ſtraight line EF, which cuts the two lines AB, CD, are = to one another (*Reſ.* 3.).

1. The ſtraight lines AB, CD, are plle. *P.* 27. *B.* 1.

Which was to be demonſtrated.



PROPOSITION XXXII. *THEOREM XXII.*

IF a ſide as (AC) of any triangle (ABC) be produced, the exterior angle (*c* + *p*) is equal to the ſum of the two interior and oppoſite angles (*n* + *m*); and the three interior angles (*n* + *m* + *r*) are equal to two right angles.

Hypotheſis.	Theſis.
ABC *is a* △, *one of whoſe ſides* AC, *is produced indefinitely to* D.	*I.* ∀ *c* + *p* is = *to* ∀ *m* + *n*. *II. the* ∀ *n* + *m* + *r are* = *to* 2∟.

Preparation.

Thro' the point C, draw the ſtraight line CE, plle to the ſtraight line AB. *P.* 31. *B.* 1.

DEMONSTRATION.

BECAUSE the ſtraight lines AB, CE, are two plles (*Prep.*) cut by the ſame ſtraight line BC.

1. The alternate ∀ *n* & *c* are = to one another. *P.* 29. *B.* 1.

Likewiſe becauſe the ſtraight line AB, CE, are two plles (*Prep.*) cut by the ſame ſtraight line AD.

2. The exterior angle *p* is = to its interior oppoſite one *m*, on the ſame ſide. *P.* 29. *B.* 1.

The ∀ *c* being therefore = to ∀ *n* (*Arg.* 1.), & ∀ *p* = ∀ *m*, (*Arg.* 2.).

3. The ∀ *c* + *p* is = to the ∀ *n* & *m* taken together. *Ax.* 2.

Which was to be demonſtrated. I.

Since then the ∀ *c* + *p* is = to ∀ *n* + *m* (*Arg.* 3); if the ∀ *r* be added to both ſides.

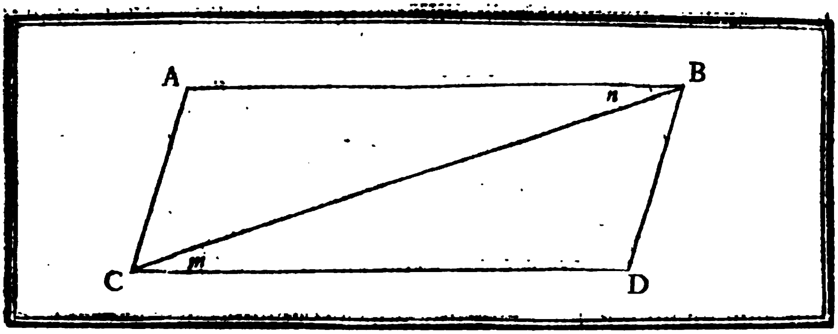
4. The ∀ *c* + *p* + *r* will be = to the three ∀ *n* + *m* + *r* of the △ ABC. *Ax.* 2.

But theſe ∀ *c* + *p* + *r* are the adjacent ∀, formed by the line BC, which meets AD at the ſame point C.

5. Conſequenly, the ∀ *c* + *p* + *r* are = to two ∟. *P.* 13. *B.* 1.

Wherefore, the three ∀ *n* + *m* + *r*, which are = to ∀ *c* + *p* + *r*, (*Arg.* 4.) are alſo = to two ∟. *Ax.* 1.

Which was to be demonſtrated. II.



PROPOSITION XXXIII. *THEOREM XXIII.*

THE ſtraight lines (AC, BD,) which join the extremities (A, C, & B, D,) of two equal and parallel ſtraight lines, towards the ſame parts, are alſo themſelves equal and parallel.

Hypotheſis.	Theſis.
AC, BD, *are two ſtraight lines, which join towards the ſame parts, the extremities of two* = & plle *ſtraight lines* AB, CD.	*I. The ſtraight lines* AC, BD, *are equal.* *II. And thoſe ſtraight lines* AC, BD, *are* plle.

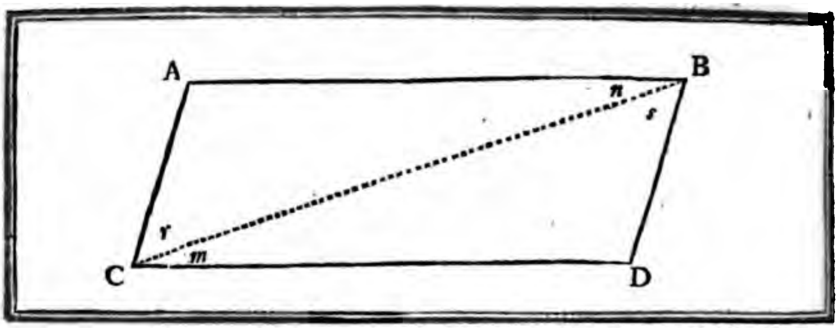
Preparation.

From the point B to the point C, draw the ſtraight line BC.

DEMONSTRATION.

BECAUSE the ſtraight lines AB, CD, are two plles *(Hyp.)* cut by the ſame ſtraight line BC *(Prep.)*.

1. The alternate ∀ *n* & *m* are = to one another. *P. 29. B. 1.*
   Since therefore in the two △ CAB, BDC, the ſide CD is = to the ſide AB *(Hyp.)*, the ſide BC is common to the two △, & the ∀ *m* is = to the ∀ *n* *(Arg. 1.)*.
2. It follows, that the baſe AC is = to the baſe BD. *P. 4. B. 1.*
   Which was to be demonſtrated. I.
3. Likewiſe that the ∀ ACB, DBC, to which the equal ſides AB, CD, are oppoſite, are alſo = to one another.
   But thoſe equal ∀ ACB, DBC, *(Arg. 3.)* are alternate ∀ formed by the ſtraight lines AC, BD, cut by the ſtraight line BC.
4. Conſequently, the ſtraight lines AC, BD, are plle. *P. 27. B. 1.*
   Which was to be demonſtrated. II.



PROPOSITION XXXIV. *THEOREM XXIV.*

THE opposite sides (AC, BD, & CD, AB,) and the opposite angles (A, D, & *m* + *r*, *n* + *s*,) of a parallelogram (AD) are equal to one another, & the diagonal (BC) divides it into two equal parts.

Hypothesis.

I. AD *is a* Pgr.
II. BC *is the diagonal of this* Pgr.

Thesis.

I. The sides AC, BD, & CD, AB, *are* = *to one another*, & ∀ A = D.
II. ∀ *m* + *r* = ∀ *n* + *s*.
III. The △ CAB, BDC, *formed by the diagonal, are* = *to one another.*

DEMONSTRATION.

BECAUSE the straight lines AB, CD, are two plles (*Hyp.* 1.) cut by the same straight line CB (*Hyp.* 2.).

1. The alternate ∀ *m* & *n* are = to one another. *P.* 29. *B.* 1.

Again, because the straight lines AC, BD, are two plles (*Hyp.* 1.) cut by the same straight line CB (*Hyp.* 2.).

2. The alternate ∀ *r* & *s* are = to one another. *P.* 29. *B.* 1.

But the △ BDC, CAB, have two ∀ *m* & *s* = to two ∀ *n* & *r*, (*Arg.* 1 & 2.), & the side BC adjacent to those equal ∀ is common to the two △.

3. Consequently, the sides AC & BD, opposite to the equal ∀ *n* & *m*, also the sides CD, AB, opposite to the equal ∀ *s* & *r*, are = to one another, & the third ∀ A is = to the third ∀ D. *P.* 26. *B.* 1.

Which was to be demonstrated. I.

But ∀ *m* being = to ∀ *n* (*Arg.* 1.), & ∀ *r* = ∀ *s* (*Arg.* 2.).

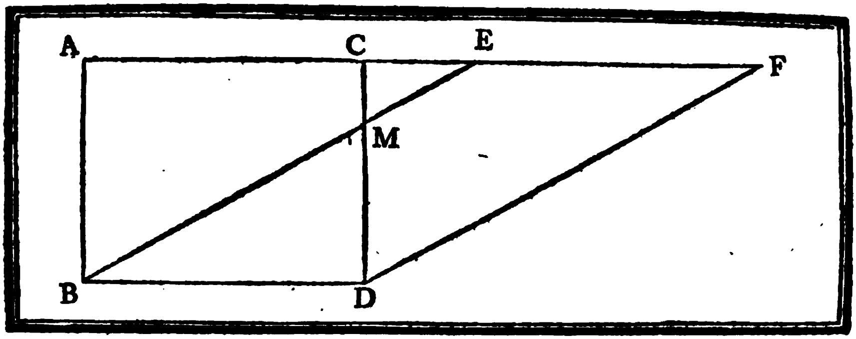
4. The whole ∀ *m* + *r* is = to the whole ∀ *n* + *s*. *Ax.* 2.

Which was to be demonstrated. II.

In fine, because in the △ CAB, BDC, the side CD is = to the side AB, (*Arg.* 3.), the side BC is common to the two △, and ∀ *m* is = to ∀ *n* (*Arg.* 1.).

5. Those two △ CAB, BDC, formed by the diagonal BC, are = to one another. *P.* 4. *B.* 1.

Which was to be demonstrated. III.



PROPOSITION XXXV. THEOREM XXV.

PARALLELOGRAMS (AD, ED,) upon the same base (BD) & between the same parallels (AF, BD,); are equal to one another.

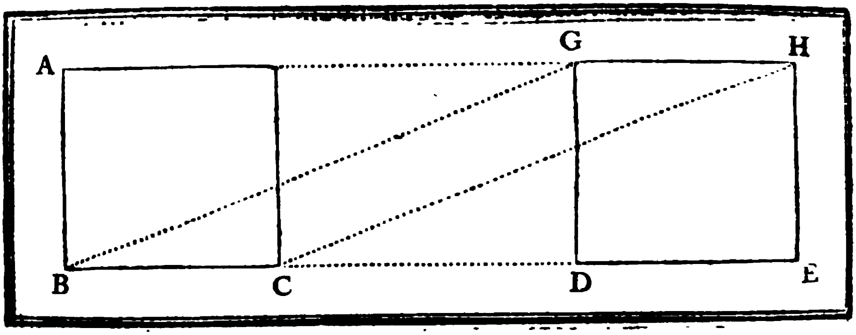
Hypothesis.	Thesis.
I. AD & ED *are two* Pgrs. II. *And those two* Pgrs, *are upon the same base* BD, & *between the same* plles AF, BD.	*The* Pgr AD *is* = *to the* Pgr ED.

DEMONSTRATION.

BECAUSE the figure AD is a Pgr (*Hyp.* 1.).

1. The opposite sides AC, BD, & AB, CD, are = to one another. *P.* 34. *B.* 1.
Likewise, because the figure ED is a Pgr (*Hyp.* 1.).
2. The opposite sides EF, BD, & BE, DF, are = to one another. *P.* 34. *B.* 1.
But the straight line AC being = to the straight line BD (*Arg.* 1.), & the straight line EF being also = to the same straight line BD (*Arg.* 2.).
3. It follows, that the straight line AC, is = to the straight line EF. *Ax.* 1.
Since therefore AC is = to EF (*Arg.* 3.); if CE be added to both.
4. The straight line AE is necessarily = to the straight line CF. *Ax.* 2.
Therefore in the △ ABE, CDF, the side AB is = to the side CD, (*Arg.* 1.), the side BE is = to the side DF (*Arg.* 2.), & the base AE is = to the base CF (*Arg.* 4.).
5. Consequently, the △ ABE is = to the △ CDF. *P.* 8. *B.* 1.
Taking away therefore from those equal △ ABE, CDF, (*Arg.* 5.) their common part CME.
6. The remaining trapeziums ABMC, MDFE, are = to one another. *Ax.* 3.
Adding in fine to those equal trapeziums ABMC, MDFE, (*Arg.* 6.) the common part MBD.
7. The Pgrs AD & ED will be = to one another. *Ax.* 2.

Which was to be demonstrated.



PROPOSITION XXXVI. *THEOREM XXVI.*

PARALLELOGRAMS (AC, GE,) upon equal bases (BC, DE,) & between the same parallels (AH, BE,), are equal to one another.

Hypothesis.	Thesis.
I. AC, GE, *are two* Pgrs.	*The* Pgr AC *is* = *to the* Pgr GE.
II. And those two pgrs *are upon equal bases* BC, DE, & *between the same* plles AH, BE.	

Preparation.

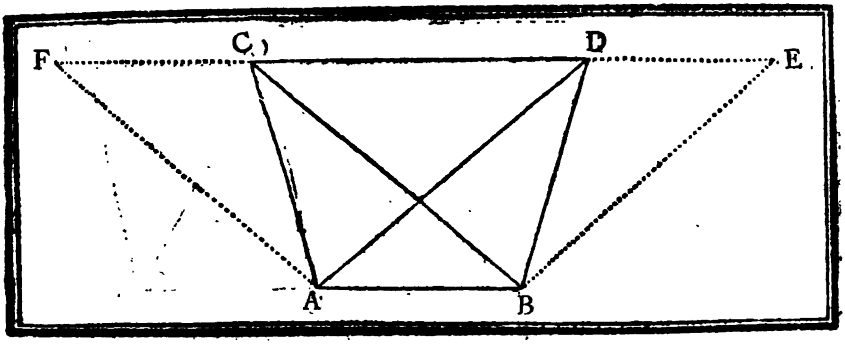
1. From the point B to the point G, draw the straight line BG. } *Pos.* 1.
2. From the point C to the point H, draw the straight line CH. }

DEMONSTRATION.

BECAUSE the figure GE is a Pgr (*Hyp.* 1.).

1. The opposite sides DE, GH, are = to one another. *P.* 34. *B.* 1.
But the straight line BC is = to DE (*Hyp.* 2.), & GH is = to the same straight line DE (*Arg.* 1.).
2. Therefore BC is = to GH. *Ax.* 1.
But since BC is = to GH (*Arg.* 2.); & they are plles (*Hyp* 2.) whose extremities are joined by the straight lines GB, HC, (*Prep.* 1 & 2.).
3. It is evident that those straight lines GB, HC, are = & plle. *P.* 33. *B.* 1.
4. Consequently, the figure GC is a Pgr. *D.* 35. *B.* 1.
Moreover, the Pgrs AC, GC, being upon the same base BC, & between the same plles AH, BE, (*Hyp.* 2.).
5. Those Pgrs AC, GC, are = to one another. *P.* 35. *B.* 1.
It will be proved after the same manner.
6. That the Pgr GC is = to the Pgr GE.
Since therefore the pgr AC is = to the pgr GC (*Arg.* 5.), & the Pgr GE is = to the same Pgr GC (*Arg.* 6.).
7. It follows, that the Pgr AC is = to the Pgr GE. *Ax.* 1.

Which was to be demonstrated.



PROPOSITION XXXVII. THEOREM XXVII.

TRIANGLES (ACB, ADB,) upon the same base (AB) & between the same parallels (AB, CD,) are equal to one another.

Hypothesis.	Thesis.
I. ACB, ABD, *are two* △.	*The* △ ACB *is* = *to the* △ ADB,
II. And those two △ *are upon the same* AB, & *between the same* plles AB, CD.	

Preparation.

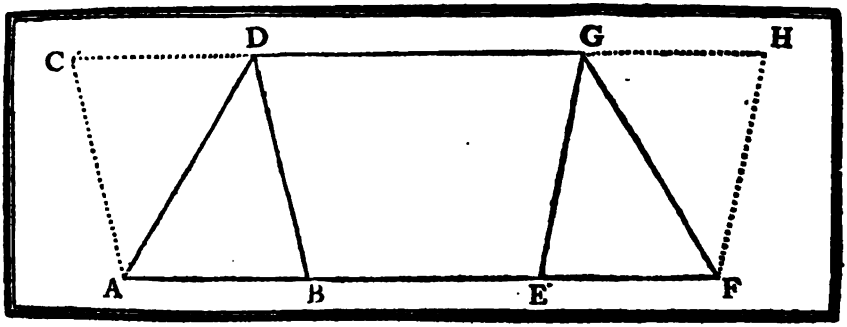
1. Produce the straight line CD both ways to E & F. *Pos.* 2.
2. Thro' the points A & B, draw the straight lines AF, BE, plle to the sides BC, AD; which will meet the produced CD somewhere in F & in E. *P.* 31. *B.* 1.

DEMONSTRATION.

BECAUSE in the figure BF the opposite sides AB, FC, & AF, BC, are plle (*Hyp.* 2 & *Prep.* 2.).

1. The figure BF is a Pgr. *D.* 35. *B.* 1.
It will be proved after the same manner,
2. That the figure AE is a Pgr.
But the Pgrs BF, AE, are upon the same base AB and between the same plles AB, FE, (*Hyp.* 2 & *Prep.* 1.).
3. Consequently, the Pgr BF is = to the Pgr AE. *P.* 35. *B.* 1.
But the straight lines AC, BD, are the diagonals of the Pgrs BF, AE, (*Prep.* 1 & 2.).
4. Wherefore those diagonals AC, BD, divide the Pgrs BF, AE, into two equal parts. *P.* 34. *B.* 1.
5. Consequently, the △ ACB is the half of the Pgr BF, & the △ ADB the half of the pgr AE.
Since then the whole Pgrs BF, AE, are equal to one another (*Arg.* 3.), & the △ ACB, ADB, are the halves of those Pgrs (*Arg.* 5.).
6. It is evident that the △ ACB, ADB, are also = to one another. *Ax.* 7.

Which was to be demonstrated.



PROPOSITION XXXVIII. *THEOREM XXVIII.*

TRIANGLES (ADB, EGF,) upon equal bases (AB, EF,) & between the same parallels (AF, DG,) are equal to one another.

Hypothesis.
I. ADB, EGF, *are two* △.
II. And those two △ *are upon* = *bases* AB, EF, *& between the same* plles AF, DG.

Thesis.
The △ ADB *is* = *to the* △ EGF.

Preparation.

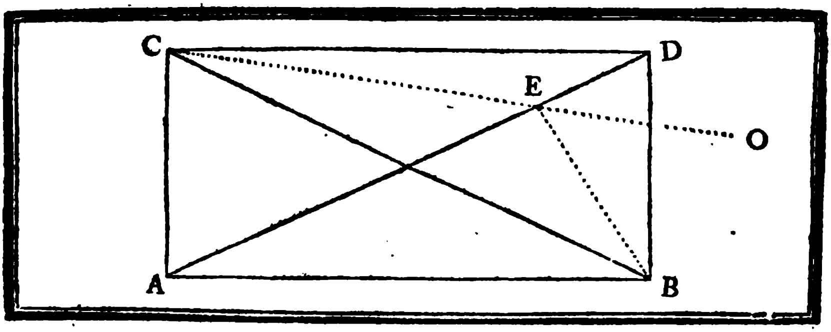
1. Produce the straight line DG both ways to the points H, C. *Pos.* 2.
2. Thro' the points A & F, draw the straight lines AC, FH, plle to the sides BD, EG; which will meet the produced line DG, somewhere in C & in H. *P.* 31. *B.* 1.

DEMONSTRATION.

BECAUSE in the figure BC, the opposite sides AB, CD, & AC, BD, are plle (*Hyp.* 2 & *Prep.* 2.).

1. The figure BC is a Pgr. *D.* 35. *B.* 1.
   It may be proved after the same manner.
2. That the figure EH is a Pgr.
   But the pgrs BC, EH, (*Arg.* 1 & 2.) are upon = bases AB, EF, & between the same plles AF, CH, (*Hyp.* 2.).
3. Consequently, the Pgr BC, is = to the Pgr EH. *P.* 36. *B.* 1.
   But the straight lines AD, FG, being the diagonals of the Pgrs BC, EH, (*Prep.* 1 & 2.).
4. Those straight lines AD, FG, divide the Pgrs BC, EH, into two equal parts. *P.* 34. *B.* 1.
5. Wherefore, the △ ADB, is half of the Pgr BC, & the △ EGF is the half of the Pgr EH.
   Since then the whole Pgrs BC, EH, are = to one another (*Arg.* 3.). and the △ ADB, EGF, are the halves of those Pgrs (*Arg.* 5.).
6. It follows, that those △ ADB, EGF, are also = to one another. *Ax.* 7.

Which was to be demonstrated.



PROPOSITION XXXIX. *THEOREM XXIX.*

EQUAL triangles (ACB, ADB,) upon the fame bafe (AB) & upon the fame fide of it, are between the fame parallels (AB, CD,).

Hypothefis.	Thefis.
I. The △ ACB, ADB, *are equal.*	*The* △ ACB, ADB, *are between*
II. And thofe △ *are upon the fame bafe* AB.	*the fame* plles AB, CD.

Demonstration.

If not,

The ftraight lines AB, CD, are not plle, & there may be drawn thro' the point C, fome other ftraight line CO, plle to AB.

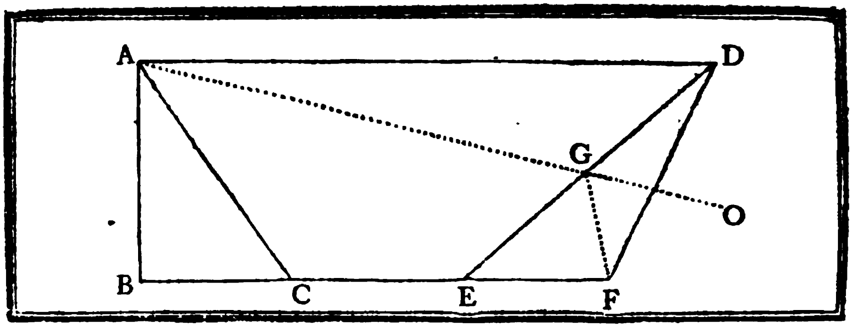
Preparation.

1. Draw then thro' the point C, the ftraight line CO plle to AB; which will cut the ftraight line AD, fomewhere in E. *P.* 31. *B.* 1.
2. From the point B, to the point of interfection E, draw the ftraight line BE. *Pof.* 1.

BECAUSE the two △ ACB, AEB, are upon the fame bafe AB, (*Hyp.* 2.), & between the fame plles AB, CO, (*Prep.* 1.).

1. The △ ACB is = to the △ AEB. *P.* 37. *B.* 1.
   But the △ ADB being = to the △ ACB (*Hyp.* 1.), & the △ AEB being = to the fame △ ACB (*Arg.* 1.).
2. The △ ADB is = to the △ AEB. *Ax.* 1.
   But the △ ADB being the whole, & the △ AEB its part.
3. It follows, that the whole is equal to its part.
4. Which is impoffible. *Ax.* 8.
5. Confequently, the ftraight line CO is not plle to AB.
   It may be proved after the fame manner, that no other ftraight line but CD, can be plle to AB.
6. Confequently, the ftraight line CD, drawn thro' the vertices of the △ ACB, ADB, is plle to the bafe AB.

Which was to be demonftrated.



PROPOSITION XL. *THEOREM. XXX.*

EQUAL triangles (BAC, EDF,) upon equal bafes (BC, EF,) & upon the fame fide, are between the fame parallels (BF, AD,).

Hypothefis.	Thefis.
I. The △ BAC, EDF, *are equal.*	*The* △ BAC, EDF, *are between*
II. And thofe △ *are upon* = *bafes* BC, EF.	*the fame* plles BF, AD.

Demonstration.

If not,

The ftraight lines BF, AD, are not plle; & there may be drawn thro' the point A fome other ftraight line AO plle to BF.

Preparation.

1. Draw then thro' the point A the ftraight line AO plle to BF, which will cut the ftraight line ED fomewhere in G. *P.* 31. *B.* 1.
2. From the point F to the point of interfection G, draw the ftraight line FG. *Pof.* 1.

BECAUSE the △ BAC, EGF, are upon the equal bafes BC, EF, (*Hyp.* 2.), & between the fame plles BF, AO, (*Prep.* 1.).

1. The △ BAC is = to the △ EGF. *P.* 38. *B.* 1.

But the △ EDF is = to the △ BAC (*Hyp.* 1.), & the △ EGF is = to the fame △ BAC (*Arg.* 1.).

2. Wherefore the △ EDF is = to the △ EGF. *Ax.* 1.

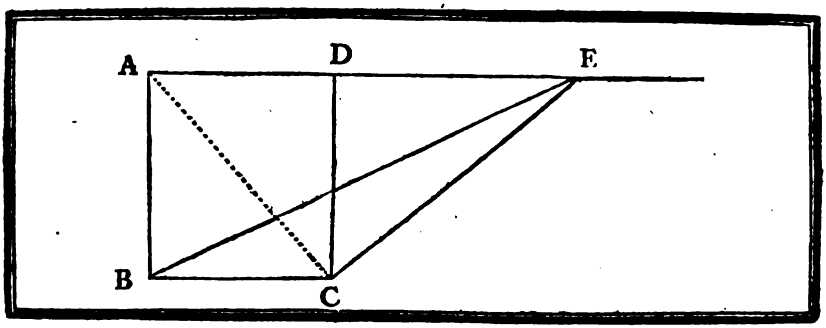
But the △ EDF being the whole & the △ EGF its part.

3. It follows, that the whole is = to its part.
4. Which is impoffible. *Ax.* 8.
5. Confequently, AO is not plle to BF.

It will be proved after the fame manner that no other ftraight line but AD can be plle to BF.

6. Confequently, the ftraight line AD, drawn thro' the fummets of the △ BAC, EDF, is plle to the ftraight line BF.

Which was to be demonftrated.



PROPOSITION XLI. *THEOREM XXXI.*

IF a parallelogram (BD) and a triangle (BEC) be upon the ſame baſe (BC), and between the ſame parallels (BC, AE,); the parallelogram ſhall be double of the triangle.

Hypotheſis.	Theſis.
I. BD is *a* Pgr & BEC *a* △.	*The* Pgr BD *is double of the* △ BEC.
II. Thoſe figures are upon the ſame baſe BC, & *between the ſame* plles BC, AE.	

Preparation.

From the point A to the point C, draw the ſtraight line AC. *Poſ.* 1.

DEMONSTRATION.

BECAUSE the △ BAC, BEC, are upon the ſame baſe BC, & between the ſame plles BC, AE *(Hyp.* 2.).

1. The △ BAC is = to the △ BEC. *P.* 37. *B.* 1.

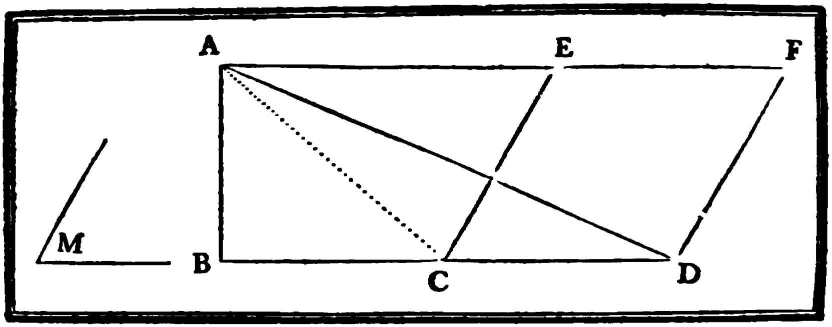
But the ſtraight line AC being the diagonal of the Pgr BD *(Prep.)*.

2. This diagonal divides the Pgr into two equal parts. *P.* 34. *B.* 1.
3. Conſequently, the Pgr BD is double of the △ BAC.

But this △ BAC being = to the △ BEC *(Arg.* 1.).

4. The Pgr BD is alſo double of the △ BEC. *Ax.* 1.

Which was to be demonſtrated.



PROPOSITION XLII. *PROBLEM XI.*

TO describe a parallelogram (ED), that ſhall be equal to a given triangle (BAD), & have one of its angles (DCE) equal to a given rectilineal angle (M).

Given	Sought
I. The △ BAD.	*The conſtruction of a* Pgr = *to the* △ BAD,
II. A rectilineal ∀ M.	*& having an* ∀ DCE = *to the given* ∀ M.

Reſolution.

1. Divide the baſe BD into two equal parts, at the point C. *P.* 10. *B.* 1.
2. Upon the ſtraight line BD at the point C, make an ∀ DCE = to the given ∀ M. *P.* 23. B. 1.
3. Thro' the point A, draw the ſtraight line AF plle to BD. *P.* 31. *B.* 1.
4. Produce the ſide CE of the ∀ DCE, until it meets the ſtraight line AF in a point E. *Poſ.* 2.
5. Thro' the point D, draw DF plle to CE, & produce it until it meets AF in a point F. *Poſ.* 2.

Preparation.

From the point A to the point C, draw the ſtraight line AC. *Poſ.* 1.

DEMONSTRATION.

BECAUSE the △ BAC, CAD, are upon equal baſes BC, CD, (*Ref.* 1.), & between the ſame plles BD, AF, (*Ref.* 3.).

1. The △ BAC is = to the △ CAD. *P.* 38. *B.* 1.
2. Conſequently, the △ BAD is double of the △ CAD. *C. N.*

But in the figure ED the ſides CD, EF, & CE, DF, are plle (*Ref.* 3 & 5.).

3. Conſequently, ED is a Pgr. *D.* 35. *B.* 1.

But this Pgr ED & the △ CAD, are upon the ſame baſe CD, & between the ſame plles BD, AF, (*Ref.* 1. 3. & *Prep.*).

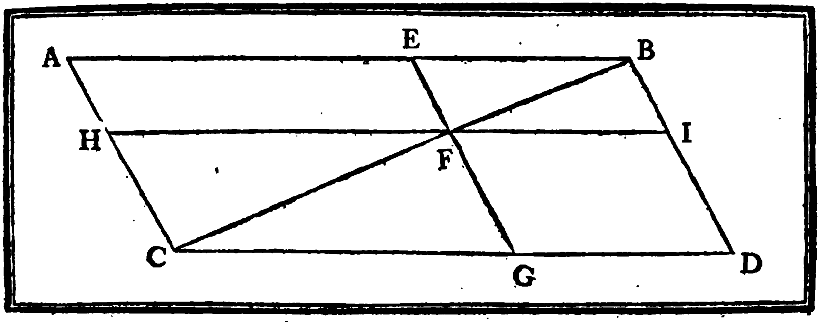
4. From whence it follows, that the Pgr ED is double of the △ CAD. *P.* 41. *B.* 1.

Since then the Pgr ED is double of the △ CAD (*Arg.* 4.), & the △ BAD is alſo double of the ſame △ CAD (*Arg.* 2.).

5. It is evident, that the Pgr ED is = to the △ BAD. *Ax.* 6.

And as its ∀ DCE is alſo = to the given ∀ M (*Ref.* 2.).

6. This Pgr ED is = to the given △ BAD, & has an ∀ DCE = to the given ∀ M. Which was to be demonſtrated.



PROPOSITION XLIII. *THEOREM XXXII.*

THE complements (AF, FD,) of the parallelograms (HG, EI,) about the diagonal (BC) of any parallelogram (AD), are equal to one another.

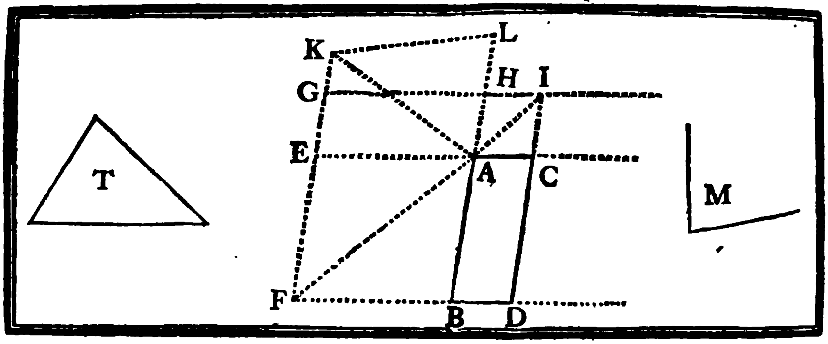
Hypothesis.	Thesis.
I. AD *is a* Pgr, *whose diagonal is* BC.	*The* Pgrs AF, FD, *which are the complements of the* Pgrs HG, EI, *are* = *to one another.*
II. HG, EI, *are the* Pgrs *about the diagonal.*	

DEMONSTRATION.

BECAUSE AD is a Pgr, whose diagonal is BC (*Hyp.* 1.).

1. This diagonal divides the Pgr into two equal parts. *P.* 34. *B.* 1.
2. Consequently, the △ CAB is = to the △ BDC.
 Likewise, EI being a Pgr, whose diagonal is BF (*Hyp.* 2.).
3. It divides also the Pgr into two equal parts. *P.* 34. *B.* 1.
4. Wherefore the △ FEB is = to the △ BIF.
 In fine, HG is a Pgr, whose diagonal is FC (*Hyp.* 2.).
5. Which consequently divides it into two equal parts. *P.* 34. *B.* 1.
6. Consequently, the △ CHF is = to the △ FGC.
 Since then the △ FEB is = to the △ BIF (*Arg.* 4.), & the △ CHF = to the △ FGC (*Arg.* 6.).
7. The △ FEB, together with the △ CHF is = to the △ BIF, together with the △ FGC. *Ax.* 2.
 But the whole △ CAB, BDC, being = to one another (*Arg.* 2.); if there be taken away from both, the △ FED + CHF, & the △ BIF + FGC, which are equal (*Arg.* 7.).
8. The remaining Pgrs AF, FD, which are the complements of the Pgrs HG, EI, will be also = to one another. *Ax.* 3.

Which was to be demonstrated.



PROPOSITION XLIV. *PROBLEM XII.*

UPON a given ſtraight line (AB), to make a parallelogram (BC) which ſhall be equal to a given triangle (T), and have one of its angles as (BAC) equal to a given rectilineal angle (M).

Given	Sought
I. The ſtraight line AB.	*A* Pgr *made upon a ſtraght line* AB
II. The △ T.	= *to the* △ T, *having one of its* ∀
III. The rectilineal ∀ M.	BAC = *to the given* ∀ M.

Reſolution.

1. Produce the ſtraight line AB indefinitely. *Poſ.* 2.
2. Take AL = to one of the ſides of the given △ T. *P.* 3. *B.* 1.
3. Make the △ AKL = to the given △ T. *P.* 22. *B.* 1.
4. Deſcribe the Pgr EH = to the △ AKL, having an ∀ HAE = to the given ∀ M. *P.* 42. *B.* 1.
5. Thro' the point B, draw a ſtraight line BF plle to EA or GH. *P.* 31. *B.* 1.
6. Produce GH indefinitely, as alſo GE, until it meets BF in F. *Poſ.* 2.
7. Thro' the points F & A, draw the ſtraight line FA, which when produced will meet GH produced, ſomewhere in I. *Poſ.* 1.
8. Thro' the point I, draw the ſtraight line ID plle to HB or GF. *P.* 31. *B.* 1.
9. Produce FB, EA, until they meet ID in the points D & C. *Poſ.* 2.

DEMONSTRATION.

BECAUSE in the figure DG the oppoſite ſides GI, FD, & GF, ID, are plle (*Reſ.* 5. 6. 8. & 9.).

1. The figure DG is a Pgr. *D.* 35. *B.* 1.

Again, the opposite sides EA, FB, & EF, AB; also HI, AC, & HA, IC, of the figures EB, HC, being plle (*Ref.* 5. 6. 8. & 9.).

2. Those figures EB, HC, are Pgrs. *D.* 35. *B.* 1.

But the straight line FI is the diagonal of the Pgr DG (*Ref.* 7.), & EB, HC, are Pgrs about this diagonal (*Arg.* 2, & *Ref.* 7.).

3. Consequently, the Pgrs BC, EH, which are the compliments, are = to one another. *P.* 43. *B.* 1.

But the Pgr EH is = to the △ AKL (*Ref.* 4.), & the given △ T is = to the same △ AKL (*Ref.* 3.).

4. From whence it follows, that the Pgr EH is = to the given △ T. *Ax.* 1.

The Pgr EH being therefore = to the given △ T (*Arg.* 4.), & this same Pgr EH being = to the Pgr BC (*Arg.* 3.).

5. The Pgr BC is = to the given △ T. *Ax.* 1.

Moreover, because the ∀ HAE, BAC, are vertically opposite.

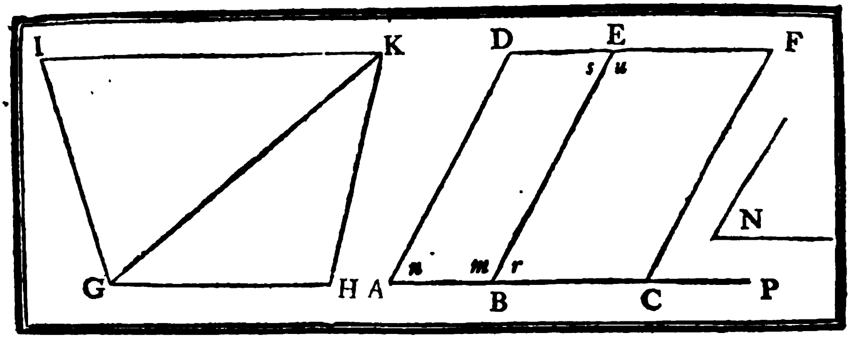
6. Those ∀ are = to one another. *P.* 15. *B.* 1.

Wherefore, ∀ HAE being = to the given ∀ M (*Ref.* 4.).

7. The ∀ BAC is also = to this given ∀ M. *Ax.* 1.

8. Therefore, upon the given straight line AB, there has been made a Pgr BC = to the given △ T (*Arg.* 5.), & which has an ∀ BAC = to the given ∀ M (*Arg.* 7.).

Which was to be done.



PROPOSITION XLV. *PROBLEM XIII.*

TO deſcribe a parallelogram (AF), equal to a rectilineal figure (IH); and having an angle *(n)* equal to a given rectilineal angle (N).

Given	Sought
I. A rectilineal figure IH.	*The conſtruction of a* Pgr = *to the rectilineal figure* IH, *& having an* ∀ *n* = *to a given* ∀ N.
II. A rectilineal ∀ N.	

Reſolution.

1. Draw the diagonal GK. *Poſ.* 1.
2. Upon an indefinite ſtraight line AP, make the Pgr AE = to the △ GHK, having an ∀ *n* = to the given ∀ N. *P.* 42. *B.* 1.
3. Upon the ſide BE of the Pgr AE, make the Pgr DF = to the △ GIK; having an ∀ *r* = to the given ∀ N. *P.* 44. *B.* 1.

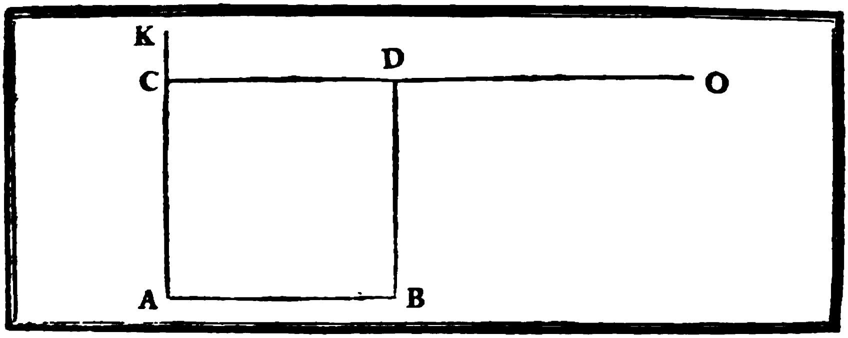
DEMONSTRATION.

BECAUSE ∀ N is = to each of the ∀ *n* & *r* *(Ref.* 2 & 3.).

1. The ∀ *n* is = to the ∀ *r*. *Ax.* 1.
   If the ∀ *m* be added to both.
2. The ∀ *n* + *m* will be = to the ∀ *r* + *m*. *Ax.* 2.
   But becauſe the ſides AD, BE, are plles *(Ref.* 2,) cut by the ſame ſtraight line AB.
3. The two interior ∀ *n* + *m*, are = to two ∟. *P.* 29. *B.* 1.
4. Conſequently, the adjacent ∀ *r* + *m*, which are = to them *(Arg.* 2.), are alſo = to two ∟. *Ax.* 1.
   The ſtraight lines AB, BC, which meet on the oppoſite ſides of the line BE at the point B, making with this ſtraight line BE the ſum of the adjacent ∀ *r* + *m* = to two ∟ *(Arg.* 4.).
5. Thoſe ſtraight lines AB, BC, form but one & the ſame ſtraight line AC. *P.* 14. *B.* 1.
   Moreover, the ſtraight lines DE, AC, being two plles *(Ref.* 2.) cut by the ſame ſtraight line BE.

6. The alternate ∀ *r* & *s*, are = to one another. *P.* 29. *B.* 1.
And if the ∀ *u* be added to both.
7. The ∀ *r* + *u*, will be = to ∀ *s* + *u*. *Ax.* 2.
But becauſe the ſides EF, BC, are two plles (*Ref.* 3.) cut by the ſame ſtraight line BE.
8. The interior ∀ *r* + *u*, are = to two ∟. *P.* 29. *B.* 1.
9. From whence it follows, that the adjacent ∀ *s* + *u*, which are = to them (*Arg.* 7.), are alſo = to two ∟. *Ax.* 1.
The ſtraight lines DE, EF, which meet on the oppoſite ſides of the line BE at the point E, making with this ſtraight line BE, the ſum of the adjacent ∀ *s* + *u* = to two ∟ (*Arg.* 9.).
10. Thoſe ſtraight lines DE, EF, form but one and the ſame ſtraight line DF. *P.* 14. *B.* 1.
But ſince the ſtraight lines AD, BE, & BE, CF, are the oppoſite ſides of the Pgrs AE, BF, (*Ref.* 2 & 3.). *P.* 34. *B.* 1.
11. The ſtraight line AD is = & plle to BF, & BE is = & plle to CF. { *P.* 30. *B.* 1.
12. Conſequently, AD is = & plle to to CF. - - - - - - - { *Ax.* 1.
Moreover, thoſe = and plle ſtraight lines AD, CF, are joined by the ſtraight lines AC, DF, (*Arg.* 5 & 10.).
13. Conſequently, the figure AF is a Pgr. - - - - - - - - { *P.* 33. *B.* 1. { *D.* 35. *B.* 1.
And becauſe the Pgr BF is = to the △ GIK (*Ref.* 3.), the Pgr AE is = to the △ GHK, & ∀ *n* = to the given ∀ N (*Ref.* 2.).
14. The whole Pgr AF is = to the rectilineal figure IH; & has an ∀ *n* = to the given ∀ N. *Ax.* 2.

Which was to be demonſtrated.



PROPOSITION. XLVI. *PROBLEM XIV.*

UPON a given ſtraight line (AB) to deſcribe a ſquare (AD).

Given	Sought
The ſtraight line AB.	*A ſquare made upon the ſtraight line* AB.

Reſolution.

1. At the point A, erect upon the ſtraight line AB the perpendicular AK. *P.* 11. *B.* 1.
2. From the ſtraight line AK cut off a part AC = to AB. *P.* 3. *B.* 1.
3. Thro' the point C, draw the ſtraight line CO plle to AB. } *P.* 31. *B.* 1.
4. And thro' the point B, draw the ſtraight line BD plle to AC, } which will cut CO ſomewhere in D.

DEMONSTRATION.

BECAUSE in the figure AD the oppoſite ſides AB, CD, & AC, BD, are plle *(Ref.* 3 & 4.).

1. The figure AD is a Pgr. *D.* 35. *B.* 1.
2. Conſequently, the oppoſite ſides AB, CD, & AC, BD, are = to one another. *P.* 34. *B.* 1.

   But AC is = to AB *(Ref.* 2.).
3. Conſequently, the four ſides AB, CD, AC, BD, are = to one another. *Ax.* 1.

   Again, becauſe the ſtraight lines AB, CD, are plle *(Ref.* 3.).
4. The interior oppoſite ∀ A & ACD, are = to two ∟. *P.* 29. *B.* 1.

   But the ∀ A being a ∟ *(Ref.* 1.).
5. It is evident, that ∀ ACD is alſo a ∟. *C. N.*

   Moreover, becauſe AD is a Pgr *(Arg.* 1.).
6. The oppoſite ∀ are = to one another. *P.* 34. *B.* 1.
7. Wherefore, the ∀ BDC & B oppoſite to the right ∀ A & ACD, are alſo ∟.

   The figure AD being therefore an equilateral Pgr *(Arg.* 3.), & rectangular *(Arg.* 7.).
8. It follows, that this figure AD deſcribed upon the ſtraight line AB, is a ſquare. *D.* 30. *B.* 1.

Which was to be done.

COROLLARY I.

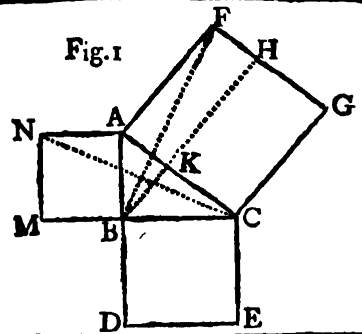
EVERY parallelogram, that has two equal sides AB, AC, *including a right angle, is a square; for drawing thro' the points* C & B *the straight lines* CD, BD, *parallel to the two sides* AB, AC, *the square* AD *will be described (D. 30. B. 1.).*

COROLLARY II.

EVERY parallelogram that has one right angle, has all its angles right angles. For since the opposite angles A & BDC, *are equal (P. 34. B. 1.), & the angle* A *is a right angle, the angle* BDC *will be also a right angle: moreover, the lines* AB, CD, & AC, BD, *being parallels; the interior angles* A & ACD, *likewise* A & B, *are equal to two right angles (P. 29. B. 1.); but the angle* A *being a right angle, it is manifest that the angles* ACD & B, *are also right angles.*

COROLLARY III.

THE squares described on equal straight lines, are equal to one another, & reciprocally, equal squares are described on equal straight lines.



PROPOSITION XLVII. *THEOREM XXXIII.*

IN any right angled triangle (ABC); the ſquare which is deſcribed upon the ſide (AC) ſubtending the right angle, is equal to the ſquares made upon the ſides (AB, BC,) including the right angle.

Hypotheſis.	Theſis.
The △ ABC *is* Rgle, *or* ∀ ABC *is a* ∟.	*The* □ *of the ſide* AC *is* = *to the* □ *of* AB, *together with the* □ *of* BC.

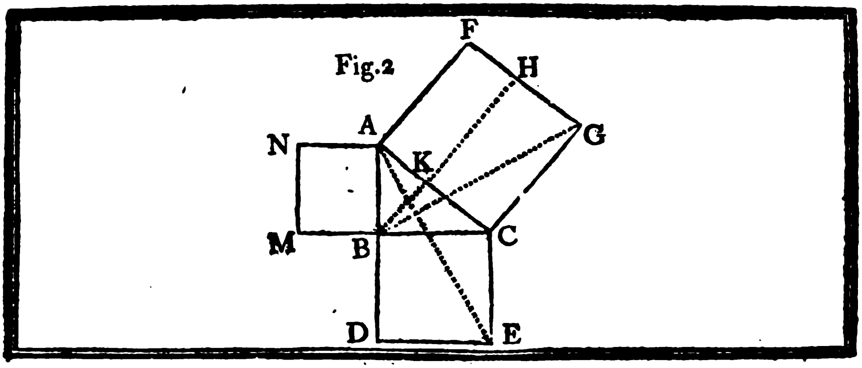
Preparation.

1. On the three ſides AC, AB, BC, deſcribe (*Fig.* 1.) the □ AG, AM, CD. *P.* 46. *B.* 1.
2. Thro' the point B, draw the ſtraight line BH plle to CG. *P.* 31. *B.* 1.
3. From the point B to the point F, draw the ſtraight line BF. } *Poſ.* 1.
4. From the point C to the point N, draw the ſtraight line CN. }

DEMONSTRATION.

BECAUSE the figure AM is a □ (*Prep.* 1.).

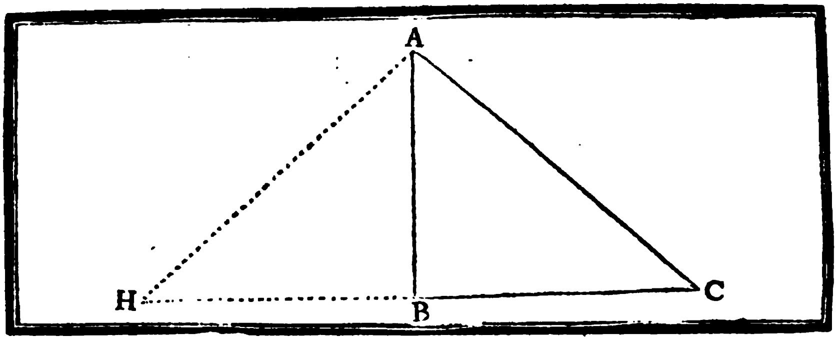
1. The ∀ ABM is a ∟. *D.* 30. *B.* 1.
   But ∀ ABC being alſo a ∟ (*Hyp.*).
2. The two adjacent ∀ ABM, ABC, are = to two ∟. *Ax.* 2.
   The ſtraight lines MB, BC, which meet on the oppoſite ſides of the line AB at the point B, making with this ſtraight line AB the ſum of the adjacent ∀ ABM, ABC, = to two ∟ (*Arg.* 2.).
3. Theſe ſtraight lines MB, BC, are in one and the ſame ſtraight line MC, *P.* 14. *B.* 1.
   which is plle to NA. *P.* 28. *B.* 1.
   In like manner it may be demonſtrated.
4. That AB, BD, are in one & the ſame ſtraight line AD, which is plle to CE.
   Moreover, becauſe AG, AM, are □ (*Prep.* 1.).
5. The ∀ FAC, NAB, are = to one another, (being right angles) & the ſides AF, AC, & AB, AN, are alſo = to one another. *D.* 30. *B.* 1.
   Therefore, if to thoſe equal ∀ FAC, NAB, ∀ CAB be added.



6. The whole ∀ FAB will be = to the whole ∀ NAC. *Ax.* 2.
Since then in the △ AFB, ACN, the ſides AF, AB, & AC, AN, are = each to each (*Arg.* 5.), & the ∀ FAB is = to the ∀ NAC, (*Arg.* 6.).
7. The △ AFB will be = to the △ ACN. *P.* 4. *B.* 1.
But the △ AFB & the Pgr AH, are upon the ſame baſe AF & between the ſame plles AF, BH, (*Prep.* 2.).
8. From whence it follows, that the Pgr AH is double of the △ AFB. *P.* 41. *B.* 1.
Likewiſe, the △ ACN & the □ AM being upon the ſame baſe AN, and between the ſame plles AN, MC, (*Arg.* 3.).
9. The □ AM is double of the △ ACN. *P.* 41. *B.* 1.
The △ AFB, ACN, being therefore = to one another (*Arg.* 7.). and the Pgr AH & the □ AM their doubles (*Arg.* 8 & 9.).
10. It follows, that the Pgr AH is = to the □ AM. *Ax.* 6.

In the ſame manner, by drawing (*Fig.* 2.) the lines BG, AE, it is demonſtrated, that the Pgr CH is = to the □ CD.
11. But the Pgr AH, together with the Pgr CH, form the □ AG.
12. Wherefore, this □ AG is = to the ſum of the □ AM & CD. *Ax.* 1.
But ſince the □ AG is the □ made upon the ſide AC, & the □ AM and CD the □ upon the ſides which include the ∟ ABC.
13. The □ made upon the ſide AC is = to the □ made upon AB & BC taken together.

Which was to be demonſtrated.



PROPOSITION XLVIII. *THEOREM XXXIV.*

IF the ſquare deſcribed upon one of the ſides (CA) of a triangle (CBA) be equal to the ſquares deſcribed upon the other two ſides of it (AB, BC,); the angle (ABC) included by theſe two ſides (AB, BC,), is a right angle.

Hypotheſis.	Theſis.
The □ *of* CA *is* = *to the* □ *of* AB, *together with the* □ *of* BC.	*The* ∀ ABC *included by the ſides* AB, BC, *is* ∟.

Preparation.

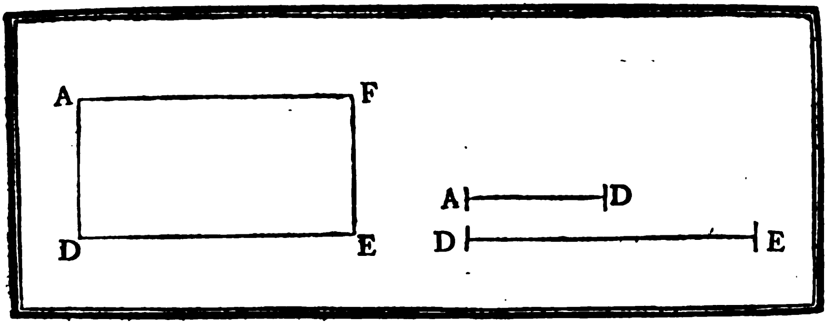
1. At the point B, in the ſtraight line BA, erect the perpendicular BH. *P.* 11. *B.* 1.
2. Make BH = BC. *P.* 3. *B.* 1.
3. From the point H to the point A, draw the ſtraight line HA. *Poſ.* 1.

DEMONSTRATION.

BECAUSE BH is = to BC (*Prep.* 2.).

1. The □ of BH will be = to the □ of BC. - - - - - - - *P.* 46. *B.* 1. *Cor.* 3.
   If the □ of AB be added to both,
2. The □ of AB & BH, will be = to the □ of AB & BC, *Ax.* 2.
   But the △ HBA being Rgle in B (*Prep.* 1.).
3. It follows, that the □ of HA is = to the □ of AB & BH. *P.* 47. *B.* 1.
   Since then the □ of CA is = to the □ of AB & BC (*Hyp.* 1.), the □ of HA = to the □ of AB & BH (*Arg.* 3.), & the □ of AB & BH, are = to the □ of AB & BC, (*Arg.* 2.).
4. The □ of CA muſt neceſſarily be = to the □ of HA. *Ax.* 1.
5. Conſequently, CA is = to HA. - - - - - - - - - - *P.* 46. *B.* 1. *Cor.* 3.
   But in the △ CBA, HBA, the ſide CA is = to the ſide HA, (*Arg.* 5.), AB is common to the two △, & the baſe BC is = to the baſe BH (*Prep.* 2.).
6. Wherefore, the ∀ ABC, ABH, included by the equal ſides AB, BC, and AB, BH, are = to one another. *P.* 8. *B.* 1.
   But the ∀ ABH is a ∟ (*Prep.* 1.).
7. Conſequently, the ∀ ABC will be alſo a ∟.

Which was to be demonſtrated.



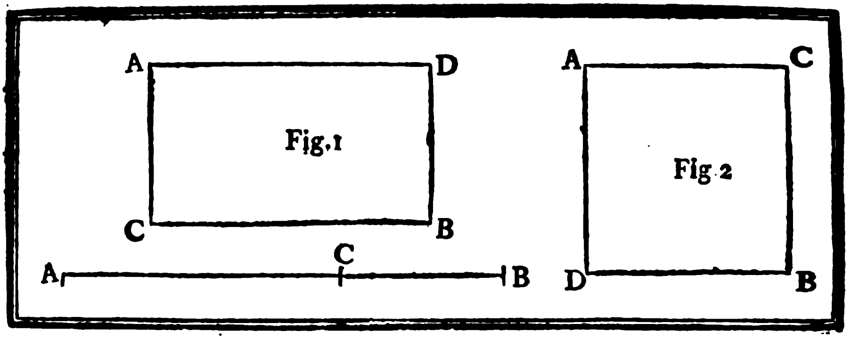
DEFINITIONS.

I.

EVERY right angled parallelogram (DF), is said to be contained by any two of the straight lines (AD, DE,) which include one of the right angles (ADE).

1. *A right angled parallelogram may be thus denoted, because a right angle & the two sides which include it, are what determine this figure. When the length of the sides* AD, DE, *including the right angle is fixed, the magnitude of the rectangle is determined, its construction being compleated by drawing thro' the extremities* A & E *of those sides, the lines* (AD, DE,) *parallel to them, according to D.* 35 & *P.* 31. *B.* 1.

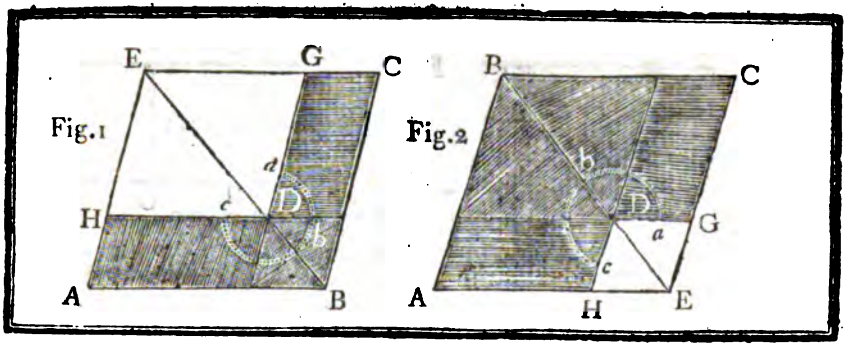
2. *A right angled parallelogram* DF *is for brevity sake often denoted by the three letters about the right angle, in this manner; the* Rgle Pgr ADE. *It is also represented thus: The* Rgle Pgr AD, DE, *that is, the* Rgle Pgr *resulting from the two sides* AD & DE, *which form a right angle; & is expressed thus: The* Rgle Pgr *under* AD & DE, *or the* Rgle Pgr *of* AD & DE.



DEFINITIONS.

3. SOMETIMES the parts of a straight line serve to denote a right angled parallelogram, for example (Fig. 1.), the straight line AB being divided in C, there may be described (P. 31. B. 1.), with those two lines AC, CB, a right angled parallelogram, by joining them at right angles, & this parallelogram is expressed thus: The RglePgr AC, CB, or simply the Rgle Pgr ACB, the letter that marks the point which is common to the two lines, being put between the other two letters; in like manner, by the Rgle Pgr ABC, is to be understood the parallelogram described according to the same rules, one of whose sides is AB & the other BC.

4. When the lines AD & DB, including the right angle, are equal (Fig. 2.), the parallelogram DC is a square (D. 30. B. 1.). As in this case one of the sides DB with the right angle, determine the square, which may be described from those data by P. 31. B. 1. This square may be expressed thus: The □ of DB, or the □ of AD..

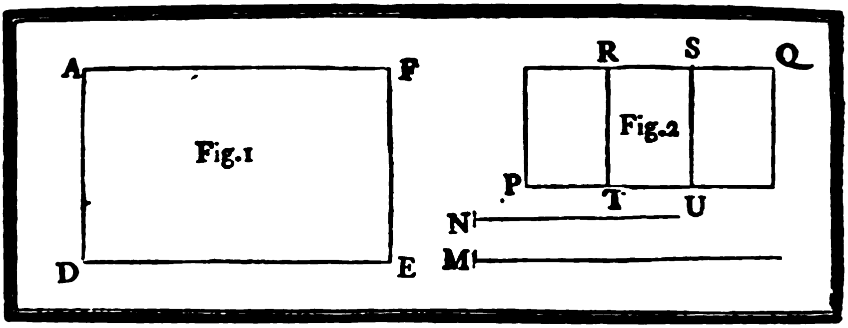


DEFINITIONS.

II.

THE figure (ABCGDH) composed of a parallelogram (DB) about the diagonal (BE), together with the two complements (AD, DC,) is called a Gnomon.

The Gnomon is marked by an arc of a circle (abc), which passes thro' the two complements (AD, DC,) *& the* Pgr *about the diagonal. There may be formed in every parallelogram two different gnomons; one, by taking away (Fig. 1.) from the whole* Pgr, *the greater* Pgr ED *about the diagonal; the other, by taking away (Fig. 2.) the lesser* Pgr ED *about the diagonal.*



AXIOMS.

I.

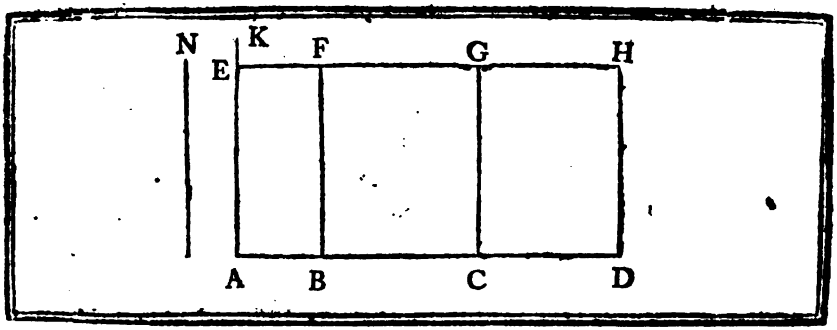
THE whole is equal to all its parts taken together.

The whole Pgr PQ *(Fig. 2.) is equal to all its parts, the* Pgrs PR, TS, VQ, *taken together.*

II.

RIGHT angled parallelograms contained by equal ſides, are equal.

The Rgle Pgr DF *(Fig. 1.) is contained by the ſtraight lines* AD, DE; *conſequently, if the ſtright line* N *is equal to* AD, *& the ſtraight line* M *is equal to* DE, *the* Rgle *formed by the ſtraight lines* N *&* M, *will be neceſſarily equal to the* Rgle DF.



PROPOSITION I. THEOREM I.

If there be two ſtraight lines (AD & N), one of which (AD) is divided into any number of parts (AB, BC, CD,); the rectangle contained by theſe ſtraight lines (AD & N) is equal to the rectangles contained by the undivided line (N), and the ſeveral parts (AB, BC, CD,) of the divided line (AD).

Hypotheſis.

AD & N *are two ſtraight lines, one of which* AD *is divided into ſeveral parts* AB, BC, CD.

Theſis.

The Rgle AD . N *is* = *to the* Rgles AB . N + BC . N + CD . N.

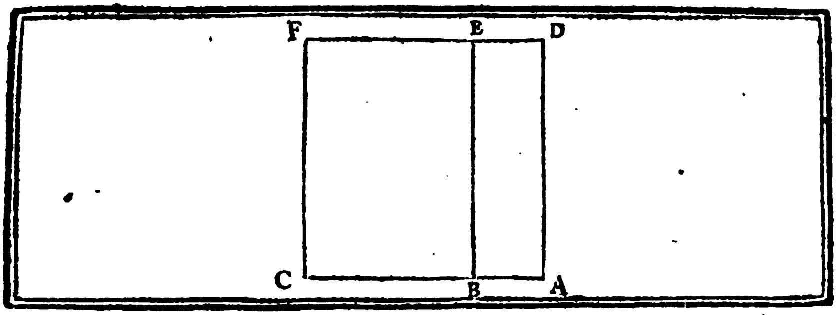
Preparation.

1. At the point A in the ſtraiaght line AD, erect the ⊥ AK. *P.* 11. *B.* 1.
2. From AK, cut off a part EA = N. *P.* 3. B. 1.
3. Thro' the points D & E, draw the ſtraight lines DH, EH, plle to AE, AD.
4. And Thro' the points of diviſion B & C, draw the ſtraight lines BF, CG, plle to AE or DH. } *P.* 31. *B.* 1.

DEMONSTRATION.

1. The Rgle AH is = to the Rgles AF, BG, CH, taken together. *Ax.* 1. *B.* 2.
But becauſe the Rgle AH is contained by the ſtraight lines EA, AD, (*Prep.* 3.), & AE is = to N (*Prep.* 2.).
2. This Rgle AH is contained by the ſtraight lines AD & N. *Ax.* 2. *B.* 2.
Likewiſe, becauſe the Rgle AF is contained by the ſtraight lines EA, AB, (*Prep.* 4.), & EA is = to N (*Prep.* 2.).
3. This Rgle AF is contained by the ſtraight lines AB & N. *Ax.* 2. *B.* 2.
4. In like manner, the Rgle BG is contained by the ſtraight lines BC & N, becauſe it is contained by the ſtraight lines FB & BC, & that FB = N. *P.* 34. *B.* 1.
And ſo of all the others.
5. Conſequently, the Rgle contained by the ſtraight lines AD & N is = to the Rgles contained by the ſtraight lines AB & N, BC & N, CD & N, taken together.
That is the Rgle AD . N is = to the Rgles AB . N + BC . N + CD. N. *Ax.* 1. *B.* 1.

Which was to be demonſtrated.



PROPOSITION II. *THEOREM II.*

IF a ſtraight line (AC) be divided into any two parts (AB, BC,); the rectangle contained by the whole line (CA), and each of the parts (AB, BC,), are together equal to the ſquare of the whole line (AC).

Hypotheſis.	Theſis.
AC *is a ſtraight line divided into two parts* AB, BC.	*The* Rgle CAB + Rgle ACB, *are* = *to the* □ *of* AC.

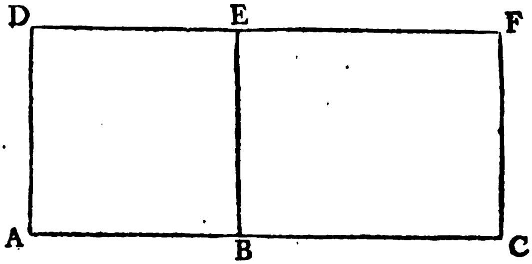
Preparation.

1. Upon the ſtraight line AC, deſcribe the □ AF. *P.* 46. B. 1.
2. Thro' the point of ſection B, draw the ſtraight line BE plle to AD or CF. *P.* 31. *B.* 1.

DEMONSTRATION.

1. THE whole Rgle AF is = to the Rgles AE, BF, taken together. *Ax.* 1. *B.* 2. But this Rgle AF is the □ of the line AC (*Prep.* 1.).
2. Conſequently, the Rgles AE, BF, taken together, are = to the □ of the line AC. *Ax.* 1. *B.* 1.
3. But the Rgle AE is contained by the ſtraight lines CA, AB, becauſe it is contained by the ſtraight lines DA, AB, of which DA = CA, (*Prep.* 1.). *Ax.* 2. *B.* 2.
4. Likewiſe, BF is a Rgle contained by the ſtraight lines AC, CB, becauſe it is contained by the ſtraight lines EB, BC, of which EB = AC, (*Prep.* 1 & 2.). *P.* 34. *B.* 1.
5. Wherefore, the Rgle contained by the ſtraight lines CA, AB, together with the Rgle contained by the ſtraight lines AC, CB, is = to the □ of the ſtraight line AC; or the Rgle CAB + the Rgle ACB, are = to the □ of AC. *Ax.* 1. *B.* 1.

Which was to be demonſtrated.



PROPOSITION III. *THEOREM III.*

IF a ſtraight line (AC) be divided into two parts in (B); the rectangle contained by the whole line (AC) & of one of the parts (AB), is equal to the rectangle contained by the two parts (AB, BC,) together with the ſquare of the aforeſaid part (AB).

Hypotheſis.	Theſis.
AC *is a ſtraight line divided into any two parts* AB, BC.	*The* Rgle CAB *is* = *to the* Rgle ABC + *the* □ *of* AB.

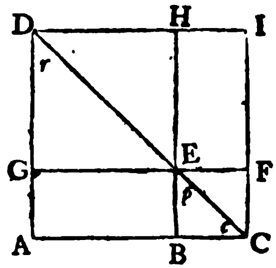
Preparation.

1. Upon the ſtraight line AB, deſcribe the □ AE. *P. 46. B. 1.*
2. Produce the line DE indefinitely to F. *Poſ. 2.*
3. Thro' the point C, draw the ſtraight line CF plle to AD or BE and produce it, until it meets DF in F. *Poſ. 2.*

DEMONSTRATION.

1. THE Rgle AF is = to the Rgles AE & BF taken together. *Ax. 1. B. 2.*
2. But the Rgle AF is contained by the ſtraight lines CA, AB; becauſe it is contained by CA & AD, of which AD = AB *(Prep. 1.)*. } *Ax. 2. B. 2.*
3. And the Rgle BF is contained by AB, BC; becauſe it is contained by EB, BC, of which EB = AB *(Prep. 1.)*.
   Moreover, the Rgle AE being the □ of the ſtraight line AB, *(Prep. 1.)*.
4. The Rgle of CA . AB, is = to the Rgle of AB . BC together with the □ of AB; or the Rgle CAB is = to the Rgle ABC + the □ of AB. *Ax. 1. B. 1.*

Which was to be demonſtrated.



PROPOSITION IV. *THEOREM IV.*

IF a straight line (AC) be divided into any two parts (AB, BC,); the square of the whole line (AC) is equal to the squares of the two parts (AB, BC,) together with twice the rectangle contained by the parts (AB, BC,).

Hypothesis.	Thesis.
AC *is a straight line divided into any two parts* AB, BC.	*The* □ *of* AC *is* = *to the* □ *of* AB + *the* □ *of* BC + 2 Rgles ABC.

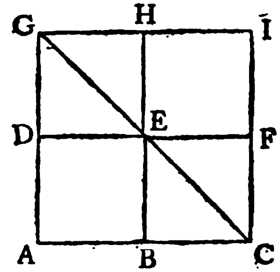
Preparation.

1. Upon AC, describe the □ AI. *P.* 46. *B.* 1.
2. Thro' the point of division B, draw BH plle to CI or AD. *P.* 31. *B.* 1.
3. Draw the diagonal CD, which will cut BH somewhere in E. *Pos.* 1.
4. Thro' the point E, draw GF plle to the opposite sides DI or AC. *P.* 31. *B.* 1.

DEMONSTRATION.

BECAUSE the lines AD, BH, CI; likewise AC, GF, DI, are plles (*Prep.* 1. 2. & 4.).

1. The four figures AE, EI, BF, GH, are Pgrs. *D.* 35. *B.* 1.
   And since each of those figures include one of the right angles of the □ AI.
2. Those Pgrs are also Rgles. { *P.* 46. *B.* 1. *Cor.* 2.
   Moreover, because the sides DA, AC, of the □ AI, are equal, (*D.* 30. *B.* 1.).
3. The ∀ *r* is = to the ∀ *c*. *P.* 5. *B.* 1.
   And because the straight lines AD, BH, are plles (*Prep.* 2.) cut by the straight line DC (*Prep.* 3.).
4. The interior ∀ *r* is = to its exterior opposite ∀ *p*. *P.* 29. *B.* 1.
5. Consequently, ∀ *c* = ∀ *p*. *Ax.* 1. *B.* 1.
6. Wherefore, the side BE is = to the side BC. *P.* 6. *B.* 1.
7. And the Rgle BF is a □, *viz.* the □ of BC. *D.* 30. *B.* 1.
8. It may be proved in the same manner, that the Pgr GH is a □, *viz.* the □ of AB, because GE = AB. *P.* 34. *B.* 1.
   Moreover, BE being = to BC (*Arg.* 6.).
9. The Rgle AE, or the Rgle of AB . BE, will be = to the Rgle of AB . BC. *Ax.* 2. *B.* 2.
   But the Rgle AE is = to the Rgle EI (*P.* 43. *B.* 1.).
   From whence it follows, that the Rgle EI is also = to the Rgle of AB . BC. *Ax.* 1. *B.* 1.



11. Consequently, the two Rgles AE, EI, taken together, are = to twice the Rgle of the parts AB, BC.
Since then the two □ GH & BF are the squares of the two parts AB & BC (*Arg.* 7. & 8.), & the Rgles AE, EI, taken together, are = to twice the Rgle of the parts AB, BC.

12. It follows, that the □ of the whole line AC is = to the □ of AB + the □ of BC + 2 Rgles ABC.

Which was to be demonstrated.

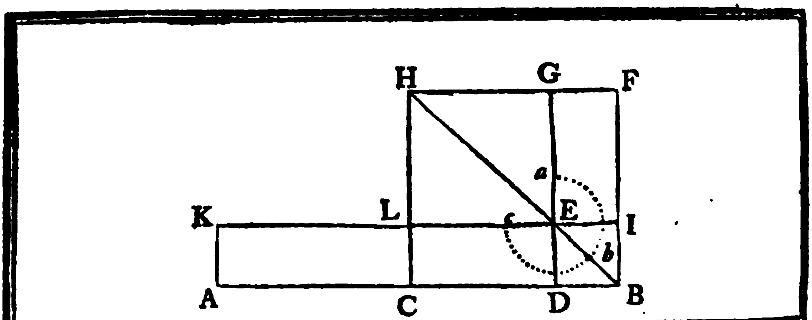
COROLLARY. I.

WHEN two straight lines HB, DF, plle *to the sides of a square intersect each other in a point* E *of the diagonal, the* Rgles BF, DH, *formed about the diagonal, are squares.*

COROLLARY II.

IF the line AC *be divided into two equal parts in* B, *the complements* AE, EI, *are squares, & those complements equal to one another, are also equal to the squares about the diagonal, & the the square of the whole line* AC *is four times the square of one of the parts* AB *or* BC.

For BF, DH, *are squares* (by the precedent Corollary), & *are equal to one another, because* BC = AB = DE. *Moreover,* AE *being = to* BF, & EI *being = to* BF (*P.* 36. *B.* 1.), *the complements* AE, EI, *are also squares;* & *since they are equal to one another, the* □ *of* AC = 4 □ *of* AB = 4 □ *of* BC.



PROPOSITION V. *THEOREM V.*

IF a ſtraight line (AB) be divided equally in (C) & unequally in (D); the rectangle contained by the unequal parts (AD, DB,) together with the ſquare of the part (CD), between the points of ſection (C & D), is equal to the ſquare of the half (AC or CB) of the whole ſtraight line (AB).

Hypotheſis.	Theſis.
AB *is a ſtraight line divided equally in* C, *&* *unequally in* D.	*The* Rgle ADB + *the* □ *of* CD, *are* = *to the* □ *of* CB.

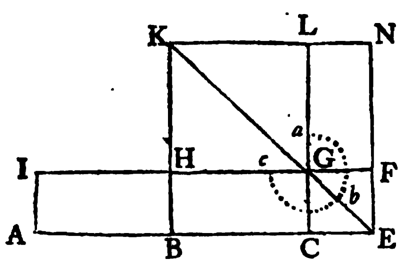
Preparation.

1. Upon the ſtraight line CB, deſcribe the □ CF. *P.* 46. *B.* 1.
2. Thro' the point of ſection D, draw DG plle to BF or CH. *P.* 31. *B.* 1.
3. Draw the diagonal BH. *Poſ.* 1.
4. Thro' the point of ſection E, draw IL plle to BC or FH, & thro' the point A, the ſtraight line AK plle to CL. *P.* 31. *B.* 1.

DEMONSTRATION.

BECAUSE the figure CF is a ſquare (*Prep.* 1.). { *P.* 4. *B.* 2. *Cor.* 1.
1. The Rgles LG, DI, about the diagonal are □.
2. Namely DI the □ of DB, & LG the □ of CD; becauſe LE = CD. *P.* 34. *B.* 1.
3. Moreover, the complement CE is = to the complement EF. *P.* 43. *B.* 1.
Let the ſquare DI be added to both.
4. The Rgle CI will be = to the Rgle DF. *Ax.* 2. *B.* 1.
But becauſe AC is = to CB (*Hyp.*).
5. The Rgle AL is = to the Rgle CI. *Ax.* 2. *B.* 2.
6. Conſequently, the Rgle AL is = to the Rgle DF. *Ax.* 1. *B.* 1.
Therefore, if the Rgle CE be added to both.
7. The Rgle AE will be = to the Rgles DF, CE, *i. e.* to the Gnomon *abc*. *Ax.* 2. *B.* 1.
8. But the Rgle AE is contained by AD, DB; becauſe it is contained by AD, DE, of which DE = DB (*Arg.* 1.). *Ax.* 2. *B.* 2.
9. Conſequently, the Rgle of AD. DB, is alſo = to the Gnomon *abc*. *Ax.* 1. *B.* 1.
Adding to both the □ LG, which is the □ of CD (*Arg.* 2.).
10. The Rgle AD. DB, together with the □ of CD, will be = to the Gnomon *abc*, together with the □ LG. *Ax.* 2. *B.* 1.
But this Gnomon *abc* together with the □ LG, is = to the □ CF, which is the □ of the half CB, of the whole line AB (*Prep.* 1.).
11. Wherefore, the Rgle ADB + the □ of CD, are = to the □ of CB. *Ax.* 1. *B.* 1.

Which was to be demonſtrated.



PROPOSITION VI. *THEOREM VI.*

If a ſtraight line (AC) be biſected in (B), & produced to any point E; the rectangle contained by the whole line thus produced (AE), & the part of it produced (EC), together with the ſquare of the half (BC), is equal to the ſquare of the ſtraight line (BE) made up of the half (BC) & the part produced (CE).

Hypotheſis.	Theſis.
I. AC *is a ſtraight line biſected in* B.	*The* Rgle AEC + *the* □ *of* BC,
II. And which is produced to the point E.	*is* = *to the* □ *of* BE.

Preparation.

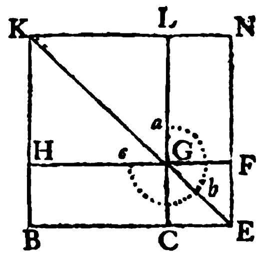
1. Upon the ſtraight line BE, deſcribe the □ BN. *P.* 46. *B.* 1.
2. Thro' the point C, draw CL plle to EN or BK. *P.* 31. *B.* 1.
3. Draw the diagonal EK. *Poſ.* 1.
4. Thro' the point G, draw FH plle to EB or NK. } *P.* 31. *B.* 1.
5. And thro' the point A, draw the ſtraight line AI plle to BK. }

DEMONSTRATION.

BECAUSE the figure BN is a ſquare (*Prep.* 1.).

1. The Rgles CF, HL, about the diagonal are ſquares. - - - - { *P.* 4. *B.* 2. *Cor.* 1.
   And becauſe HG is = to BC (*P.* 34. *B.* 1.).
2. The □ HL is = to the □ of BC. - - - - - - - - - - - { *P.* 46. *B.* 1. *Cor.* 3.
   Moreover, AB being = to BC (*Hyp.* 1.).
3. The Rgle AH is = to the Rgle BG. *Ax.* 2. *B.* 2.
   But the Rgle BG is = to the Rgle GN (*P.* 43. *B.* 1.).
4. Therefore, the Rgle AH is alſo = to the Rgle GN. *Ax.* 1. *B.* 1.
   And if the Rgle BF be added to both.
5. The Rgle AF will be = to the Rgles GN, BF, *i. e.* to the Gnomon *abc*. *Ax.* 2. *B.* 1.
6. But this Rgle AF is contained by AE, EC; becauſe EC = EF (*Arg.* 1.).
7. Conſequently, the Rgle AE. EC, is alſo = to the Gnomon *abc*. *Ax.* 1. *B.* 1.
   Therefore, if the □ HL, which is the □ of BC (*Arg.* 2.), be added to both.
8. The Rgle AE. EC, together with the □ of BC, will be = to the Gnomon *abc*, together with the □ HL. *Ax.* 2. *B.* 1.
   But the Gnomon *abc* & the □ HL form the □ of BE, (*Prep.* 1.).
9. Conſequently, the Rgle AEC + the □ of BC is = to the □ of BE. *Ax.* 1. *B.* 1.

Which was to be demonſtrated.



PROPOSITION VII. *THEOREM VII.*

IF a ſtraight line (BE) be divided into any two parts (BC, CE,); the ſquares of the whole line (BE) & of one of the parts as (CE), are equal to twice the rectangle contained by the whole (BE) & that part (EC), together with the ſquare of the other part (BC).

Hypotheſis.	Theſis.
BE *is a ſtraight line divided unequally in* C.	*The* □ *of* BE + *the* □ *of* CE, *are* = *to* 2 Rgles BEC + *the* □ *of* BC.

Preparation.

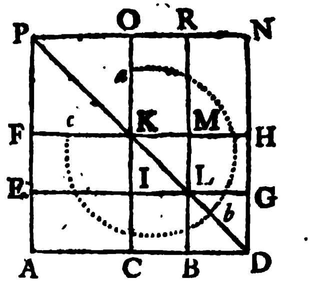
1. Upon BE, deſcribe the □ BN. *P.* 46. *B.* 1.
2. Thro' the point C, draw the ſtraight line CL plle to EN or BK. *P.* 31. *B.* 1.
3. Draw the diagonal EK. *Poſ.* 1.
4. Thro' the point G, draw the ſtraight line FH plle to EB or NK. *P.* 31. *B.* 1.

DEMONSTRATION.

BECAUSE the figure BN is a ſquare (*Prep.* 1.). *P.* 4. *B.* 2.

1. The Rgles about the diagonal CF, HL, are □. *Cor.* 1.
2. Namely CF the □ of CE, & HL the □ of BC; becauſe HG = BC. *P.* 34. *B.* 1.
   But the Rgle BG being = to the Rgle NG (*P.* 43. *B* 1.); if the □ CF be added to both.
3. The Rgle BF will be = to the Rgle NC. *Ax.* 2. *B.* 1.
4. Conſequently, twice the Rgle BF is = to the Rgles BF & NC.
   And becauſe the Rgles BF, NC, are = to the Gnomon *abc* together with the □ CF.
5. This Gnomon *abc* together with the □ CF, will be alſo double of the Rgle BF. *Ax.* 1. *B.* 1.
   But the Rgle BF is = to the Rgle contained by BE, EC, becauſe EF = EC (*Arg.* 1.).
6. Wherefore, the Gnomon *abc* together with the □ CF is = to twice the Rgle contained by BE . EC. *Ax.* 1. *B.* 1.
   If the □ HL which is = to the □ of BC (*Arg.* 2.) be added to both.
7. The Gnomon *abc* + the □ CF + the □ HL will be = to twice the Rgle BE . EC + the □ of BC. *Ax.* 2. *B.* 1.
   Since then the Gnomon *abc* + the □ HL are = to the □ of BE, and the □ CF is the □ of CE (*Arg.* 2.).
8. It is manifeſt that the □ of BE + the □ of CE, are = to 2 Rgles BEC + the □ of BC. *Ax.* 1. *B.* 1.

Which was to be demonſtrated.



PROPOSITION VIII. *THEOREM VIII.*

IF a straight line (AB) be divided into any two parts (AC, CB,); four times the rectangle contained by the whole line (AB) & one of the parts (BC), together with the square of the other part (AC), is equal to the square of the straight line (AD), which is made up of the whole (AB), & the part produced (BD) equal to the part (BC).

Hypothesis.	Thesis.
AB *is a straight line divided in* C, & *produced to* D, *so that* BD = BC.	*Four times the* Rgle ABC + *the* □ *of* AC *are* = *to the* □ *of* AD.

Preparation.

1. Upon AD, describe the □ AN. — *P.* 46. *B.* 1.
2. Thro' the points B & C, draw BR & CO plle to DN or AP. — *P.* 31. *B.* 1.
3. Draw the diagonal DP. — *Pos.* 1.
4. Thro' the points L & K, draw GE & HF plle to DA or NP. — *P.* 31. *B.* 1.

DEMONSTRATION.

BECAUSE the figure AN is a square (*Prep.* 1.).

1. The Rgles about the diagonal CH, ER, FO, are squares. — *P.* 4. *B.* 2. *Cor.* 1.

And because in the □ CH, the side CD is bisected in B (*Hyp.*).

2. The Rgles BG, CL, LH, IM, are four equal squares. — *P.* 4. *B.* 2. *Cor.* 2.
3. And the □ CH is = to four times the □ CL.

Moreover, because ER is a square (*Arg.* 1.).

4. The Rgle EK is = to the Rgle KR. — *P.* 43. *B.* 1.

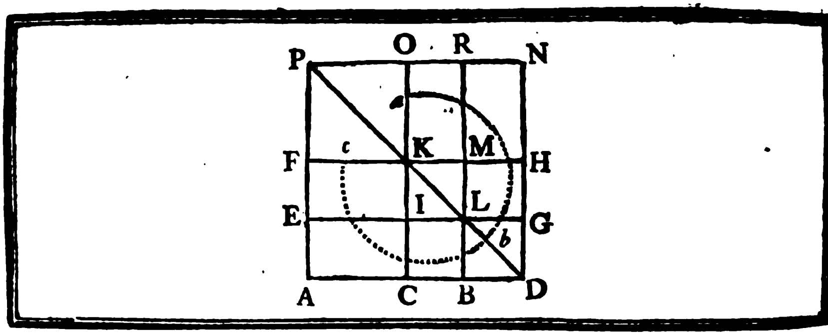
But since IK = IC (*Arg.* 2.), & CO plle to AP (*Prep.* 2.).

5. The Rgle AI is = to the Rgle EK. — *P.* 36. *B.* 1.
6. Consequently, the Rgle AI is also = to the Rgle KR. — *Ax.* 1. *B.* 1.

Likewise, because KM = MH (*Arg.* 2.), & HF plle to NP (*Prep.* 4.).

7. The Rgle KR is = to the Rgle MN. — *P.* 36. *B.* 1.
8. Wherefore, the Rgles AI, EK, KR, MN, are = to one another. — *Ax.* 1. *B.* 1.

L

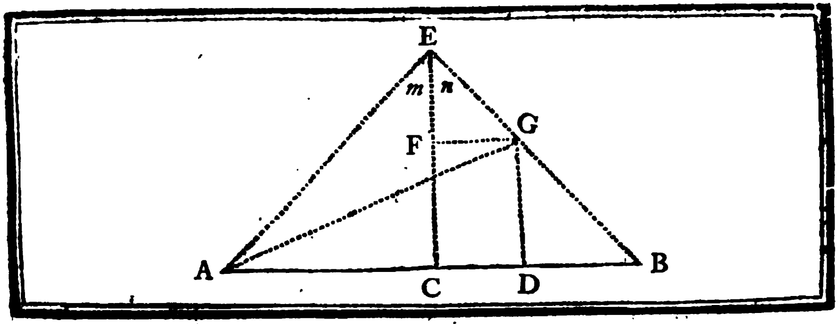


9. Consequently, their sum is = to four times the Rgle AI.
If the □ CH which is = to four times the □ CL (*Arg*. 3) be added to both.

10. The Gnomon *abc* which results on one side, is = to four times the Rgle AI & to four times the □ CL, *i. e.* to four times the Rgle AL, the Rgle AI + the □ CL being = to the Rgle AL. *Ax*. 2. *B*. 1.
Adding to both the □ of AC, which is = to the □ FO, because AC = FK (*P*. 34. *B*. 1.).

11. Four times the Rgle AL & the □ of AC will be = to the □ AN. *Ax*. 2. *B*. 1.
But the Rgle AL is = to the Rgle contained by AB, BC, because BC = BL (*Arg*. 2.), & the □ AN is = to the □ of AD (*Prep*. 1.).

12. Wherefore, four times the Rgle ABC + the □ of AC, are = to the □ of AD. *Ax*. 1. *B*. 1.



PROPOSITION IX. *THEOREM IX.*

IF a ſtraight line (AB) be divided into two equal parts (AC, CB,), & into two unequal parts (AD, DB,); the ſquares of the two unequal parts (AD, DB,) are together double of the the ſquare of the half (AC) of the whole line (AB) & of the ſquare of the part (CD) between the points of ſection (C & D).

Hypotheſis.	Theſis.
AB *is a ſtraight line divided equally in* C *&* *unequally in* D.	*The* □ *of* AD + *the* □ *of* DB, *are double of the* □ *of* AC + *the* □ *of* CD.

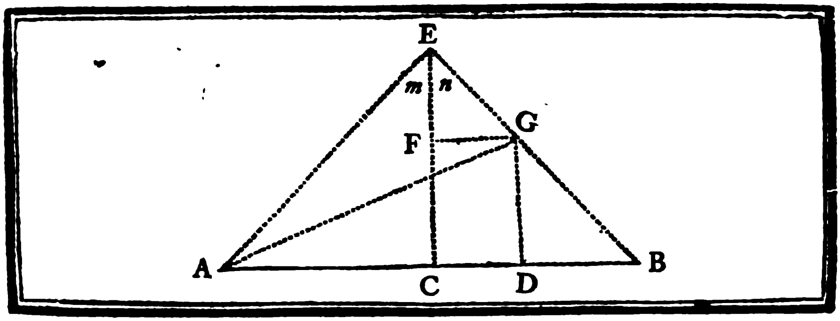
Preparation.

1. At the point C in the line AB, erect the ⊥ CE. *P.* 11. *B.* 1.
2. Make CE = to AC or BC. *P.* 3. *B.* 1.
3. From the points A & B to the point E, draw AE, BE. *Poſ.* 1.
4. Thro' the points D & G, draw the ſtraight lines DG & GF plle to CE & AB. *P.* 31. *B.* 1.

DEMONSTRATION.

BECAUSE CE is = to AC (*Prep.* 2.).

1. The ∀ CAE is = to the ∀ *m*. *P.* 5. *B.* 1.
   But the ∀ ECA is a ∟ (*Prep.* 1.).
2. Wherefore, the two other ∀ CAE & *m* together, make alſo a ∟. *P.* 32. *B.* 1.
3. Conſequently, each of them is half a ∟; becauſe they are = to one another (*Arg.* 1.).
   It may be proved after the ſame manner that:
4. Each of the ∀ CBE & *n* is half a ∟.
5. Conſequently, the whole ∀ *m* + *n* is = to a ∟. *Ax.* 2. *B.* 1.
   Again, ∀ *n* being half a ∟ (*Arg.* 4.), & ∀ EFG a ∟; being = to its interior oppoſite one ECB (*P.* 29. *B.* 1.), which is a ∟, (*Prep.* 1.).
6. The ∀ EGF is alſo half a ∟. *P.* 32. *B.* 1.
7. Conſequently, EF is = to FG. *P.* 6. *B.* 1.
   It is proved in the ſame manner that:
8. The ∀ BGD is = to half a ∟, & DG = DB.
   Since then the □ of AE is = to the □ of AC together, with the □ of CE (*P.* 47. *B.* 1.), & AC = CE (*Prep.* 2.).
9. The □ of AE is double of the □ of AC.



For the fame reafon :

10. The □ of EG is double of the □ of FG, *i. e.* of the □ of CD, becaufe FG = CD. *P.* 34. *B.* 1.

11. Confequently, the □ of AE & the □ of EG taken together, are double of the □ of AC & of the □ of CD. *Ax.* 2. *B.* 1.
And becaufe the □ of AE & the □ of EG taken together, are = to the □ of AG (*P.* 47. *B.* 1. & *Arg.* 5.).

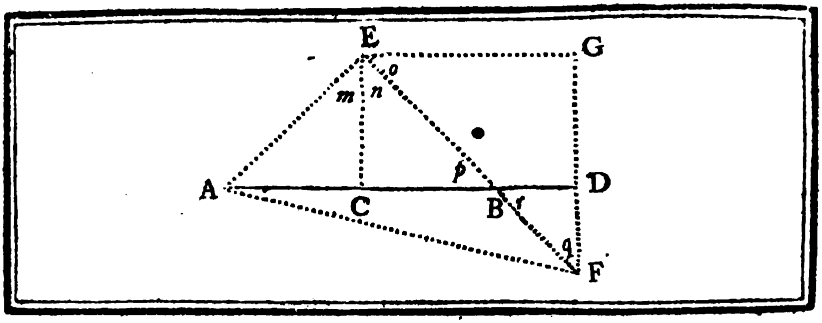
12. The □ of AG is alfo double of the □ of AC & of the □ of CD. *Ax.* 1. *B.* 1.
But ∠ ECA being = to a ∟ (*Prep.* 1.), & ∠ GDC = to ∠ ECA, (*P.* 29. *B.* 1.).

13. The □ of AG is = to the □ of AD & to the □ of DG. *P.* 47. *B.* 1.

14. Or the □ of AG is = to the □ of AD & to the □ of DB taken together, becaufe DB is = to DG (*Arg.* 8.).

15 Wherefore, the □ of AD & the □ of DB taken together, are double of the □ of AC & of the □ CD ; or the □ of AD + the □ of DB, are double of the □ of AC + the □ of CD. *Ax.* 1. *B.* 1.

Which was to be demonftrated.



PROPOSITION X. *THEOREM X.*

IF a ſtraight line (AB) be biſected in (C) & produced to any point (D), the ſquare of the whole line thus produced (AD) & the ſquare of the part of it produced (BD), are together double of the ſquare of the half (AC) of the whole line (AB), & of the ſquare of the line (CD) made up of the half (CB) & the part produced (BD).

Hypotheſis.	Theſis.
AB *is a ſtraight line biſected in* C *and produced to the point* D,	*The* □ *of* AD + *the* □ *of* BD, *are double of the* □ *of* AC + *the* □ *of* CD.

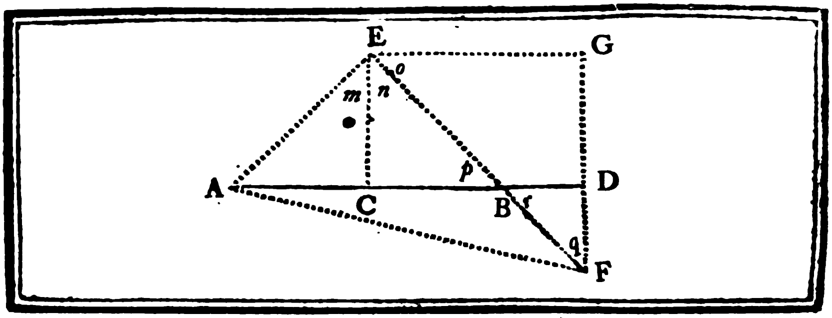
Preparation.

1. At the point C in the line AB, erect the ⊥ CE. — *P.* 11. *B.* 1.
2. Make CE = AC or BC. — *P.* 3. *B.* 1.
3. From the points A & B to the point E, draw AE & BE. — *Poſ.* 1.
4. Thro' the points E & D, draw EG, DG, plle to AD & CE, — *P.* 31. *B.* 1.
   and produce DG until it meets EB produced, in F. — *Poſ.* 2.

DEMONSTRATION.

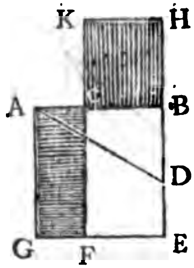
BECAUSE in the △ ACE the ſide AC is = to CE (*Prep.* 2.).

1. The ∀ CAE is = to ∀ *m*. — *P.* 5. *B.* 1.
   But ∀ ACE is a ∟ (*Prep.* 1.).
2. Hence each of the ∀ CAE & *m* is half a ∟. — *P.* 32. *B.* 1.
   It is proved in the ſame manner that:
3. Each of the ∀ *p* & *n* is half a ∟.
4. Conſequently, ∀ *m* + *n* will be = to ∟. — *Ax.* 2. *B.* 1.
   Moreover, ∀ *p* being half a ∟ (*Arg.* 3.).
5. The ∀ *r* will be alſo half a ∟. — *P.* 15. *B.* 1.
   But the ∀ BDF being a ∟ (*P.* 29. *B.* 1.), becauſe it is the alternate of ∀ ECD which is a ∟ (*Prep.* 1.).
6. The ∀ *q* is alſo half a ∟. — *P.* 32. *B.* 1.
7. Conſequently, the ſide BD is = to the ſide DF. — *P.* 6. *B.* 1.
   Likewiſe, ∀ *q* being half a ∟ (*Arg.* 6.), & ∀ G a ∟, as being diagonally oppoſite to ∀ ECD (*P.* 34. *B.* 1.).
8. The ∀ *o* is half a ∟. — *P.* 32. *B.* 1.
9. Therefore EG = GF. — *P.* 6. *B.* 1.



Alſo AC being = to CE (*Prep.* 2.). { *P.* 46. *B.* 1.
10. The □ of AC is = to the □ of CE. *Cor.* 3.
11. Conſequently, the □ of AC & of CE are double of the □ of AC.
And thoſe □ of AC & CE being = to the □ of AE (*P.* 47. *B.* 1.).
12. The □ of AE will be alſo double of the □ of AC. *Ax.* 6. *B.* 1.
It is proved after the ſame manner that:
13. The □ of EF is double of the □ of EG; *i. e.* of the □ of CD, becauſe EG = CD. *P.* 34. *B.* 1.
14. Conſequently, the □ of AE together with the □ of EF, are double of the □ of AC & of the □ of CD.
But the □ of AE & the □ of EF being = to the □ of AF, (*P.* 47. *B.* 1.).
15. The □ of AF is double of the □ of AC & of the □ of CD.
And this ſame □ of AF being alſo = to the □ of AD & to the □ of DF (*P.* 47. *B.* 1.), or of BD, ſince DF = BD (*Arg.* 7.).
16. It follows, that the □ of AD + the □ of BD, are double of the □ of AC + the □ of CD.

Which was to be demonſtrated.



PROPOSITION. XI. *PROBLEM I.*

TO divide a given ſtraight line (AB) into two parts, ſo that the rectangle contained by the whole (BA) & one of the parts (AC) ſhall be equal to the ſquare of the other part (CB).

Given	Sought
The ſtraight line AB.	*The point of interſection* C, *ſuch that the* Rgle BAC *ſhall be* = to *the* ☐ *of* CB.

Reſolution.

1. Upon the ſtraight line AB, deſcribe the ☐ AE. *P.* 46. *B.* 1.
2. Biſect the ſide BE in D, & draw thro' the point D to the point A the ſtraight line DA. *P.* 10. *B.* 1. *Poſ.* 1.
3. Upon EB produced, take DH = DA. *P.* 3. *B.* 1.
4. Upon the ſtraight line BH, deſcribe the ☐ CH. *P.* 46. *B.* 1.
5. And produce the ſide KC to F. *Poſ.* 2.

DEMONSTRATION.

BECAUSE the ſtraight line BE is biſected in D & produced to the point H.

1. The Rgle EH . HB + the ☐ of BD is = to the ☐ of DH. *P.* 6. *B.* 2.
2. And this ☐ of DH is = to the ☐ of DA, becauſe DH = DA (*Ref.* 3.). { *P.* 46. *B.* 1. *Cor.* 3.
3. Conſequently, the Rgle EH . HB + the ☐ of BD is = to the ☐ of DA. *Ax.* 1. *B.* 1.

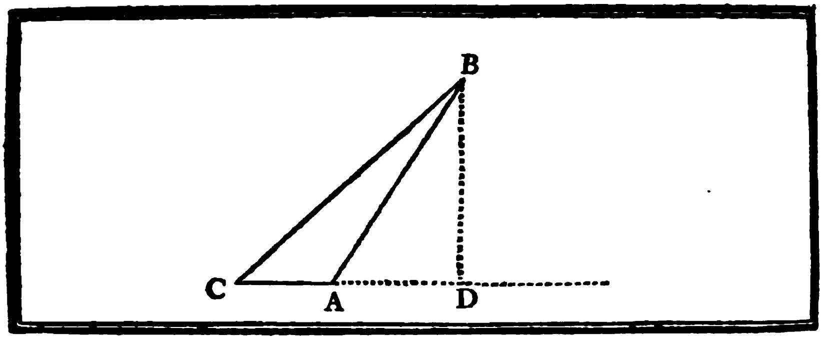
   But this ſame ☐ of DA is = to the ☐ of AB + the ☐ of BD (*P.* 47. *B.* 1.).
4. Wherefore, the Rgle EH . HB + the ☐ of BD is = to the ☐ of AB + the ☐ of BD. *Ax.* 1. *B.* 1.

   Therefore if the ☐ of BD be taken away from both ſides.
5. The Rgle EH . HB will be = to the ☐ of AB. *Ax.* 3. *B.* 1.

   And if from the Rgle EH . HB which is = to the Rgle FH (*Reſ.* 4. 5.) and from the ☐ of AB which is = to the ☐ AE (*Reſ.* 1.) the Rgle FB be taken away.
6. There will remain the ☐ CH = to the Rgle GC. *Ax.* 3. *B.* 1.

   This ☐ CH being therefore = to the ☐ of BC (*Reſ.* 4.), & the Rgle GC = to the Rgle BA . AC ; becaue AG = AB (*Reſ.* 1.).
7. It follows, that the ſtraight line AB is divided in C, ſo that the Rgle BAC is = to the ☐ of CB. *Ax.* 1. *B.* 1.

Which was to be done.



PROPOSITION XII. *THEOREM XI.*

IN any obtuse angled triangle (CBA); if a perpendicular be drawn from one of the acute angles (B) to the opposite side (CA) produced; the square of the side (BC) subtending the obtuse angle (A), is greater than the squares of the sides (AB, CA,) containing the obtuse angle, by twice the rectangle contained by the side (CA), upon which when produced the perpendicular falls, & the straight line (AD) intercepted between the perpendicular & the obtuse angle (A).

Hypothesis.	Thesis.
I. CBA *is an obtuse angled* △.	*The* □ *of* BC *is* = *to the* □ *of* AB + *the* □ *of* AC. + 2 Rgles CAD.
II. BD *the* ⊥ *drawn from the vertex of the* ∀B *to the opposite side* CA *produced.*	

DEMONSTRATION.

BECAUSE the straight line CD is divided into two parts CA, AD, (*Hyp.* 2.).

1. The □ of CD is = to double the Rgle CA . AD together with the □ of CA & of AD. *P.* 4. *B.* 2.

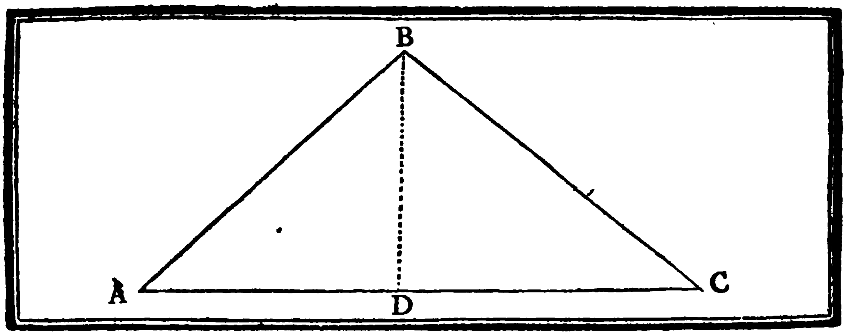
   Therefore if the □ of BD be added to both sides.

2. The □ of CD + the □ of BD, will be = to double the Rgle CA . AD + the □ of CA + the □ of AD + the □ of BD. *Ax.* 2. *B.* 1.

   But the □ of CD together with the □ of BD is = to the □ of BC, and the □ of AD together with the □ of BD is = to the □ of AB, (*P.* 47. *B.* 1.).

3. Consequently, the □ of BC is = to double the Rgle CAD + the □ of CA + the □ of AB. *Ax.* 1. *B.* 1.

Which was to be demonstrated.



PROPOSITION XIII. *THEOREM XII.*

IN every acute angled triangle (CBA); the ſquare of the ſide (BA) ſubtending one of the acute angles (C), is leſs than the ſquares of the ſides (CB, CA,) containing that angle, by twice the rectangle contained by one of thoſe ſides (AC) & the ſtraight line (CD) intercepted between the perpendicular (BD) let fall upon it from the oppoſite angle (B), & the acute angle (C).

Hypotheſis.	Theſis.
I. CBA *is an acute angled* △.	*The* □ *of* BA + *twice the* Rgle ACD
II. BD *the* ⊥ *let fall upon* AC *from the oppoſite angle* B.	*is* = *to the* □ *of* CA + *the* □ *of* CB.

DEMONSTRATION.

BECAUSE the ſtraight line CA is divided into two parts CD, DA, *(Hyp. 2.)*.

1. The □ of CA together with the □ of CD is = to twice the Rgle AC . CD together with the □ of AD. *P. 7. B. 2.*

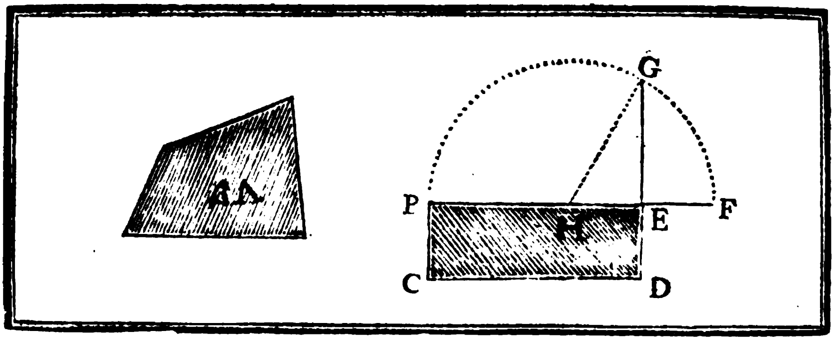
Therefore if the □ of DB be added to both ſides:

2. The □ of CA + the □ of CD + the □ of DB will be = to twice the Rgle AC . CD + the □ of AD + the □ of DB. *Ax. 2. B. 1.*

But the □ of CD + the □ of DB is = to the □ of CB, & the □ of AD + the □ of DB is = to the □ of BA *(P. 47. B. 1.)*.

3. Wherefore the □ of BA + twice the Rgle ACD is = to the □ of CA + the □ of CB. *Ax. 1. B. 1.*

Which was to be demonſtrated.



PROPOSITION XIV. *PROBLEM II.*

TO describe a square that shall be equal to a given rectilineal figure (A).

Given	Sought
The rectilineal figure A.	*The construction of a square* = *to a given rectilineal figure* A.

Resolution.

1. Describe the Rgle Pgr CE = to the figure A. *P.* 45. *B.* 1.
2. Produce the side BE, & make EF = to ED. *P.* 3. *B.* 1.
3. Bisect the straight line BF in H. *P.* 10. *B.* 1.
4. From the center H at the distance HB, describe the ⊙ BGF. *Pos.* 3.
5. Produce the side DE, until it cuts the ○ BGF in G. *Pos.* 1.

Preparation.

From the point H to the point G, draw the straight line HG. *Pos.* 1.

DEMONSTRATION.

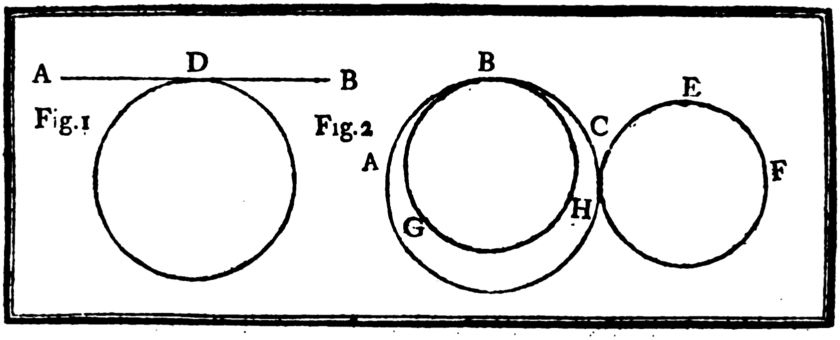
BECAUSE BF is divided equally in H & unequally in E (*Ref.* 3 & 2.).

1. The Rgle BE . EF together with the □ of HE is = to the □ of HF. *P.* 5. *B.* 2.
2. And because HF = HG (*D.* 15. *B.* 1.), the □ of HF = the □ of HG, the Rgle BE . EF + the □ HE is = to the □ of HG. { *P.* 46. *B* 1. *Cor.* 3.
   But the □ of HG being = to the □ HE + the □ of EG (*P.* 47. *B.* 1.).
3. The Rgle BE . EF + the □ of HE is also = to the □ of HE + the □ of EG. *Ax.* 1. *B.* 1.
   Therefore, if the □ of HE be taken away from both sides:
4. The Rgle BE . EF will be = to the □ of EG. *Ax.* 3. *B.* 1.
   And this Rgle BE EF being moreover = to the Rgle BE . ED; because EF = ED (*Ref.* 2.).
5. The Rgle BE . ED will be also = to the □ of EG. *Ax.* 1. *B.* 1.
   But the Rgle BE . ED is = to the given figure A (*Ref.* 1.).
6. Consequently, the □ of EG will be also = to this given figure A. *Ax.* 1. *B.* 1.

Which was to be done.

REMARK.

IF the *point* H *falls upon the point* E, *the straight lines* BE, EF, ED, *will be each equal to* EG, & *the* Rgle Pgr CE *itself, will be the square sought* (*Cor.* 1. & 3. *of P.* 46. *B.* 1.).



DEFINITIONS.

I.

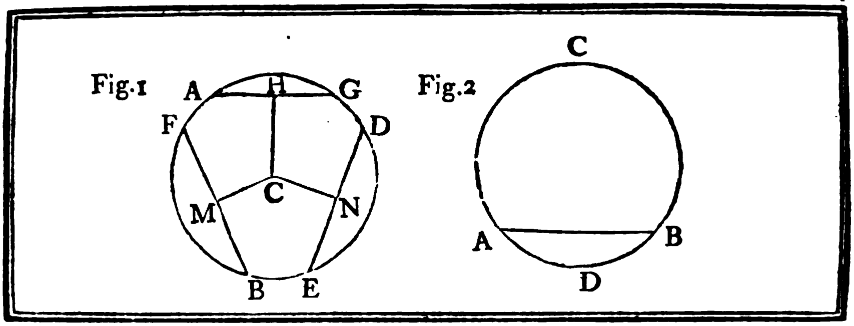
A Straight line (ADB) is said to *touch a circle* when it meets the circle & being produced does not cut it. *Fig.* 1.

II.

Circles are said to *touch one another* when their circumferences (ABC, CEF, or ABC, GBH) meet but do not cut one another. *Fig.* 2.

III.

Two circles touch each other externally, *when one* (CEF) *falls without the other* (ABC): *but* two circles touch each other internally, *when one* (GBH) *falls within the other* (ABC). *Fig.* 2.



DEFINITIONS.

IV.

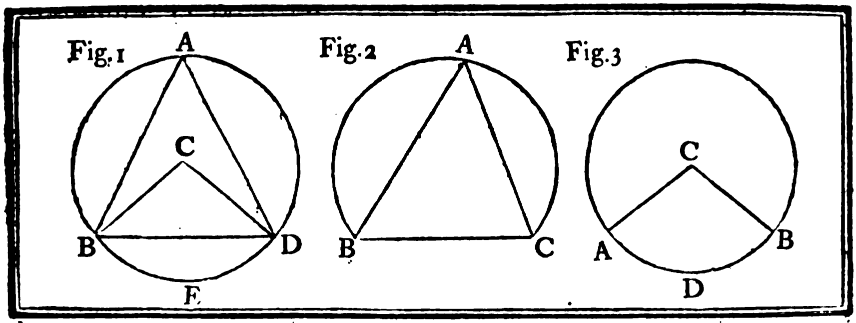
THE diſtance of a ſtraight line (FB) from the center of a circle, *is the perpendicular* (CM) *let fall from the center of the circle* (C) *upon this ſtraight line* (FB); for which reaſon *two ſtraight lines* (FB, DE,) *are ſaid to be equally diſtant from the center of a circle*, when the perpendiculars (CM, CN,) let fall upon thoſe lines (FB, DE,) from the center (C), are equal. *Fig.* 1.

V.

And *a ſtraight line* (AG) *is ſaid to be farther from the center of the circle than* (BF *or* ED), when the perpendicular (CH) drawn to this line from the center (C), is greater than (CM or CN). *Fig.* 1.

VI.

The angle of a ſegment, is the angle (CAB or DAB) formed by the arch (CA or DA) of the ſegment (ACB or ADB) & by its chord (AB). *Fig.* 2.



DEFINITIONS.

VII.

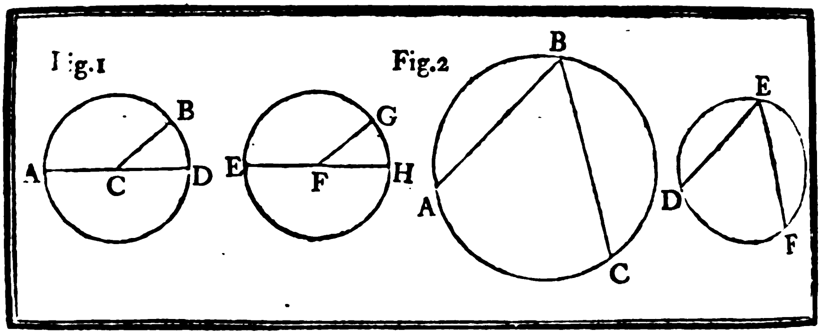
AN angle in a ſegment, is the angle (BAC) contained by two ſtraight lines (AB, AC,) drawn from any point (A) of the arch of the ſegment, to the extremities (B & C) of the chord (BC) which is the baſe of the ſegment. *Fig.* 2. *When the ſtraight lines* (AB, AD,) *are drawn from a point* (A) *in the circumference of the circle, the angle* (BAD) is an angle at the circumference: *but when the ſtraight lines* (CB, CD,) *are drawn from the center, the angle* (BCD) *is* an angle at the center. *Fig.* 1.

VIII.

An angle is ſaid *to inſiſt or ſtand upon the arch of a circle*, when the ſtraight lines (AB, AD, or CB, CD,) which form this angle (BAD, or BCD,), are drawn; either from the ſame point (A) in the circumference; or from its center (C), to the extremities (B & D) of the arch (BED). *Fig.* 1.

IX.

A ſector of a circle, is the figure contained by two rays (CA, CB,) & the arch (ADB) between thoſe two rays. *Fig.* 3.



AXIOMS.

I.

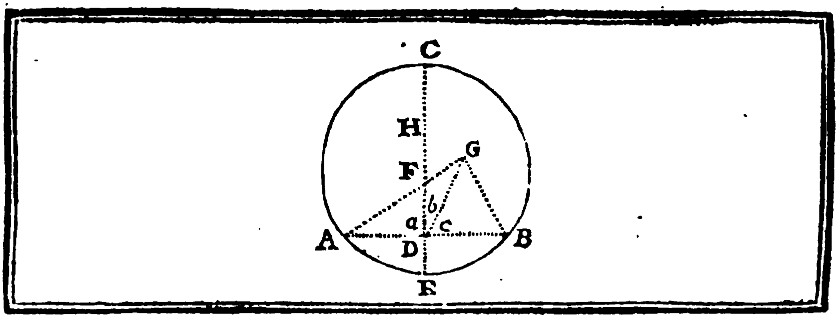
EQUAL circles (ABD, FGH,), are thoſe of which the diameters (AD, EH,) or the rays (CB, FG,) are equal. *Fig.* 1.

If the circles be applied to one another, ſo that their centers coincide, when their rays are equal, the circles muſt likewiſe coincide.

II.

Similar ſegments of circles (ABC, DEF,), are thoſe in which the angles (ABC, DEF,), are equal. *Fig.* 2.

Circles are ſimilar figures. If then the two ſegments (ABC, DEF,) *be taken away by ſubſtituting the equal angles* (ABC, DEF,), *thoſe ſegments are ſimilar.*



PROPOSITION I. *PROBLEM I.*

TO find the center (F) of a given circle (ACBE).

Given	Sought
The ⊙ ACBE.	*The center* F *of this* ⊙.

Resolution.

1. Draw the chord AB.	*Pof.* 1.
2. Bisect it in the point D.	*P.* 10. *B.* 1.
3. At the point D in AB, erect the ⊥ DE & produce it to E.	*P.* 11. *B.* 1.
4. Bisect CE in F.	*P.* 10. *B.* 1.

The point F will be the center sought of the given ⊙ ACBE.

Demonstration.

If not,

Some other point as H or G taken in the line, or without the line EC, will be the center sought of the ⊙ ACBE.

Case I.

Suppose the center to be in EC at a point H different from F.

BECAUSE the center of the ⊙ is in the line EC, at a point H different from F (*Sup.* 1.).

1. The rays HE & HC are = to one another. *D.* 15. *B.* 1.
   But FE being = to FC (*Ref.* 4.) & HC < FC (*Ax.* 8. *B.* 1.).
2. HC will be also < FE, & a fortiori < HE.
3. Therefore HE is not = to HC.
4. Consequently, the point H taken in the line EC different from the point F, cannot be the center of the ⊙ ACBE.

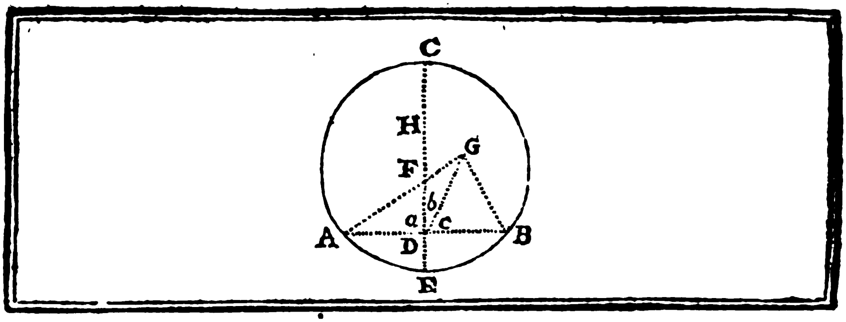
Case II.

Suppose the center to be without the line EC in the point G.

Preparation.

Draw from the center G, the straight lines GA, GD, GB. *Pof.* 1.

BECAUSE in the △ AGD, DGB, the side GA is = to side GB (*Prep.* & *D.* 15. *B.* 1.), the side GD common to the two △, & the base AD = to the base DB (*Ref.* 2.).

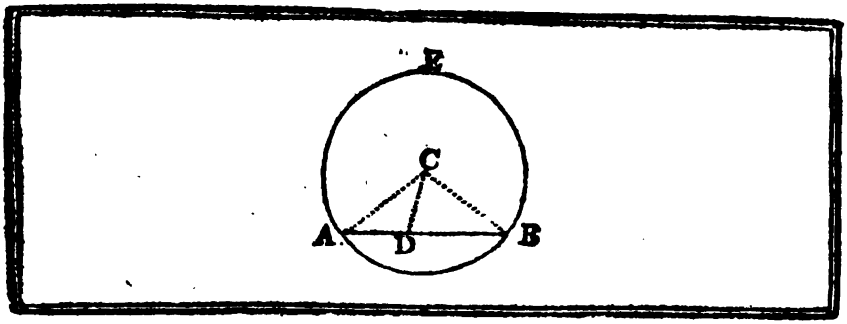


1. The adjacent ∀ a + b & c to which the equal sides GA, GB, are opposite, are = to one another. *P.* 8. *B.* 1.
2. Therefore ∀ a + b is a ∟. *D.* 10. *B.* 1.
   But ∀ a being also a ∟ (*Ref.* 3.).
3. It follows, that ∀ a + b is = to ∀ a, which is impossible. *Ax.* 8. *B.* 1.
4. Therefore the point G taken without the line EC, cannot be the center of the ⊙ ACBE.
   Consequently, since the center is not in the line EC, at a point H different from F (*Case* 1.) nor without the line EC in a point G (*Case.* 11.)
5. The center sought of the ⊙ ACBE, will be necessarily in F.

Which was to be done.

COROLLARY.

IF in a circle ACBE, *a chord* EC *bisects another chord* AB *at right angles; this chord* CE *is a diameter, & consequently passes thro' the center of the circle,* (*D.* 17. *B.* 1.).



PROPOSITION II. THEOREM I.

IF any two points (A & B) be taken in the circumference of a circle (AEB); the ſtraight line (AB) which joins them, ſhall fall within the circle.

Hypotheſis.	Theſis.
The two points A & B *are taken in the* ○ AEB.	*The ſtraight line* AB *falls within the* ⊙ AEB.

Preparation.

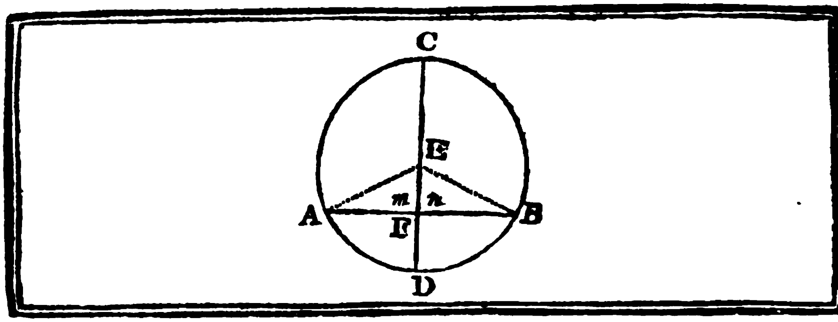
1. Find the center C of ⊙ AEB. — *P.* 1. *B.* 3.
2. Draw the ſtraight lines CA, CD, CB. — *Poſ.* 1.

DEMONSTRATION.

BECAUSE in the △ ACB, the ſide CA is = to the ſide CB, (*Prep.* 2. & *D.* 15. *B.* 1.).

1. The ∀ CAD, CBD, are = to one another. — *P.* 5. *B.* 1.
   But ∀ CDA being an exterior ∀ of △ CDB.
2. It is > than its interior CBD. — *P.* 16. *B.* 1.
   And becauſe the ∀ CBD is = to the ∀ CAD (*Arg.* 1.).
3. This ∀ CDA will be alſo > than ∀ CAD.
4. Conſequently, the ſide CA oppoſite to the greater ∀ CDA, is > the ſide CD oppoſite to the leſſer ∀ CAD. — *P.* 19. *B.* 1.
5. From whence it follows, that the extremity D of this ſide CD falls within the ⊙ AEB.
   And as the ſame may be demonſtrated with reſpect to any other point in the line AB.
6. It is evident that the whole line AB falls within the ⊙ AEB.

Which was to be demonſtrated.



PROPOSITION III. *THEOREM II.*

IF a diameter (CD) bisects a chord (AB) in (F); it shall cut it at right angles, & *reciprocally* if a diameter (CD) cuts a chord (AB) at right angles, it shall bisect it.

I.

Hypothesis.	Thesis.
CD *is a diameter of the* ⊙ ACBD, *which bisects* AB *in* F.	*The diameter* CD *is* ⊥ *upon the chord* AB.

Preparation.

Draw the rays EA, EB. *Pos.* 1.

DEMONSTRATION.

IN the △ AEF, BEF, the side EA is = to the side EB *(Prep. & D.* 15. *B.* 1.), the side EF is common to the two △, & the base AF is = to the base BF *(Hyp.)*.

1. Consequently, the adjacent ∀ *m* & *n*, to which the equal sides EA, EB, are opposite, are = to one another. *P.* 8. *B.* 1.
2. Wherefore, the straight line CD, which stands upon AB making the adjacent ∀ *m* & *n* = to one another, is ⊥ upon AB. *D.* 10. *B.* 1.

Which was to be demonstrated.

II.

Hypothesis.	Thesis.
CD *is a diameter of the* ⊙ ACBD, ⊥ *upon the chord* AB; *or which makes* ∀ *m* = ∀ *n*.	AF *is* = *to* FB.

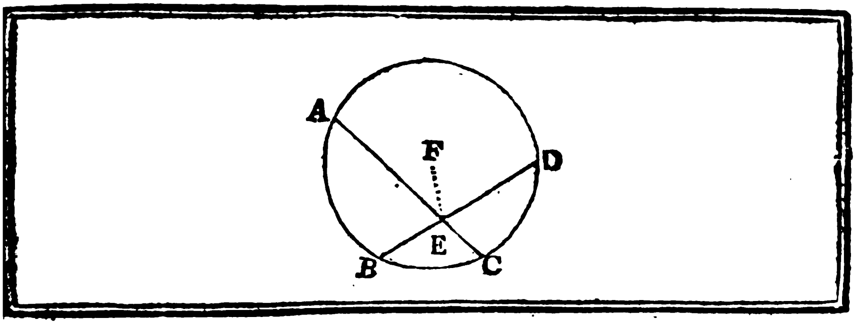
DEMONSTRATION.

THE sides EA, EB, of the △ AEB being = to one another *(Prep.* & *D.* 15. *B.* 1.).

1. The ∀ EAF, EBF, will be also = to one another. *P.* 5. *B.* 1.

   Since then in the △ AEF, BEF, the ∀ EAF, EBF, are = *(Arg.* 1.), as also the ∀ *m* & *n* *(Hyp.)*, & the side EF common to the two △.
2. The base AF will be = to the base FB. *P.* 26. *B.* 1.

Which was to be demonstrated.



PROPOSITION IV. *THEOREM III.*

IF in a circle (ADCB) two chords (AC, DB,) cut one another, they are divided into two unequal parts.

Hypothesis.
The two chords AC, DB, *of the* ⊙ ADCB *cut one another in the point* E.

Thesis.
These chords are divided into two unequal parts.

DEMONSTRATION.

IF not,
The chords AC, DB, bisect one another.

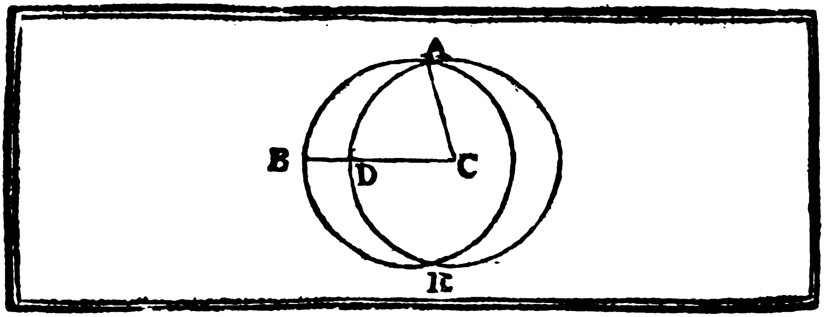
Preparation.

From the center F to the point E, draw the portion of the diameter FE. *Pos.* 1.

BECAUSE the diameter, or its part FE, bisects each of the chords AC, DB, of the ⊙ ADCB *(Sup.)*.

1. This straight line FE is ⊥ upon each of the chords AC, DB. *P.* 3. *B.* 1.
2. Consequently, the ∀ FEB, FEA, are = to one another; which is impossible. { *Ax.*10.*B.*1. *Ax.* 8. *B.* 1.
3. Wherefore, the two chords AC, DB, are divided into two unequal parts.

Which was to be demonstrated.



PROPOSITION V. *THEOREM IV.*

IF two circles (ABE, ADE,) cut one another, they ſhall not have the ſame center (C).

Hypotheſis.	Theſis.
ABE, ADE, *are two* ⊙ *which cut one another in the points* A & E.	*Theſe two* ⊙ *have different centers.*

DEMONSTRATION.

IF not,
The circles ABE, ADE, have the ſame center C.

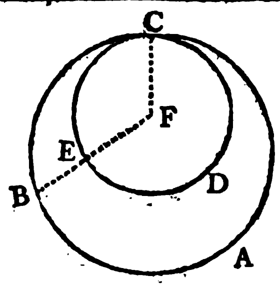
Preparation.

1. From the point C to the point of ſection A, draw the ray CA.
2. And from the ſame point C, draw the ſtraight line CB; which cuts the two ⊙ in D & B. } *Poſ.* 1.

BECAUSE the ſtraight lines CA, CD, are drawn from the center C to the ○ ADE (*Prep.* 1. & 2.).

1. Theſe ſtraight lines CA, CD, are = to one another. *D.* 15. *B.* 1.
   It is proved in the ſame manner, that:
2. The ſtraight lines CA, CB, are = to one another.
3. Conſequently, CB will be = to CD; which is impoſſible. *Ax.* 8. *B.* 1.
4. Therefore, the two circles ABE, ADE, have not the ſame center.

Which was to be demonſtrated.



PROPOSITION VI. *THEOREM V.*

IF two circles (BCA, ECD,) touch one another internally in (C); they ſhall not have the ſame center (F).

Hypotheſis.	Theſis.
The ⊙ ECD *touches the* ⊙ BCA *internally in* C.	*Thoſe two* ⊙ *have different centers.*

DEMONSTRATION.

If not,
The ⊙ BCA, ECD, have the ſame center F.

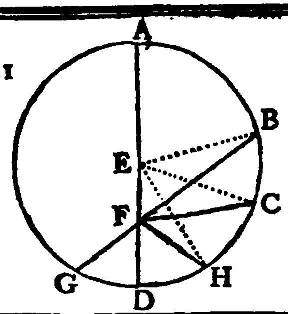
Preparation.

Draw the rays FB, FC. *Poſ.* 1.

BECAUSE the point F is the center of the ⊙ BCA *(Sup.)*.
1. The rays FB, FC, are = to one another.
   Again, the point F being alſo the center of ⊙ ECD *(Sup.)*
2. The rays FE, FC, are = to one another. } *D.* 15. *B.* 1.
3. Conſequently, FB = FE *(Ax.* 1. *B.* 1.); which is impoſſible. *Ax.* 8. *B.* 1.
4. Wherefore, the two ⊙ BCA, ECD, have not the ſame center.

Which was to be demonſtrated.

Fig. 1



PROPOSITION VII. *THEOREM VI.*

IF any point (F) be taken in a circle (AHG) which is not the center (E), of all the ſtraight lines (FA, FB, FC, FH,) which can be drawn from it to the circumference, the greateſt is (FA) in which the center is, & the part (FD) of that diameter is the leaſt, & of any others, that (FB or FC) which is nearer to the line (FA) which paſſes thro' the center is always greater than one (FC or FH) more remote, & from the ſame point (F) there can be drawn only two ſtraight lines (FH, FG,), that are equal to one another, one upon each ſide of the ſhorteſt line (FD).

Hypotheſis.	Theſis.
I. *The point* F *taken in the* ⊙ AHG *is not the center* E.	I. FA *is the greateſt of all the ſtraight lines which can be drawn from the point* F *to the* ○ AHG.
II. *The ſtraight line* FA, *drawn from the point* F, *paſſes thro' the center* E *of the* ⊙ AHG.	II. FD *is the leaſt.*
	III. *And of any others* FB *or* FC *which is nearer to* FA *is* $>$ FC *or* FH *more remote.*
III. *And the ſtraight lines* FB, FC, FH, *are drawn from the point* F *to the* ○ AHG.	IV. *From the point* F *there can be drawn only two* $=$ *ſtraight lines* FH, FG, *one upon each ſide of the ſhorteſt* FD.

I. Preparation.

Draw the rays EB, EC, EH, &c. *Fig.* 1.

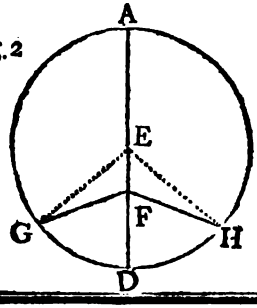
DEMONSTRATION.

1. THE two ſides FE $+$ EB of the △ FEB are $>$ the third FB. *P.* 20. *B.* 1.
But EB is $=$ to EA *(D.* 15. *B.* 1.).
2. Therefore, FE $+$ EA, or FA is $>$ FB.
It is proved in the ſame manner that:
3. The ſtraight line FA, is the greateſt of all the ſtraight lines drawn from the point F to the ○ AHG.

Which was to be demonſtrated. I.

4. Again, the two ſides FE $+$ FH of the △ FEH are $>$ the third EH. *P.* 20. *B.* 1.
And ED being $=$ to EH *(D.* 15. *B.* 1.).

Fig. 2



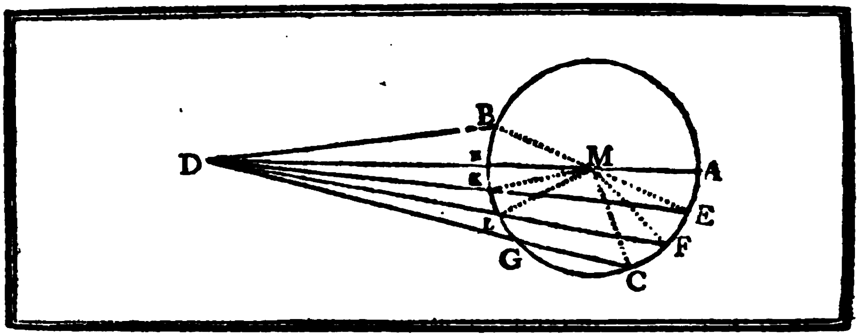
5. The ſtraight lines FE + FH are alſo > ED.
Therefore, taking away from both ſides the part FE:
6. The ſtraight line FH will be > FD; or FD < FH. *Ax.* 5. *B.* 1.
It is proved in the ſame manner that:
7. The ſtraight line FD, which is the produced part of FA, is the leaſt of all the ſtraight lines drawn from the point F to the ○ AHG.
Which was to be demonſtrated. II.

Moreover, the ſide FE being common to the two △ FEB, FEC, the ſide EB = the ſide EC (*D.* 15. *B.* 1.), & the ∀ FEB > ∀ FEC (*Ax.* 8. *B.* 1.).
8. The baſe FB will be > the baſe FC. *P.* 24. *B.* 1.
For the ſame reaſon:
9. The ſtraight line FC is > FH.
10. Conſequently, the ſtraight line FB or FC which is nearer the line FA, which paſſes thro' the center, is > FC or FH more remote.
Which was to be demonſtrated. III.

II. Preparation. Fig. 2.

1. Make ∀ FEG = to ∀ FEH, & produce EG until it meets the ○ AHG. *P.* 23. *B.* 1.
2. From the point F to the point G, draw the ſtraight line FG. *Poſ.* 1.

Then, EF being common to the two △ FEH, FEG, the ſide EH = the ſide EG (*D.* 15. *B.* 1.), & the ∀ FEH = to the ∀ FEG (II. *Prep.* 1.).
11. The baſe FH will be = to the baſe FG. *P.* 4. *B.* 1.
But becauſe any other ſtraight line, different from FG, is either nearer the line FD, or more remote from it, than FG.
12. Such a ſtraight line will be alſo < or > FG (*Arg.* 10.).
13. Wherefore, from the ſame point F, there can be drawn only two ſtraight lines FH, FG, that are = to one another, one upon each ſide of the ſhorteſt line FD.
Which was to be demonſtrated. IV.



PROPOSITION VIII. *THEOREM VII.*

IF a point (D) be taken without a circle (BGCA), & straight lines (DA, DE, DF, DC,) be drawn from it to the circumference, whereof one (DA) passes thro' the center (M); of those which fall upon the concave circumference, the greatest is that (DA) which passes thro' the center; & of the rest, that (DE or DF) which is nearer to that (DA) thro' the center, is always greater than (DF or DC) the more remote: but of those (DH, DK, DL, DG,) which fall upon the convex circumference, the least is that (DH) which produced passes thro' the center: & of the rest, that (DK or DL) which is nearer to the least (DH) is always less than (DL or DG) the more remote: & only two equal straight lines (DK, DB,) can be drawn from the point (D) unto the circumference, one upon each side of (DH) the least.

Hypothesis.

I. *The point* D *is taken without a* ⊙ BGCA *in the same plane.*

II. *The straight lines* DA, DE, DF, DC, *are drawn from this point to the concave part of the* ⊙ BGCA.

III. *And those straight lines cut the convex part in the points* H, K, L, G.

Thesis.

I. DA *which passes thro' the center* M *is the greatest of all the straight lines* DA, DE, DF, DC.

II. DE *or* DF, *which is nearer to* DA *is* > DF *or* DC, *the more remote.*

III. DH *which when produced passes thro' center* M *is the least of all the straight lines* DH, DK, DL, DG.

IV. DK *or* DL *which is nearer to the line* DH, *is* < DL *or* DG *the more remote.*

V. *From the point* D *only two equal straight lines* DK, DB, *can be drawn, one upon each side of* DH *the least.*

I. Preparation.

Draw the rays ME, MF, MC, MK, ML.

DEMONSTRATION.

1. THE two sides DM + ME of the △ DME are > the third DE. *P.* 20. *B.* 1. And because ME = MA (*D.* 15. *B.* 1.).

2. DM + MA or DA will be > DE.
It is demonſtrated after the ſame manner that:
3. The ſtraight line DA, which paſſes thro' the center M, is > any other ſtraight line drawn from the point D to the concave part of the ⊙ BGCA. Which was to be demonſtrated I.

Moreover, DM being common to the two △ DME, DMF, ME = MF (*D.* 15. *B.* 1.), & ∀ DME > ∀ DMF (*Ax.* 8. *B.* 1.).
4. The baſe DE will be alſo > the baſe DF. *P.* 24. *B.* 1.
In like manner it may be ſhewn that:
5. The ſtraight line DF is > DC, & ſo of all the others.
6. Conſequently, the ſtraight lines DE or DF, which is nearer the line DA, which paſſes thro' the center, is > DF or DC more remote. Which was to be demonſtrated. II.

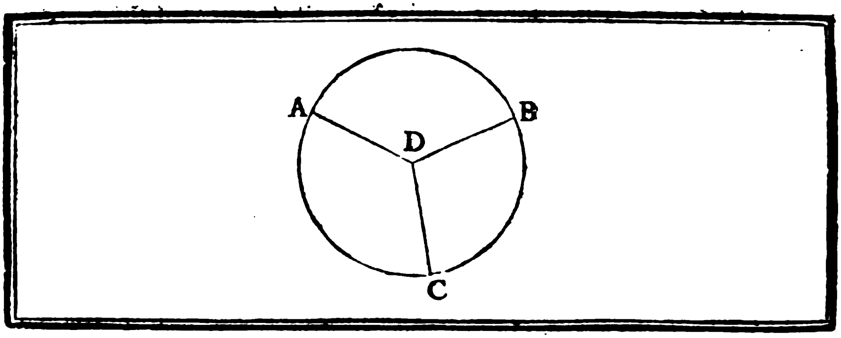
7. Again, the ſides DK + KM of the △ DKM are > the third DM. *P.* 20. *B.* 1.
If the equal parts MK, MH, (*D.* 15. *B.* 1.) be taken away.
8. The remainder DK will be > DH, or DH < DK.
It may be proved in the ſame manner, that:
9. The ſtraight line DH is < DL, & ſo of all the others.
10. Conſequently, the ſtraight line DH, which produced paſſes thro' the center M, is the leaſt of all the ſtraight lines drawn from the point D to the convex part of the ⊙ BGCA.
Which was to be demonſtrated. III.

Alſo, DK, MK, being drawn from the extremities D & M of the ſide DM of the △ DLM to a point K, taken within this △ (*Hyp.* 3.).
11. It follows, that DK + MK < DL + ML. *P.* 21. *B.* 1.
And taking away the equal parts MK, ML, (*D.* 15. *B.* 1.).
12. The ſtraight line DK will be < DL.
In like manner it may be ſhewn, that:
13. The ſtraight line DL is < DG, & ſo of all the others.
14. Conſequently, the ſtraight lines DK or DL, which are nearer the line DH, which produced paſſes thro' the center, are < DL or DG the more remote. Which was to be demonſtrated. IV.

II. Preparation.

1. Make ∀ DMB = ∀DMK, & produce MB 'till it meets the ○. *P.* 23. *B.* 1.
2. From the point D to the point B, draw the ſtraight line DB. *Poſ.* 1.

Then, the ſide DM being common to the two △ DKM, DBM, the ſide MK = the ſide MB (*D.* 15. *B.* 1.), & ∀ DMK = ∀ DMB (II. *Prep.* 1.).
15. The baſe DK will be = to the baſe DB. *P.* 4. *B.* 1.
But becauſe any other ſtraight line different from DB, is either nearer the line DH or more remote from it, than DB.
16. Such a ſtraight line will be alſo < or > BD (*Arg.* 14.).
17. Wherefore, from the point D, only two = ſtraight lines DK, DB, can be drawn, one upon each ſide of DH.
Which was to be demonſtrated. V.



PROPOSITION IX. *THEOREM VIII.*

IF a point (D) be taken within a circle (ABC), from which there fall more than two equal ſtraight lines (DA, DB, DC,) to the circumference; that point is the center of the circle.

Hypotheſis.
From the point D, *taken within a* ⊙ ABC, *there fall more than two equal ſtraight lines* DA, DB, DC, *to the* ○ ABC.

Theſis.
The point D *is the center of the* ⊙ ABC.

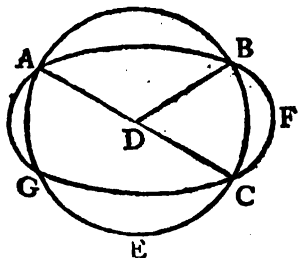
DEMONSTRATION.

IF not,
Some other point will be the center.

BECAUSE the point D is not the center *(Sup.)*, & from this point D there fall more than two equal ſtraight lines DA, DB, DC, to the ○ ABC *(Hyp.)*.

1. It follows, that from a point D, which is not the center, there can be drawn more than two equal ſtraight lines; which is impoſſible. *P. 7. B. 3.*
2. Conſequently, the point D is the center of the ⊙ ABC.

Which was to be demonſtrated



PROPOSITION X. *THEOREM IX.*

ONE circumference of a circle (ABCEG) cannot cut another (ABFCG) in more than two points (A & B).

Hypothesis.	Thesis.
The two ⊙ ABCEG, ABFCG, *cut one another.*	*They cut one another only in two points* A & B.

DEMONSTRATION.

If not,
They cut each other in more than two points, as A, B, C, &c.

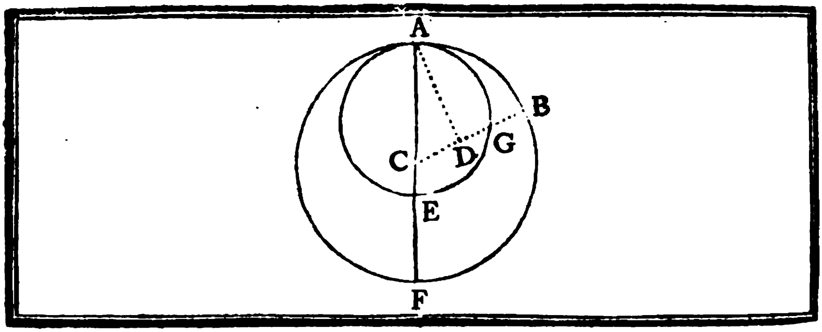
Preparation.

1. Find the center D of the ⊙ ABCEG. *P.* 1. *B.* 3.
2. From the center D to the points of section A, B, C, &c. draw the rays DA, DB, DC. *Pos.* 1.

BECAUSE the point D is taken within the ⊙ ABFCG, & that more than two straight lines DA, DB, DC, drawn from this point to the circumference of the ⊙ ABFCG, are equal to one another, (*Prep. I.* & *D.* 15. *B.* 1.).

1. The point D is the center of this ⊙. *P.* 9. *B.* 3.
   But this point D being also the center of the ⊙ ABCEG (*Prep.* 1.).
2. It would follow, that two ⊙ ABFCG, ABCEG, which cut one another, have a common center D; which is impossible. *P.* 5. *B.* 3.
3. Consequently, two ⊙ ABCEG, ABFCG, cannot cut one another in more than two points.

Which was to be demonstrated.



PROPOSITION XI. *THEOREM X.*

IF two circles touch each other internally in (A); the ſtraight line which joins their centers being produced, ſhall paſs thro' the point of contact (A).

Hypotheſis.
The ſtraight line CA *joins the centers of the two* ⊙ AGE, ABF, *which touch each other internally in* A.

Theſis.
This ſtraight line CA *being produced, paſſes thro' the point of contact* A *of thoſe two* ⊙.

DEMONSTRATION.

IF not,
The ſtraight line which joins the centers, will fall otherwiſe, as the ſtraight line CGB.

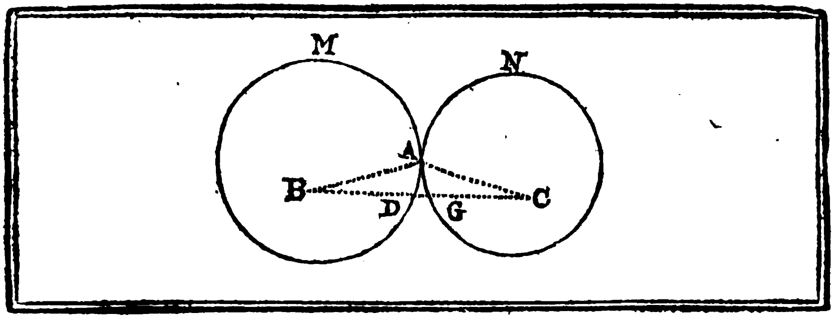
Preparation.

From the centers C & D to the point of contact A, draw the lines CA, DA. *Poſ.* 1.

BECAUSE in the △ CDA, the two ſides CD & DA taken together, are < the third CA (*P.* 20. *B.* 1.), & that CA = CB (*D.* 15. *B.* 1.).

1. The ſtraight lines CD + DA will be alſo > CB.
Therefore, if the common part CD be taken away from both ſides.
2. The ſtraight line DA will be > DB. *Ax.* 5. *B.* 1.
But the ſtraight line DA being = to DG (*Prep.* & *D.* 15. *B.* 1.).
3. DG will be alſo > DB, which is impoſſible. *Ax.* 8. *B.* 1.
4. Wherefore, the ſtraight line CA, which joins the centers of the ⊙ AGE, ABF, which touch each other internally, being produced, will paſs thro' the point of contact A.

Which was to be demonſtrated.



PROPOSITION XII. THEOREM XI.

IF two circles (DAM, GAN,) touch each other externally; the ſtraight line (BC), which joins their centers, ſhall paſs thro' the point of contact (A).

Hypotheſis.	Theſis.
The ſtraight line BC *joins the centers of the two* ⊙ DAM, GAN, *which touch each other externally in* A.	*This ſtraight line* BC *paſſes thro' the point of contact of the two* ⊙.

DEMONSTRATION.

If not,

This ſtraight line, which joins the centers, will paſs otherwiſe, as BDGC.

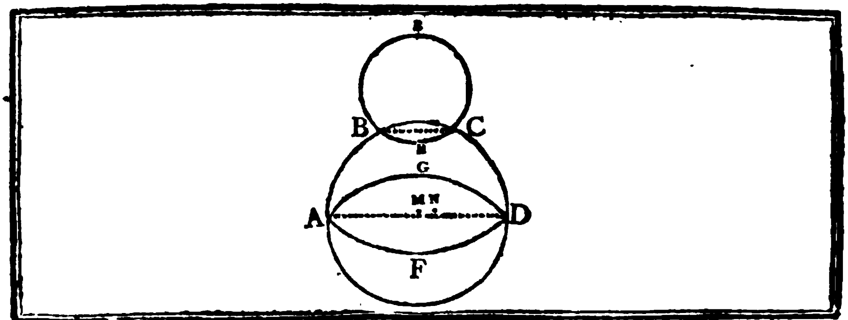
Preparation.

Draw from the centers B & C to the point of contact A, the rays BA, CA. *Poſ.* 1.

BECAUSE BA is = to BD, & CA = to CG (*D.* 15. *B.* 1.).

1. The ſtraight lines BA + CA are = to the ſtraight lines BD + CG. *Ax.* 2. *B.* 1.
   And if the part DC be added to the ſtraight lines BD + CG.
2. BD + DG + CG, or the baſe BC of the △ BAC is > the two ſides BA + CA, which is impoſſible. *P.* 20. *B.* 1.
3. Therefore, the ſtraight line BC, which joins the centers, will paſs thro' the point of contact A.

Which was to be demonſtrated.



PROPOSITION XIII. *THEOREM XII.*

TWO circles (ABCD, AGDF or ABCD, BECH,) which touch each other; whether internally; or externally: cannot touch in more points than one.

Hypothesis.	Thesis.
I. ⊙ ABCD *touches* ⊙ AGDF *internally.*	*The* ⊙ ABCD, AGDF, *or* ABCD,
II. ⊙ ABCD *touches* ⊙ BECH *externally.*	BECH, *touch only in one point.*

DEMONSTRATION.

If not,

1. Either the ⊙ ABCD, AGDF, touch each other internally in more points than one, as in A & in D.
2. Or the ⊙ ABCD, BECH, touch each other externally in more points than one, as in B & in C.

I. Preparation.

1. Find the centers M & N of the ⊙ ABCD, AGDF. *P.* 1. *B.* 3.
2. Thro' the centers, draw the line MN, & produce it to the O. *Pos.* 1. & 2.

BECAUSE MN joins the centers M & N of the two ⊙ ABCD, AGDF, (*Prep.* 2.) which touch on the inside (*Sup.* 1.).

1. This straight line will pass thro' the points of contact A & D. *P.* 11. *B.* 3.
   But AM is = to MD (*I. Prep.* 2. & *D.* 15. *B.* 1.).
2. Therefore, the straight line AM is > ND, & AN is much > ND. *Ax.* 8. *B.* 1.
   But since AN is = to ND (*I. Prep.* 2. & *D.* 15. *B.* 1.).
3. The line AN will be > ND & = to ND; which is impossible.
4. Consequently, two ⊙ ABCD, AGDF, which touch each other internally, cannot touch each other in more points than one.

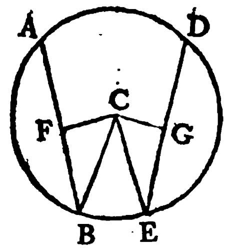
II. Preparation.

Thro' the points of contact B & C of the ⊙ ABCD, BECH, draw the straight line BC. *Pos.* 1.

BECAUSE the line BC joins the two points B & C in the O of the ⊙ ABCD, BECH, (*II. Prep.*).

1. This straight line will fall within the two ⊙ ABCD, BECH. *P.* 2. *B.* 3.
   But the ⊙ BECH touching externally the ⊙ ABCD (*Sup.* 2.).
2. BC, drawn in the ⊙ BECH, will fall without the ⊙ ABCD. *D.* 3. *B.* 3.
3. Consequently, BC will, at the same time, fall within the ⊙ ABCD (*Arg.* 1,), & without the same ⊙ (*Arg.* 2.); which is impossibe.
4. Wherefore, two ⊙ ABCD, BCEH, which touch each other externally, cannot touch each other in more points than one.

Which was to be demonstrated.



PROPOSITION XIV. *THEOREM XIII.*

IN a circle (ABED) the equal chords (AB, DE,) are equally distant from the center (C); & the chords (AB, DE,) equally distant from the center (C), are equal to one another.

Hypothesis. CASE I. Thesis.

The chords AB, DE, *are equal.* *They are equally distant from the center* C.

Preparation.

1. Find the center C of the ⊙ ABED. *P.* 1. *B.* 3.
2. Let fall upon the chords AB, DE, the ⊥ CF, CG. *P.* 12. *B.* 1.
3. From the center C to the points E & B, draw the rays CE, CB. *Pos.* 1.

DEMONSTRATION.

THE chords AB, DE, being = to one another *(Hyp.)* & bisected in F & G *(Prep.* 2. & *P.* 3. *B.* 3.).

1. Their halves FB, GE, are also equal. *Ax.* 7. *B.* 1.
2. Consequently, the □ of FB is = to the □ of GE. { *P.* 46. *B.* 1. *Cor.* 3.
   But because □ of CB = □ of CE *(Prep.* 3. & *P.* 46. *Cor.* 3.).
3. It follows, that □ of FB + □ of FC is = to □ of GE + □ of CG. { *P.* 47. *B.* 1. *Ax.* 1. *B.* 1.
   Therefore the equal □ of FB & of GE *(Arg.* 2.) being taken away.
4. The □ of FC will be = the □ of GC *(Ax.* 3. *B.* 1.), or FC = GC. { *P.* 46. *B.* 1. *Cor.* 3.
5. Consequently, the chords AB, DE, are equally distant from the center C of the ⊙ ABED. Which was to be demonstrated. *D.* 4. *B.* 3.

Hypothesis. CASE II. Thesis.

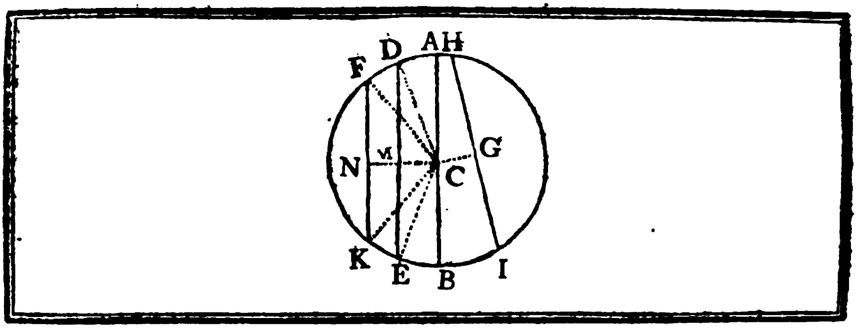
The chords AB, DE, *are equally distant from the center* C *of the* ⊙ ABED. *These chords are equal.*

DEMONSTRATION.

BECAUSE FC = GC *(Hyp.* & *D.* 4. *B.* 3.), & CB = CE *(Prep.* 3. & *D.* 15, *B.* 1.).

1. The □ of FC = the □ of CG, & the □ of CB = the □ of CE. { *P.* 46. *B.* 1. *Cor.* 3.
2. Consequently, □ of FC + □ of FB, = □ of CG + □ of GE. { *P.* 47. *B.* 1. *Ax.* 1. *B.* 1.
   Therefore, the equal □ of FC & of CG *(Arg.* 1.) being taken away.
3. The □ of FB will be = the □ of GE *(Ax.* 3. *B.* 1.) or FB = GE. { *P.* 46. *B.* 1. *Cor.* 3.
4. Consequently, FB, GE, being the semichords *(Prep.* 2. *P.* 3. *B.* 3.), the whole chords AB, DE, are also = to one another. *Ax.* 6. *B.* 1.

Which was to be demonstrated.



PROPOSITION XV. *THEOREM XIV.*

THE diameter (AB) is the greateſt ſtraight line in a circle (AIK); & of all others that (HI), which is nearer the diameter, is always greater than one (FK) more remote.

Hypotheſis.	Theſis.
I. AB *is the diameter of the* ⊙ AIK.	*I. The diameter* AB *is* $>$ *each of the chords* HI, FK.
II. The chord HI *is nearer the diameter than the chord* FK.	*II. The chord* HI *is* $>$ *the chord* FK.

Preparation.

1. From the center C let fall upon HI & FK the $\perp$ CG, CN. *P.* 12. *B.* 1.
2. From CN, the greateſt of thoſe $\perp$, take away a part CM $=$ to CG. *P.* 3. *B.* 1.
3. At the point M in CN, erect the $\perp$ DM & produce it to E. *P.* 11. *B.* 1.
4. Draw the rays CD, CF, CE, CK. *Poſ.* 1.

DEMONSTRATION.

BECAUSE the ſtraight lines CD, CE, CA, CB, are $=$ to one another *(Prep.* 4. & *D.* 15. *B.* 1.).

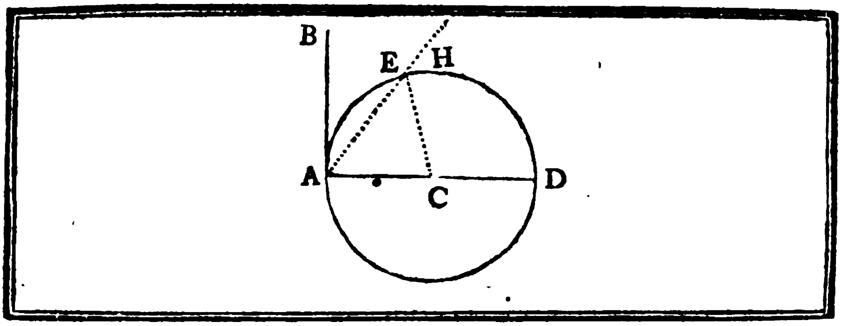
1. It follows, that CD $+$ CE is $=$ to CA $+$ CB or AB. *Ax.* 2. *B.* 1.
   But CD $+$ CE is $>$ DE *(P.* 20. *B.* 1.).
2. Wherefore, AB is alſo $>$ DE or $>$ HI, becauſe HI $=$ DE *(Prep.* 2.). { *D.* 4. *B.* 3. *P.* 14. *B.* 3.
3. It may be proved after the ſame manner, that AB is alſo $>$ FK.

Which was to be demonſtrated. I.

Moreover, the △ CDE, CFK, having two ſides CD, CE, $=$ to the two ſides CF, CK, each to each *(Prep.* 4. & *D.* 15. *B.* 1.), & the $\angle$ DCE $>$ $\angle$ FCK *(Ax.* 8. *B.* 1.).

4. The baſe DE will be $>$ the baſe FK. *P.* 24. *B.* 1.
5. And becauſe HI is $=$ to DE *(Prep.* 2.), HI is alſo $>$ FK. { *D.* 4. *B.* 3. *P.* 14. *B.* 3.

Which was to be demonſtrated. II.



PROPOSITION XVI. *THEOREM XV.*

THE ſtraight line (AB) perpendicular to the diameter of a circle (AHD) at the extremity of it (A), falls without the circle; & no ſtraight line can be drawn between this perpendicular (AB) & the circumference from the extremity, ſo as not to cut the circle; alſo the angle (HAD) formed by a part of the circumference (HEA) & the diameter (AD), is greater than any acute rectilineal angle; & the angle (HAB) formed by the perpendicular (AB) & the ſame part of the circumference (HEA), is leſs than any acute rectilineal angle.

Hypotheſis.	Theſis.
I. AB *is drawn perpendicular to the extremity* A *of the diametr.*	*I. The* ⊥ AB *falls without the* ⊙ AHD.
II. And makes with the arch HEA *the mixtilineal* ∀ HAB.	*II. No ſtraight line can be drawn between the* ⊥ AB *&* *the arch* HEA.
III. The diameter AD *makes with the ſame arch* HEA *the mixtilineal* ∀ HAD.	*III. The mixtilineal* ∀ HAD *is* $>$ *any acute rectilineal* ∀.
	IV. The mixtilineal ∀ HAB *is* $<$ *any acute rectilineal* ∀.

DEMONSTRATION.

I. IF not,

The ⊥ AB will fall within the ⊙ AHD, & will cut it ſomewhere in E, as AE.

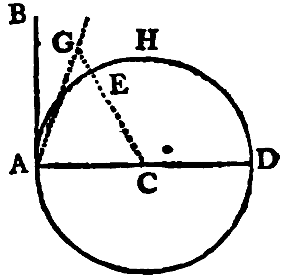
Preparation.

From the center C to the point of ſection E, draw the ray CE. *Poſ.* 1.

BECAUSE CA is = to CE (*D.* 15. *B.* 1.).

1. The ∀ CAE will be = to the ∀ CEA. *P.* 5. *B.* 1.
2. And becauſe the ∀ CAE is a ∟ (*Sup.*); ∀ CEA is alſo a ∟. *Ax.* 1. *B.* 1.
3. Wherefore, the two ∀ CAE + CEA, of the △ AEC will not be $<$ 2 ∟; which is impoſſible. *P.* 17. *B.* 1.
4. Therefore, the ⊥ AB falls without the circle.

Which was to be demonſtrated. I.



II. If not,

There may be drawn a straight line, as AG, between the ⊥ AB & the circumference of the ⊙ AHD.

Preparation.

From the center C, let fall upon AG, the ⊥ CG. *P. 12. B. 1.*

BECAUSE ∀ CGA is a ∟; & ∀ CAG < a ∟ *(Ax. 8. B. 1.)* as being but a part of the ∟ CAB *(Hyp. 1.)*.

1. It follows, that the side CA is > the side CG. *P. 19. B. 1.*
   But CA being = to CE *(D. 15. B. 1.)*.
2. The straight line CE will be also > CG; which is impossible. *Ax. 8. B. 1.*
3. Therefore, no straight line can be drawn between the ⊥ AB & the ○ of the ⊙ AHD.

Which was to be demonstrated. II.

III. & IV. If not,

There may be drawn a straight line, as AG, which makes with the diameter AD & with the ⊥ AB, an acute rectilineal ∀ GAD > the mixtilineal ∀ HAD, & an acute rectilineal ∀ GAB < the mixtilineal ∀ EAB.

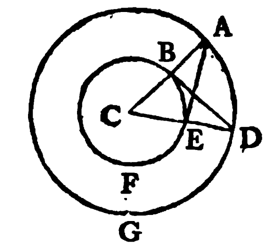
BECAUSE then the straight line AG, drawn from the extremity A of the diameter AD, makes with the diameter & with the ⊥ AB, an acute rectilineal ∀ GAD > the mixtilineal ∀ HAD, & a rectilineal ∀ GAB < the mixtilineal ∀ EAB *(Sup.)*.

1. This straight line AG will necessarily fall on the extremity A of the diameter AD, between the ⊥ AB & the circumference of the ⊙ AHD; which is impossible. *II. Case.*
2. Therefore, the mixtilineal ∀ HAD is >, & the mixtilineal ∀ HAB < any acute rectilineal ∀.

Which was to be demonstrated. III. & IV.

COROLLARY.

A Straight line, drawn at right angles to the diameter of a circle from the extremity of it, touches the circle only in one point.



PROPOSITION XVII. *PROBLEM II.*

FROM a given point (A) without a circle (BEF), to draw a tangent (AE) to this circle.

Given	Sought
The point A *without the* ⊙ BEF.	*The tangent* AE, *drawn from the point* A *to the* ⊙ BEF.

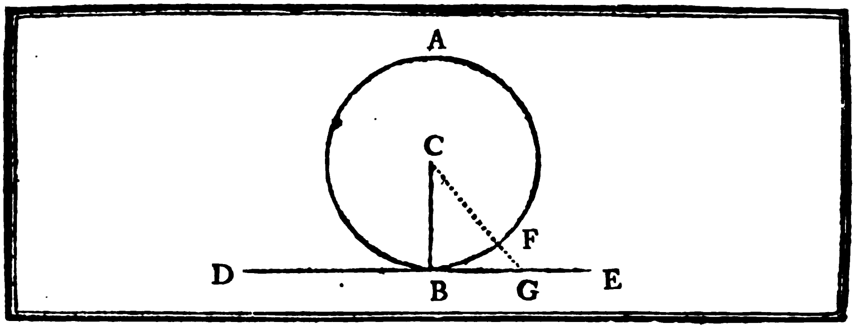
Resolution.

1. Find the center C of the ⊙ BEF, & draw CA. *P.* 1. *B.* 3.
2. From the center C at the diftance CA, defcrbe the ⊙ ADG. *Pof.* 3.
3. At the point B in the line CA, where it cuts the O BEF, erect the ⊥ BD. *P.* 11. *B.* 1.
4. From the center C to the point D, where the ⊥ BD cuts the O ADG, draw the ray CD. *Pof.* 1.
5. From the point A to the point E, where CD cuts the O BEF, draw the ftraight line AE, which will be the tangent fought.

DEMONSTRATION.

BECAUSE in the △ CBD, CEA, the fide CB is = to the fide CE, the fide CA = to the fide CD (*D.* 15. *B.* 1.), & the ∀ BCD common to the two △.

1. The ∀ CBD, CEA, oppofite to the equal fides CD, CA, are = to one another. *P.* 4. *B.* 1.
2. Wherefore, ∀ CBD being a ∟ (*Ref.* 3.), ∀ CEA will be alfo a ∟. *Ax.* 1. *B.* 1.
3. Confequently, the ftraight line AE, drawn from the given point A, is a tangent of the ⊙ BEF. { *P.* 16. *B.* 1. *Cor.* *D.* 1. *B.* 3.



PROPOSITION XVIII. *THEOREM XVI.*

IF a ſtraight line (DE) touches a circle (AFB) in a point (B); the ray (CB), drawn from the center to the point of contact (B), ſhall be perpendicular to the tangent (DE).

Hypotheſis.

I. *The ſtraight line* DE *touches the* ⊙ AFB *in the point* B.

II. *And the ray* CB *paſſes thro' the point of contact* B.

Theſis.

The ray CB *is* ⊥ *upon the tangent* DE.

DEMONSTRATION.

IF not,

There may be let fall from the center C, another ſtraight line CG ⊥ upon the tangent DE.

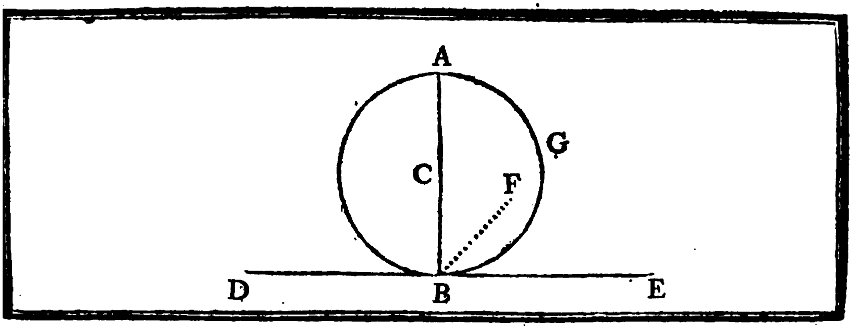
Preparation.

Let fall then from the center C upon the tangent DE, the ⊥ CG. *P.* 12. *B.* 1.

BECAUSE the ∠ BGC of the △ BCG is a ∟ *(Prep.)*.

1. The ∠ CBG will be < a ∟. *P.* 17. *B.* 1.
2. Conſequently, CB is > CG. *P.* 19. *B.* 1.
   And CF being = CB *(D.* 15. *B.* 1.).
3. The ſtraight line CF is alſo > CG; which is impoſſible. *Ax.* 8. *B.* 1.
4. Wherefore, the ray CB is ⊥ upon the tangent DE.

Which was to be demonſtrated.



PROPOSITION XIX, *THEOREM XVII.*

IF a ſtraight line (DE) touches a circle (AGB in B), & from the point of contact (B) a perpendicular (BA) be drawn to the touching line; the center (C) of the circle, ſhall be in that line,

Hypotheſis.	Theſis.
I. The ſtraight line DE *touches the* ⊙ AGB. *II. And* BA *is the* ⊥ *erected from the point of contact* B *in this line.*	*The ſtraight line* BA *paſſes thro' the center* C *of the* ⊙ AGB.

Demonstration.

If not,

The center will be in a point F without the ſtraight line BA.

Preparation.

Draw then from the point of contact B to the center F, the ſtraight line BF. *Poſ.* 1.

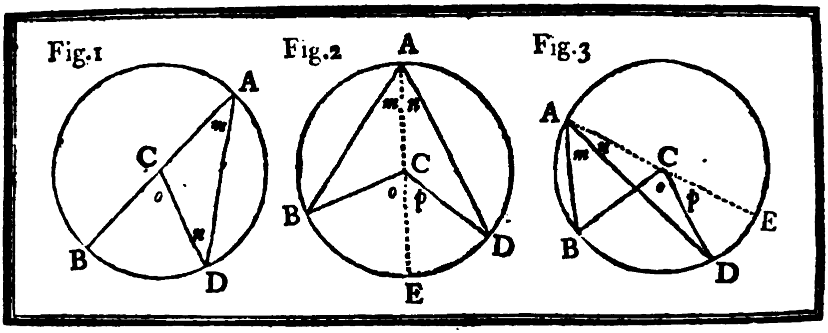
BECAUSE the ſtraight line BF is drawn from the point of contact B to the center F of the ⊙ AGB *(Prep.)*.

1. The ∀ FBE is a ∟. *P.* 18. *B.* 3.

   But ∀ ABE being alſo a ∟ *(Hyp.* 2.).

2. The ∀ ABE is = to the ∀ FBE; which is impoſſible *Ax.* 10. *B.* 1.
3. Wherefore, the center C will be neceſſarily in the ſtraight line BA. *Ax.* 8. *B.* 1.

Which was to be demonſtrated.



PROPOSITION XX. *THEOREM XVIII.*

THE angle (BCD) at the center of a circle, is double of the angle (BAD) at the circumference, when both angles stand upon the same arch (BD).

Hypothesis.

I. The ∀ BCD *is at the center &* ∀ BAD *at the* ○.
II. The sides BC, CD, *&* BA, AD, *of those* ∀, *stand upon the same arch* BD.

Thesis.

The ∀ BCD *at the center is double of the* ∀ BAD *at the* ○.

Demonstration.

CASE I.

If the center C, is in one of the sides AB of the ∀ at the ○ (*Fig.* 1.).

BECAUSE in the △ CAD the side CA is = to the side CD (*D.* 15. *B.* 1.).

1. The ∀ *m* is = to the ∀ *n*, & ∀ *m* + *n* is double of ∀ *m*. { *P.* 5. *B.* 1. *Ax.* 2. *B.* 1.
   But ∀ *o* is = to ∀ *m* + *n* (*P.* 32. *B.* 1.).
2. Therefore, ∀ *o* is double of ∀ *m*, or ∀ BCD is double of ∀ BAD. *Ax.* 6. *B.* 1.

CASE II.

If the center C falls within the ∀ at the ○ (*Fig.* 2.).

Preparation.

Draw the diameter ACE. *Pos.* 1.

IT may be proved as in the first case.

1. That the ∀ *o* is double of the ∀ *m*, & ∀ *p* double of the ∀ *n*.
2. From whence it follows, that ∀ *o* + *p* is double of the ∀ *m* + *n*, or ∀ BCD is double of the ∀ BAD. *Ax.* 8. *B.* 1.

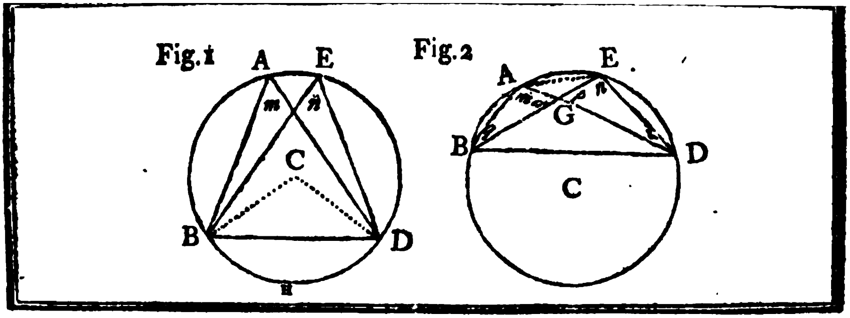
CASE III.

If the center C falls without the ∀ at the ○ (*Fig.* 3.).

THE diameter ACE being drawn, it is demonstrated as in the first case, that:

1. The ∀ *p* is double of the ∀ *n*, & ∀ *o* + *p* is double of the ∀ *m* + *n*.
   Therefore, the ∀ *p* being taken away from one side, & the ∀ *n* from the other.
2. The ∀ *o* will be double of the ∀ *m*, or ∀ BCDis double of ∀ BAD. *Ax.* 3. *B.* 1.

Which was to be demonstrated.



PROPOSITION XXI. *THEOREM XIX.*

THE angles (*m* & *n*) in the ſame ſegment of a circle (BAED), are equal to one another.

Hypotheſis.	Theſis.
The ∠ *m* & *n are in the ſame ſegment of the* ⊙ BAED.	∠ *m is* = *to* ∠ *n*.

DEMONSTRATION.

CASE I.

If the ſegment BAED is > the ſemi ⊙ (*Fig.* 1.).

Preparation.

1. Find the center C of the ⊙ BAED. *P.* 1. *B.* 3.
2. And draw the rays CB, CD. *Poſ.* 1.

BECAUSE ∠BCD is double of each of the ∠ *m* & *n* (*P.* 20. *B.* 3.).

1. It follows, that ∠ *m* is = to ∠ *n*. *Ax.* 7. *B.* 1.

CASE II.

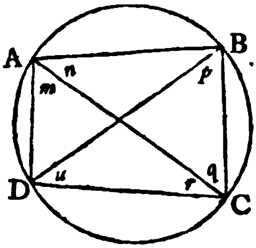
If the ſegment BAED is < the ſemi ⊙ (*Fig.* 2.).

Preparation.

Draw the ſtraight line AE. *Poſ.* 1.

1. THE three ∠ *m* + *o* + *q* of the △ BAG, are = to the three ∠ *p* + *n* + *r* of the △ GED. { *P.* 32. *B.* 1. *Ax.* 1. *B.* 1.
But ∠ *q* is = to ∠ *r* (*Caſe* 1.), & ∠ *o* = to ∠ *p* (*P.* 15. *B.* 1.).
Therefore, the ∠ *q* + *o* being taken away from one ſide, & their equals ∠ *p* + *r* from the other.
2. The remaining ∠ *m* & *n* will be = to one another, *Ax.* 3. *B.* 1.

Which was to be demonſtrated.



PROPOSITION XXII. *THEOREM XX.*

THE opposite angles (BAD, BCD, or ABC, ADC,) of any quadrilateral figure (DABC) inscribed in a circle, are together, equal to two right angles.

Hypothesis.	Thesis.
The figure DABC *is a quadrilateral figure inscribed in a* ⊙.	*The opposite* ∠ BAD + BCD, or ABC + ADC, *are* = *to* 2 ∟.

Preparation.

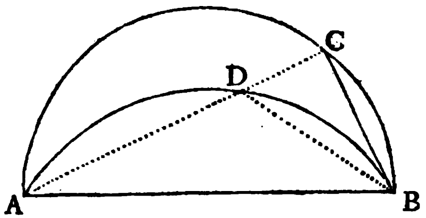
Draw the diagonals AC, BD. *Pos.* 1.

DEMONSTRATION.

BECAUSE the ∠ *u* + *n* are the ∠ at the O, in the same segment DABC.

1. These ∠ *u* & *n* are = to one another. *P.* 21. *B.* 3.
   It is proved in the same manner, that :
2. The ∠ *p* & *m* are = to one another.
3. Wherefore, the ∠ *u* + *p* are = to the ∠ *n* + *m* or to the ∠ BAD. *Ax.* 2. *B.* 1.
   Therefore, if the ∠ *r* + *q* or BCD be added to both sides.
4. The ∠ *u* + *p* + *(r* + *q)* are = to the ∠ BAD + BCD. *Ax.* 2. *B.* 1.
   But the three ∠ *u* + *p* + *(r* + *q)* of the △ DBC being = to 2 ∟ *(P.* 32. *B.* 1.).
5. The two opposite ∠ BAD + BCD of the quadrilateral figure DABC, are also = to 2 ∟. *Ax.* 1. *B* 1.
   It may be demonstrated after the same manner, that :
6. The ∠ ABC + ADC are = to 2 ∟.

Which was to be demonstrated



PROPOSITION XXIII. *THEOREM XXI.*

UPON the ſame ſtraight line (AB) & upon the ſame ſide of it, there cannot be two ſimilar ſegments of circles (ADB, ACB,) not coinciding with one another.

Hypotheſis.	Theſis.
The ſegments ADB, ACB, *of circles, are upon the ſame ſtraight line & upon the ſame ſide of it.*	*Theſe ſegments are diſſimilar*

DEMONSTRATION.

If not,

The ſegments ADB, ACB, upon the ſame chord AB, & upon the ſame ſide of it, are ſimilar.

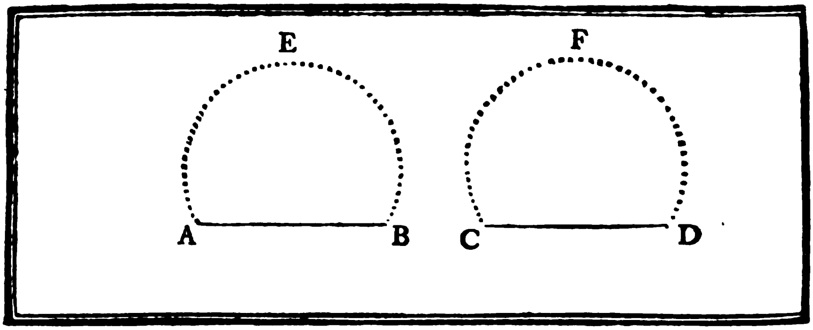
Preparation.

1. Draw any ſtraight line AC, which cuts the ſegments ADB, ACB, in the points D & C.
2. Draw the ſtraight lines BD, BC. } *Poſ.* 1.

BECAUSE the ∀ BDA, BCA, are contained in the ſimilar ſegments ADB, ACB, (*Hyp. & Prep.* 1. & 2.).

1. Theſe ∀ are = to one another. *Ax.* 2. *B.* 1.
2. Therefore, the exterior ∀ ADB of the △ BDC, will be = to its interior oppoſite one BCD; which is impoſſible. *P.* 16. *B.* 1.
3. Conſequently, there cannot be two ſimilar ſegments of ⊙ ADB, ACB, upon the ſame ſide of the ſame ſtraight line AB, which do not coincide.

Which was to be demonſtrated.



PROPOSITION XXIV. *THEOREM XXII.*

SIMILAR segments of circles (AEB, CFD,) subtended by equal chords (AB, CD,), are equal to one another.

Hypothesis.

I. The segments of ⊙ AEB, CFD, *are similar.*

II. These segments are subtended by equal chords AB, CD.

Thesis.

The segments AEB, CFD, *are* = *to one another.*

DEMONSTRATION.

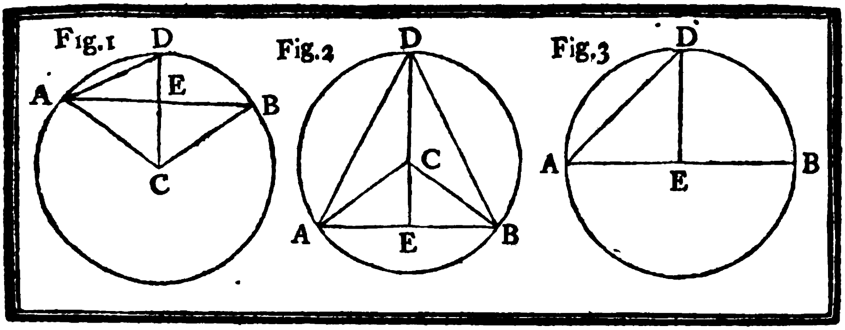
If not,

The segments AEB, CFD, are unequal.

BECAUSE the segment AEB is not = to the segment CFD (*Sup.*), & the chord AB is = to the chord CD (*Hyp.* 2.).

1. Upon the same straight line AB or its equal CD, there could be two similar segments of ⊙, AEB, CFD; which is impossible. *P. 23. B. 3.*
2. Therefore, these segments are = to one another.

Which was to be demonstrated.



PROPOSITION. XXV. *PROBLEM III.*

A Segment of a circle (ADB) being given; to describe the circle of which it is the segment,

Given	Sought
The segment of ⊙ ADB.	*The center* C *of the* ⊙, *of which* ADB *is the segment.*

Resolution.

1. Divide the chord AB into two equal parts in the point E. *P.* 10. *B.* 1.
2. At the poinnt E in AB, erect the ⊥ ED. *P.* 11. *B.* 1.
3. Draw the straight line AD, *Pos.* 1.
   And ∀ ADE will be >, or <, or = ∀ DAE.

CASE I. & II.

If ∀ ADE be either > or < ∀ DAE (*Fig.* 1. & 2.).

4. At the point A in DA, make ∀ DAC = to ∀ ADE. *P.* 23. *B.* 1.
5. Produce DE to C (*Fig.* 1.), & draw BC (*Fig.* 1. & 2.). *Pos.* 2. & 1.

DEMONSTRATION.

BECAUSE in the △ ADC the ∀ DAC is = to ∀ ADC (*Ref.* 4.).

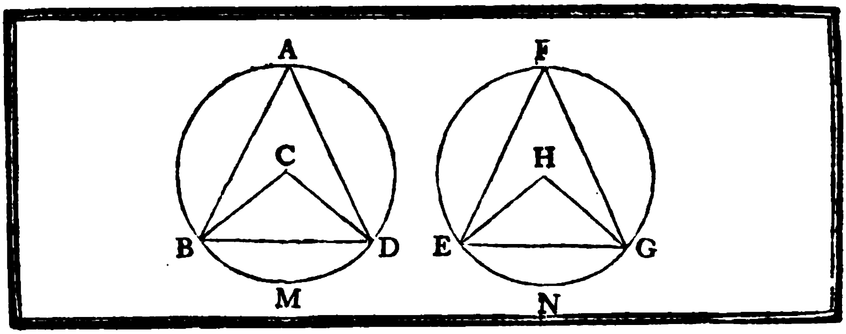
1. The side AC is = to the side DC. *P.* 5. *B* 1.
   But in the △ AEC, BEC, the side AE is = to the side EB, the side EC common to the two △, & the ∀ AEC = to the ∀ BEC (*Ref.* 2. & *Ax.* 10. *B.* 1.).
2. The base AC will be = to the base BC. *P.* 4. *B.* 1
3. Consequently, the three straight lines AC, DC, BC, drawn from the point C to the ○ ADB, are = to one another. *Ax.* 1. *B.* 1.
4. Wherefore, the point C is the center of the ⊙, of which ADB is the segment. *P.* 9. *B.* 3.

CASE III.

If ∀ ADE be = to ∀ DAE (*Fig.* 3.).

1. THEN the side AE is = to the side ED. *P.* 5. *B.* 1
2. Consequently, AE being = EB (*Ref.* 1.), the three straight lines AE, ED, EB, drawn from a point E to the ○ ADB, are = to one another. *Ax.* 1. *B.* 1.
3. From whence it follows, that the point E is the center of the ⊙ of which ADB is the segment. *P.* 9. *B.* 3

Which was to be demonstrated.



PROPOSITION XXVI. *THEOREM XXIII.*

IN equal circles (BADM, EFGN,), equal angles, whether they be at the centers as (C & H) or at the circumferences as (A & F), ſtand upon equal arches (BMD, ENG,).

Hypotheſis.

I. The ∀ C, H, *are* ∀ *at the centers, & equal.*
II. The ∀ A, F, *are* ∀ *at the* ○, *& equal.*
III. Theſe ∀ *are contained in the equal* ⊙ BADM, EFGN.

Theſis.

The arches BMD, ENG, *upon which theſe* ∀ *ſtand, are* = *to one another.*

Preparation.

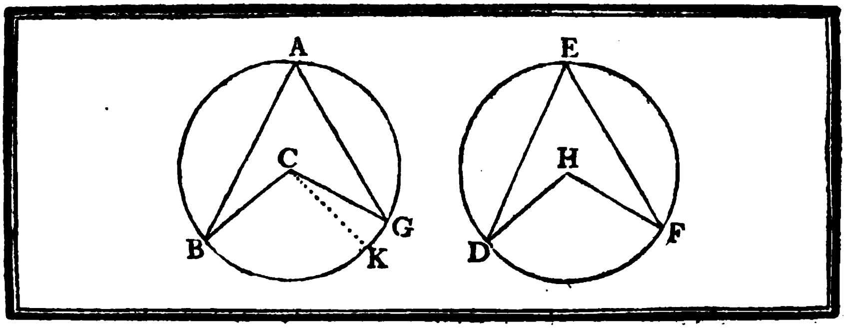
Draw the chords BD, EG.

DEMONSTRATION.

THE two ſides CB, CD, of the △ BCD being = to the two ſides HE, HG, of the △ EHG (*Hyp.* 3. & *Ax.* 1. *B.* 3.), & the ∀ C = to the ∀ H (*Hyp.* 2.).

1. The baſe BD will be = to the baſe EG. *P.* 4. *B.* 1.
   And becauſe ∀ A is = to ∀ F (*Hyp.* 1.).
2. The ſegment BAD is ſimilar to the ſegment EFG. *Ax.* 2. *B.* 3.
3. Wherefore, the baſe BD being = to the baſe EG (*Arg.* 1.), theſe ſegments will be = to one another. *P.* 24. *B.* 3.
   Therefore, if the equal ſegments BAD, EFG, (*Arg.* 3.) be taken away from the equal ⊙ BADM, EFGN, (*Hyp.* 3.).
4. The remaining arches BMD, ENG, will be alſo = to one another. *Ax.* 3. *B.* 1.

Which was to be demonſtrated.



PROPOSITION XXVII. THEOREM XXIV.

IN equal circles (BAG, DEF,) the angles, whether at the centers as (BCG & H) or at the circumferences as (A & E), which ſtand upon equal arches (BG, DF,) ; are equal to one another.

Hypotheſis.	Theſis.
I. The ⊙ BAG, DEF, *are* =, *as alſo their arches* BG, DF.	*I. The* ∀ BCG *&* H *at the centers, are* = *to one another.*
II. The ∀ BCG *&* H *at the centers, as alſo the* ∀ A *&* E *at the* ○, *ſtand upon* = *arches.*	*II. The* ∀ A *&* E *at the* ○, *are alſo* = *to one another.*

Demonstration.

If not,

The ∀ BCG & H at the centers will be unequal, & one, as BCG, will be > the other H.

Preparation.

At the point C in the line BC, make the ∀ BCK = to ∀ H. *P.* 23. *B.* 1.

1. THEREFORE the arch BK is = to the arch DF. *P.* 26. *B.* 3.

But the arch DF being = to the arch BG *(Hyp.* 2*)*

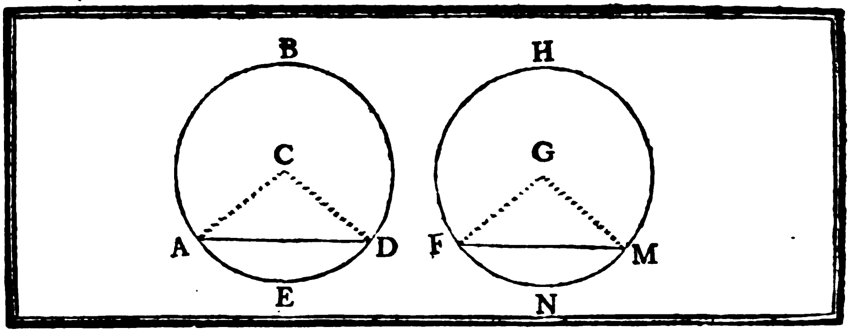
2. The arch BK will be alſo = to the arch BG ; which is impoſſible { *Ax.* 1. *B.* 1. *Ax.* 8. *B.* 1.
3. Conſequently, the ∀ BCG & H at the centers, are = to one another.

Which was to be demonſtrated. I.

And theſe ∀ being double of the ∀ A & F at the ○ *(P.* 20. *B.* 3.*)*.

4. Theſe ∀ A & E at the ○, are alſo = to one another. *Ax.* 7. *B.* 1.

Which was to be demonſtrated. II.



PROPOSITION XXVIII. *THEOREM XXV.*

IN equal circles (ABDE, FHMN,); the equal chords (AD, FM,) ſubtend equal arches (ABD, FHM or AED, FNM,).

Hypotheſis.	Theſis.
I. The ⊙ ABDE, FHMN, *are equal.*	*The chords* AD, FM, *ſubtend equal*
II. The chords AD, FM, *are equal.*	*arches* ABD, FHM *or* AED, FNM.

Preparation.

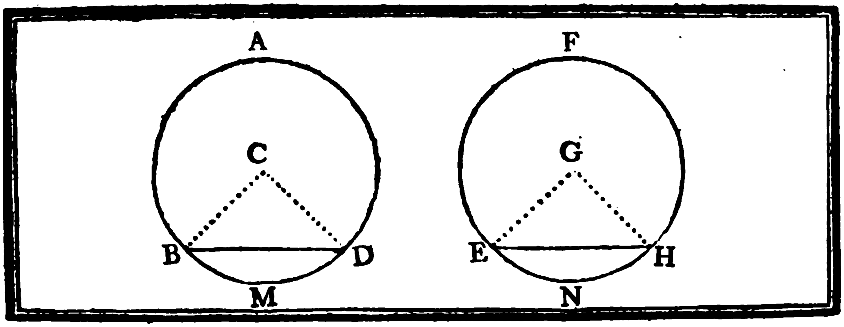
1. Find the centers C & G of the two ⊙ ABDE, FHMN. *P.* 1. *B.* 3.
2. Draw the rays CA, CD, alſo GF, GM. *Poſ.* 1.

DEMONSTRATION.

BECAUSE the ⊙ ABDE, FHMN, are equal (*Hyp.* 1.).

1. The ſides CA, CD, & GF, GM, of the △ ACD, FGM, are equal. *Ax.* 1. *B.* 3.
   And the chords AD, FM, being equal (*Hyp.* 2.).
2. The ∀ ACD, FGM, are = to one another. *P.* 8. *B.* 1.
3. Conſequently, the arches AED, FNM, ſubtended by the chords AD, FM, will be alſo = to one another. *P.* 26. *B.* 3.
4. And moreover, the whole ○ being equal (*Hyp.* 1.), the arches ABD, FHM, are alſo equal. *Ax.* 3. *B.* 1.

Which was to be demonſtrated.



PROPOSITION XXIX. *THEOREM XXVI.*

IN equal circles (BADM, EFHN,); equal arches (BMD, ENH,) are ſubtended by equal chords (BD, EH,).

Hypotheſis.	Theſis.
I. The ⊙ BADM, EFHN, *are equal.* *II. The arches* BMD, ENH, *are equal.*	*The chords* BD, EH, *which ſubtend theſe arches, are equal.*

Preparation.

1. Find the centers C & G of the two ⊙ BADM, EFHN. *P.* 1. *B.* 3.
2. Draw the rays CB, CD, GE, GH. *Poſ.* 1.

DEMONSTRATION.

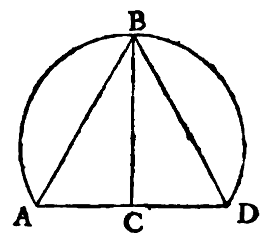
BECAUSE the ⊙ BADM, EFHN, are equal *(Hyp.* 1.).

1. The ſides CB, CD, & GE, GH, of the △ BCD, EGH, are = to one another. *Ax.* 1. *B.* 3.

   But the arches BMD, ENH, being alſo equal *(Hyp.* 2.).

2. The ∀ C & G, contained by thoſe equal ſides, will be = to one another. *P.* 27. *B.* 3.
3. Conſequently, the chord BD is = to the chord EH. *P.* 4. *B.* 1.

Which was to be demonſtrated.



PROPOSITION XXX. *PROBLEM IV.*

TO divide an arch (ABD) into two equal parts (AB, BD,).

Given	Sought
The arch A B D.	*The division of the arch* ABD *into two equal parts* AB, BD.

Resolution.

1. From the point A to the point D, draw the chord AD. *Pof.* 1.
2. Divide this chord into two equal parts at the point C. *P.* 10. *B.* 1.
3. At the point C in the ſtraight line AD, erect the ⊥ CB, which when produced, will divide the arch ABD into two equal parts at the point B. *P.* 11. *B.* 1.

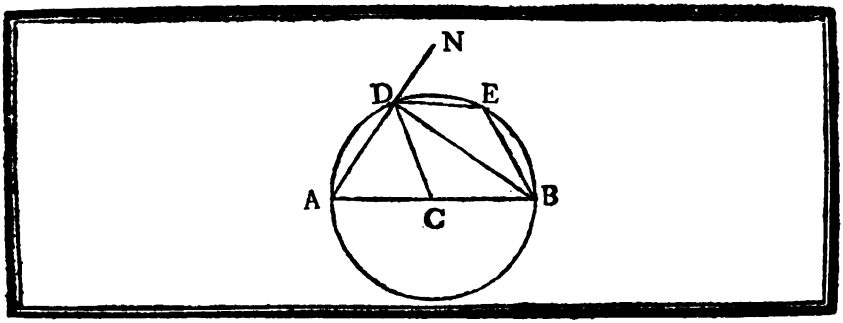
Preparation.

Draw the chords AB, DB. *Pof.* 1.

BECAUSE the ſide AC is = to the ſide CD (*Ref.* 2), CB common to the two △ ABC, DBC, & the ∀ ACB = to the ∀ DCB (*Ax.* 10. *B.* 1. & *Ref.* 3.).

1. The baſe AB is = to the baſe DB. *P.* 4. *B.* 1.
2. Conſequently, the arches AB & DB, ſubtended by the equal chords AB, DB, are = to one onother, and the whole arch ABD, is divided into two equal parts in B. *P.* 28. *B.* 3.

Which was to be done.



PROPOSITION XXXI. *THEOREM XXVII.*

IN a circle, the angle (ADB) in a ſemicircle (ADEB), is a right angle; but the angle (DAB) in a ſegment (DAB) greater than a ſemicircle, is leſs than a right angle, & the angle (DEB) in a ſegment (DEB) leſs than a ſemicircle, is greater than a right angle: alſo the mixtilineal angle (BDA) of the greater ſegment, is greater than a right angle, & that (BDE) of the leſſer ſegment, is leſs than a right angle.

CASE I.

Hypotheſis.	Theſis.
The ∀ ADB *is in the ſemi* ⊙ ADEB.	*This* ∀ ADB *is a* ∟.

Preparation.

1. Draw the ray CD. *Poſ.* 1.
2. And produce AD to N. *Poſ.* 2.

DEMONSTRATION.

BECAUSE in the △ ADC the ſide CA is = to the ſide CD (*D.* 15. *B.* 1.).

1. The ∀ CDA is = to the ∀ CAD. *P.* 5. *B.* 1.
   Again, in the △ CDB; the ſide CD being = to the ſide CB. *D.* 15. *B.* 1.
2. The ∀ CDB is = to the ∀ CBD. *P.* 5. *B.* 1.
3. Conſequently, ∀ ADB is = to ∀ CAD + CBD. *Ax.* 2. *B.* 1.
   But ∀ NDB is alſo = to ∀ CAD + CBD (*P.* 32. *B.* 1.).
4. Wherefore, this ∀ NDB is = to ∀ ADB. *Ax.* 1. *B.* 1.
5. From whence it follows, that ∀ ADB is a ∟. *D.* 10. *B.* 1.

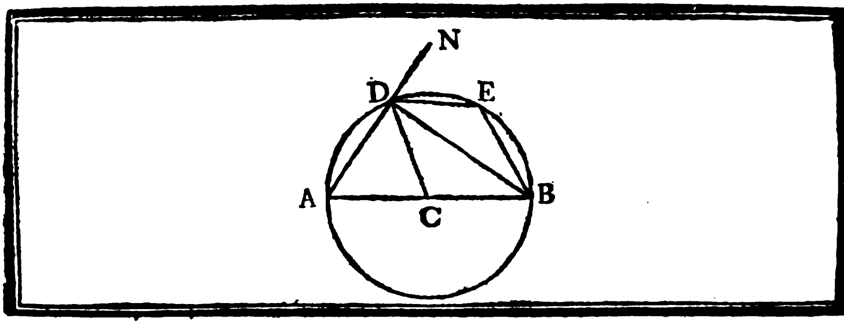
CASE II.

Hypotheſis.	Theſis.
The ∀ DAB *is in the ſegment* DAB > *a ſemi* ⊙.	*This* ∀ DAB *is* < *a* ∟.

DEMONSTRATION.

BECAUSE in the △ ADB, the ∀ ADB is a ∟ (*Caſe I.*).

1. The ∀ DAB will be < a ∟. *P.* 17. *B.* 1.



CASE III.

Hypothesis.
The ∀ DEB *is in a segment* DEB < *a semi* ⊙.

Thesis.
This ∀ DEB *is* > *a* ∟

DEMONSTRATION.

1. THE the opposite ∀ DAB + DEB of the quadrilateral figure ADEB are = to 2 ∟. *P. 22. B. 3.*
2. Wherefore, ∀ DAB being < a ∟ *(Case II.)*, DEB will be necessarily > a ∟.

CASE IV.

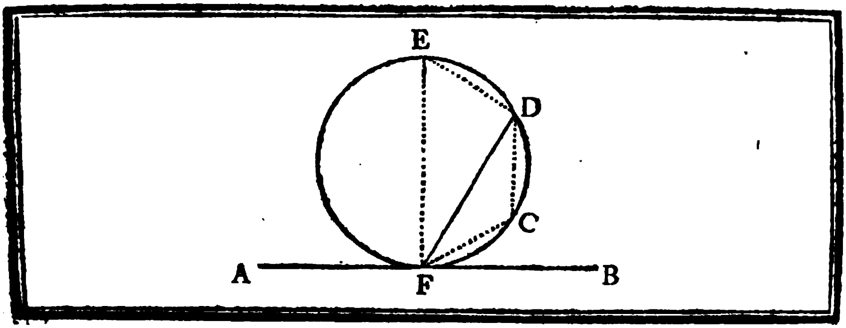
Hypothesis.
The mixtilineal ∀ BDA, BDE, *are formed by the straight line* BD & *the arches* DA, DE.

Thesis.
The ∀ BDA *is* > *a* ∟, & *the* ∀ BDE *is* < *a* ∟.

DEMONSTRATION.

BECAUSE the rectilineal ∀ ADB, NDB, are ∟ *(Case I.)*.
1. The mixtilineal ∀ BDA will be necessarily > a ∟, & the mixtilineal ∀ BDE < a ∟. *Ax. 8. B. 1.*

Which was to be demonstrated.



PROPOSITION XXXII. *THEOREM XXVIII.*

IF a straight line (AB) touches a circle (ECF), & from the point of contact (F) a chord (FD) be drawn; the angles (DFB, DFA,) made by this chord & the tangent, shall be equal to the angles (FED, FCD,) which are in the alternate segments (FED, FCD,) of the circle.

Hypothesis.	Thesis.
I. BA *is a tangent of the* $\odot$ ECF.	*I. The* $\measuredangle$ FED *is* $=$ *to* $\measuredangle$ DFB.
II. And FD *is a chord of this* $\odot$ *drawn from the point of contact.*	*II. The* $\measuredangle$ FCD *is* $=$ *to* $\measuredangle$ DFA.

Preparation.

1. At the point of contact F in AB, erect the $\perp$ FE. *P.* 11. *B.* 1.
2. Take any point C in the arch DF, & draw ED, DC, CF. *Pos.* 1.

DEMONSTRATION.

BECAUSE the straight line AB touches the $\odot$ ECF *(Hyp.* 1.), and FE is a $\perp$ erected at the point of contact F in the line AB *(Prep.* 1.).

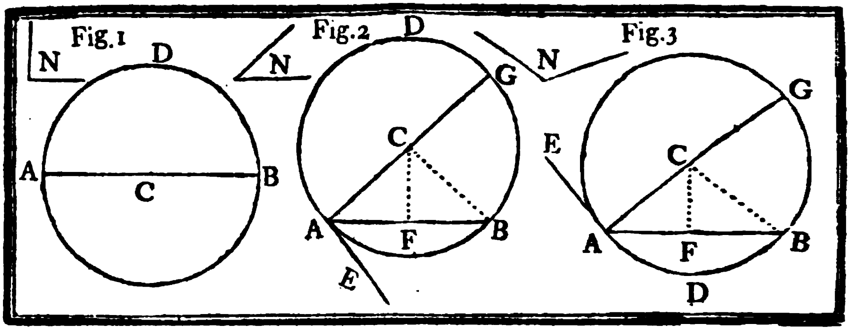
1. The straight line FE is a diameter of the $\odot$ ECF. *P.* 19. *B.* 3.
2. Consequently, $\measuredangle$ FDE is a $\llcorner$. *P.* 31. *B.* 3.
3. Wherefore, the $\measuredangle$ DEF $+$ DFE are $=$ to a $\llcorner$. *P.* 32. *B.* 1.
   But $\measuredangle$ EFB or $\measuredangle$ DFE $+$ $\measuredangle$ DFB being also $=$ to a $\llcorner$ *(Prep.* 1.).
4. The $\measuredangle$ DEF $+$ DFE are $=$ to the $\measuredangle$ DFB $+$ DFE. *Ax.* 1. *B.* 1.
5. Wherefore, the $\measuredangle$ DEF is $=$ to $\measuredangle$ DFB, or the $\measuredangle$ in the segment DEF is $=$ to the $\measuredangle$ made by the tangent BF & the chord DF. { *Ax.* 3. *B.* 1. *P.* 21. *B.* 3.

Which was to be demonstrated. I.

The $\measuredangle$ FED $+$ FCD being $=$ to 2 $\llcorner$ *(P.* 22. *B.* 3.), & the adjacent $\measuredangle$ DFB $+$ DFA being also $=$ to 2 $\llcorner$ *(P.* 13. *B.* 1.). *Ax.* 1. *B* 1.

6. The $\measuredangle$ FED $+$ FCD are $=$ to the $\measuredangle$ DFB $+$ DFA.
7. Wherefore, $\measuredangle$ FED being $=$ to the $\measuredangle$ DFB *(Arg.* 5.), the $\measuredangle$ FCD is also $=$ to the $\measuredangle$ DFA; or the $\measuredangle$ in the segment FCD is $=$ to the $\measuredangle$ contained by the tangent AF & the chord DF. { *Ax.* 3. *B.* 1. *P.* 21. *B.* 3.

Which was to be demonstrated. II.



PROPOSITION XXXIII. *PROBLEM V.*

UPON a given ſtraight line (AB), to deſcribe a ſegment of a circle (ADB) containing an angle equal to a given rectilineal angle (N).

Given	Sought
The ſtraight line AB *together with* ∀ N.	*The ſegment* ADB *deſcribed upon* AB, *containing an* ∀ = *to* ∀ N.

CASE I. If the given ∀ is a ∟. *(Fig.* 1.),

IT ſuffices to deſcribe upon AB a ſemi ⊙ ADB. *Poſ.* 3.

1. This ſemi ⊙ will contain an ∀ = to the given right ∀ N. *P.* 31. *B.* 3.

CASE II. If the given ∀ is acute *(Fig.* 2.) or obtuſe *(Fig.* 3.)

Reſolution.

1. At the point A in AB, make the ∀ BAE = to the given ∀ N. *P.* 23. B. 1.
2. At the point A in AE, erect the ⊥ AG. *P.* 11. *B.* 1.
3. Divide AB into two equal parts in the point F. *P.* 10. *B.* 1.
4. At the point F in AB, erect the ⊥ FC, which will cut AG in C. *P.* 11. *B.* 1.
5. From the center C at the diſtance CA, deſcribe the ⊙ ADG. *Poſ.* 3.

Preparation.

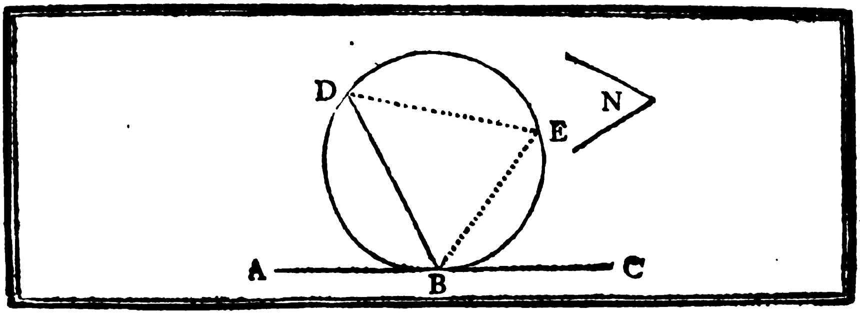
Draw the ſtraight line CB. *Poſ.* 1.

DEMONSTRATION.

BECAUSE in the △ ACF, BCF, the ſide AF is = to the ſide BF *(Reſ.* 3.), FC common to the two △, & the ∀ AFC = the ∀ BFC *(Ax.* 10. *B.* 1. & *Reſ.* 4.).

1. The baſe CA is = to the baſe CB. *P.* 4. *B.* 1.
2. Conſequently, the ⊙ deſcribed from the center C at the diſtance CA, will paſs thro' the point B, & ADB is a ſegment deſcribed upon AB. But AE touching the ⊙ ADB in A *(Reſ.* 2. & *P.* 16. *Cor. B.* 3.), and AB being a chord drawn from this point of contact A *(Arg.* 2.). { *D.* 15. *B.* 1. *D.* 19. *B.* 1.
3. The ∀ contained in the alternate ſegment ADB is = the ∀ BAE. *P.* 32. *B.* 3.
4. Wherefore, ∀ BAE being = to the given ∀ N *(Reſ.* 1.), the ∀ contained in the ſegment ADB deſcribed upon AB, is alſo = to the given ∀ N. *Ax.* 1. *B.* 1.

Which was to be done.



PROPOSITION XXXIV. *PROBLEM VI.*

TO cut off a ſegment (BED) from a given circle (BDE), which ſhall contain an angle (DEB) equal to a given rectilineal angle (N).

Given	Sought
The ⊙ BDE, *&* *the rectilineal* ∀ N.	*The ſegment* BED *cut off from this* ⊙, *containing an* ∀ DEB = *to the given* ∀ N.

Reſolution.

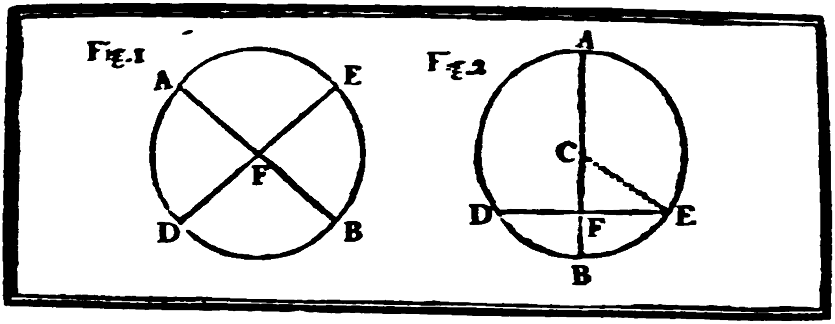
1. From any point A to the ⊙ BDE, draw the tangent ABC. *P.* 17. *B.* 3.
2. At the point of contact B in the line AB, make the ∀ DBA = to the given ∀ N. *P.* 23. *B.* 1.

DEMONSTRATION.

BECAUSE the given ∀ N is = to the ∀ DBA (*Ref.* 2.), & ∀ DEB = to the ∀ DBA (*P.* 32. *B.* 3.).

1. The ∀ DEB & N are = to one another. *Ax.* 1. *B.* 1.
2. Wherefore, the ſegment BED is cut off from the ⊙ BDE, containing an ∀ DEB = to the given ∀ N. *P.* 21. *B.* 3.

Which was to be done.



PROPOSITION XXXV. *THEOREM XXIX.*

IF in a circle (DAEB) two chords (AB, DE,) cut one another; the rectangle contained by the segments (AF, FB,) of one of them, is equal to the rectangle contained by the segments (DF, FE,) of the other.

Hypothesis.	Thesis.
I. AB, DE, *are two chords of the same* ⊙ DAEB.	*The* Rgle AF . FB *is* = *to*
II. And these chords cut one another in a point F.	*the* Rgle DF . FE.

CASE I. If the two chords pass thro' the center F of the ⊙. *Fig.* 1.

DEMONSTRATION.

1. **T**HEN, the straight lines AF, FB, DF, FE, are = to one another. *D. 15. B. 1.*
2. Consequently, the Rgle AF . FB is = to the Rgle DF . FE. *Ax. 2. B. 2.*

CASE II. If one of the chords AB, passes thro' the center & cuts the other DE which does not pass thro' the center at ∟ (*Fig.* 2.).

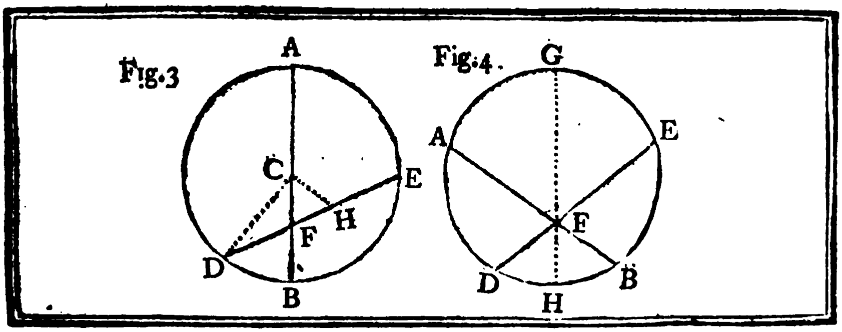
Preparation.

Draw the ray CE. *Pos.* 1.

DEMONSTRATION.

BECAUSE the straight line AB is cut equally in C & unequally in F.

1. The Rgle AF . FB + the □ of CF is = to the □ of CB, or is = to the □ of CE. *P. 5. B. 2. Ax. 1. B. 1.*
   But the □ of FE + the □ of CF is also = to the □ of CE (*P.* 47. *B.* 1.).
2. From whence it follows, that the Rgle AF . FB + the □ of CF is = to the □ of FE + the □ of CF. *Ax. 1. B. 1.*
3. Consequently, the Rgle AF . FB is = to the □ of FE. *Ax. 3. B. 1.*
   And since DF is = to FE (*P.* 3. *B.* 3.), or DF . FE = to the □ of FE (*Ax.* 2. *B.* 2.).
4. The Rgle AF . FB is also = to the Rgle DF . FE. *Ax. 1. B. 1.*



CASE III. If one of the chords AB, passes thro' the center & cuts the other DE which does not pass thro' the center, obliquely (*Fig.* 3.).

Preparation.

1. From the center C, let fall upon DE, the ⊥ CH. *P.* 12. *B.* 1.
2. And draw the ray CD. *Pos.* 1.

DEMONSTRATION.

BECAUSE DH is = to HE (*Prep.* 1. & *P.* 3. *B.* 3.).

1. The Rgle DF . FE + the □ of FH is = to the □ of DH. *P.* 5. *B.* 2.
2. Wherefore, the Rgle DF . FE + □ of FH + □ of CH is = to the □ of DH + □ of CH. *Ax.* 2. *B.* 1.
   But the □ of FH + □ of CH is = to the □ of CF, & the □ of DH + the □ of CH is = to the □ of CD (*P.* 47. *B.* 1.).
3. Therefore, the Rgle DF . FE + □ of CF is = to the □ of CD or to the □ of CB. *Ax.* 1. *B.* 1.
   Moreover, the Rgle AF . FB + □ of CF being = to the same □ of CB (*P.* 5. *B.* 2.).
4. The Rgle DF . FE + □ of CF is also = to the Rgle AF . FB + □ of CF. *Ax.* 1. *B.* 1.
5. Or taking away the common □ of CF, the Rgle DF . FE is = to the Rgle AF . FB. *Ax.* 3. *B.* 1.

CASE IV. If neither of the chords AB, DE, passes thro' the center (*Fig.* 4.).

Preparation.

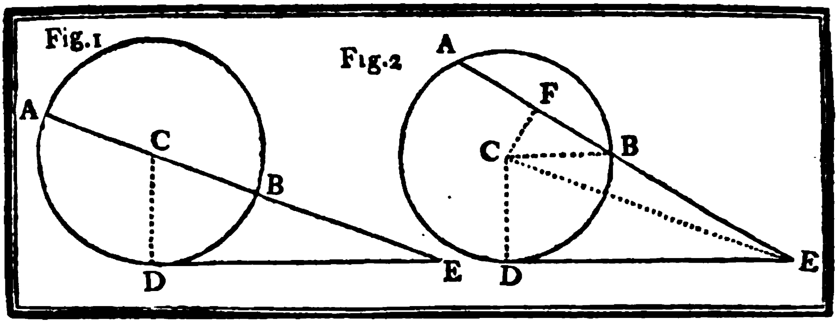
Thro' the point F, draw the diameter GH. *Pos.* 1.

DEMONSTRATION.

BECAUSE each of the Rgles AF . FB & DF . FE is = to the Rgle GF . FH (*Case III.*).

6. These Rgles AF . FB & DF . FE are also = to one another. *Ax.* 1. *B.* 1.

Which was to be demonstrated.



PROPOSITION XXXVI. *THEOREM XXX.*

IF from any point (E) without a circle (ABD) two straight lines be drawn, one of which (DE) touches the circle, & the other (EA) cuts it; the rectangle contained by the whole secant (AE), & the part of it (EB) without the circle, shall be equal to the square of the tangent (ED).

Hypothesis.	Thesis.
I. The point E *is taken without the* ⊙ ABD.	*The* Rgle AE . EB *is* = *to the* ☐ *of* ED.
II. From this point E, *a tangent* ED *&* *a secant* EA, *have been drawn.*	

CASE I. If the secant AE passes thro' the center (*Fig.* 1.).

Preparation.

From the point of contact D, Draw the ray CD. *Pos.* 1.

DEMONSTRATION.

1. THE ray CD is then ⊥ to the tangent ED. *P.* 18. *B.* 3.
And because the straight line AB is bisected in C, & produced to the point E.
2. The Rgle AE . EB + the ☐ of CB is = to the ☐ of CE. *P.* 6. *B.* 2.
Moreover, the ☐ of CE being also = to the ☐ of DE + the ☐ of CD (*P.*47.*B.*1.), or to the ☐ of DE + the ☐ of CB (*P.* 46. *Cor.* 3. *B.* 1.).
3. The Rgle AE . EB + the ☐ of CB is = to the ☐ of DE + the ☐ of CB. *Ax.* 1. *B.* 1.
The ☐ of CB being taken away from both sides.
4. The Rgle AE . EB will be = to the ☐ of DE. *Ax.* 3. *B.* 1.

CASE II. If the secant AE does not pass thro' the center. *Fig.* 2.

Preparation.

1. Let fall from the center C upon AE, the ⊥ CF. *P.* 12. *B.* 1.
2. Draw the rays CB, CD, & the straight line CE. *Pos.* 1.

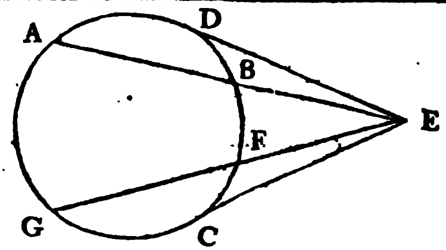


Fig. 3

DEMONSTRATION.

BECAUSE the straight line AB is bisected in F (*Prep.* 1. & *P.* 3. *B.* 3.) and produced to the point E.

1. The Rgle AE.EB + □ of FB is = to the □ of FE. *P.* 6. *B.* 2.
2. Consequently, the Rgle AE. EB + □ of FB + □ of FC is = to the □ of FE + □ of FC, or is = to the □ of CE. { *Ax.* 2. *B.* 1. *P.* 47. *B.* 1.
   But since the □ of DE + □ of CD is = to the □ of CE, and the □ of FB + □ of FC is = to the □ of CB (*P.* 47. *B.* 1.), or is = to the □ of CD (*D* 15. & *P.* 46. *Cor.* 3. *B.* 1.)
3. The Rgle AE.EB + □ of CD is = to the □ of DE + □ of CD.
4. Consequently, the Rgle AE.EB is = to the □ of DE. *Ax.* 3. *B.* 1.

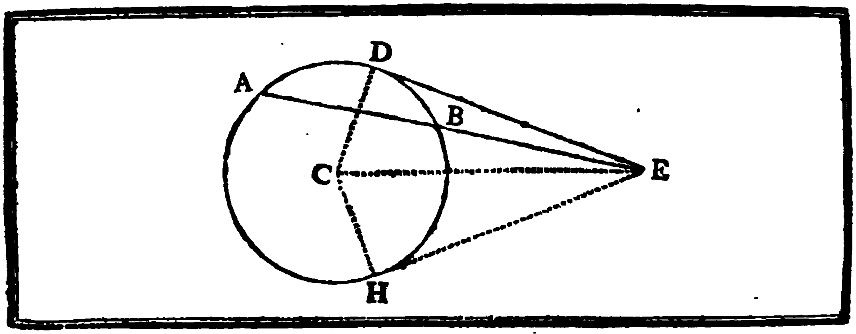
Which was to be demonstrated.

COROLLARY I.

IF (fig. 3.) *from any point* (E) *without a circle* (ADBF), *there be drawn several straight lines* (AE, EG, &c). *cutting it in* (B & F, &c): *the rectangles contained by the whole secants* (AE, GE), *and the parts of them* (EB, EF) *without the circle, are equal to one another; for drawing from the point* (E) *the tangent* (ED), *these rectangles will be equal to the square of the same tangent* (ED).

COROLLARY II.

IF *from any point* (E), *without a circle* (ADBF), *there be drawn to this circle two tangents* (ED, EC), *they will be equal to one another, because the square of each is equal to the same rectangle* (AE.EB).



PROPOSITION XXXVII. *THEOREM XXXI.*

IF from a point (E), without a circle (ADH), there be drawn two straight lines, one of which (AE) cuts the circle, and the other (ED) meets it; if the rectangle contained by the whole secant (AE) and the part of it without the circle (EB), be equal to the square of the line (ED) which meets it: the line which meets shall touch the circle in D.

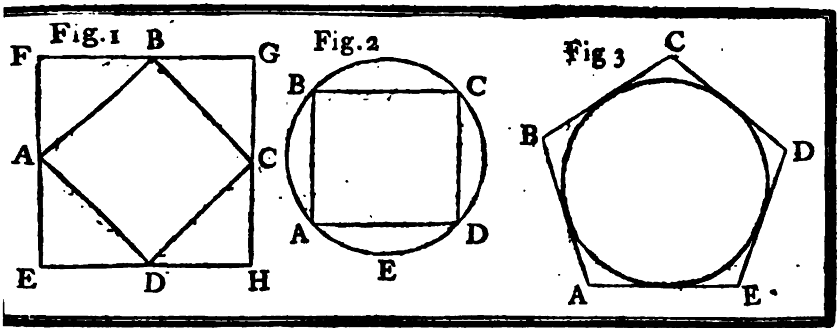
Hypothesis.	Thesis.
I. AE *cuts the* ⊙ ADH *in* B.	*The straight line* ED *touches the* ⊙ ADH *in the point* D.
II. ED *meets the* ○.	
III. The Rgle AE.EB *is* = *to the* □ *of* ED.	

Preparation.

1. From the point E to the ⊙ ADH draw the tangent EH. *P.* 17. *B.* 3.
2. Draw the rays CD, CH and the straight line CE. *Pos.* 1.

DEMONSTRATION.

BECAUSE the Rgle of AE.EB is = to the □ of ED (*Hyp.* 3.) and the Rgle AE.EB is also = to the □ of EH (*Prep.* 1 & *P.* 36. *B.* 3)
1. The □ of ED is = to the □ of EH (*Ax.* 1. *B.* 1.) or ED = EH. { *P.* 46. *B.* 1. *Cor.* 3.
And moreover, since in the △ CDE, CHE, the side CD is = to the side CH (*D.* 15. *B.* 1), and CE is common to the two △.
2. The ∀ CDE is = to the ∀ CHE. *P.* 8. *B.* 1.
3. Wherefore, ∀ CHE being a ∟ (*Prep.* 1. & *P.* 18. *B.* 3), ∀ CDE is also a ∟. *Ax* 1. *B.* 1.
4. And the straight line ED touches the ⊙ ADH in the point D. { *P.* 16. *B.* 3. *Cor.* 3.



DEFINITIONS.

I.

A *Rectilineal figure* (ABCD) is ſaid *to be inſcribed in another rectilineal igure* (EFGH), when all the angles (A, B, C, D) of the inſcribed figure, are upon the ſides of the figure in which it is inſcribed *(fig.* 1*)*.

II.

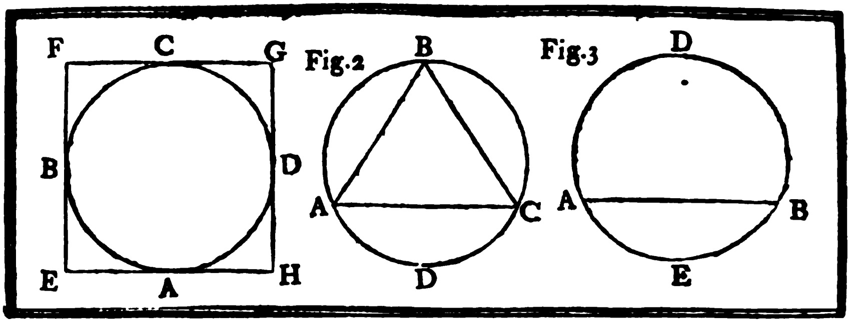
In like manner *a rectilineal figure* (EFGH) is ſaid *to be deſcribed about another rectilineal figure* (ABCD); when all the ſides (EF, FG, GH, HE) of the circumſcribed figure paſs thro' the angular points (A, B, C, D) of the figure about which it is deſcribed, each thro' each *(Fig.* 1).

III.

A rectilineal figure (ABCD) is ſaid *to be inſcribed in a circle*, when all the angles (A, B, C, D) of the inſcribed figure are upon the circumference of the cricle (ABCDE) in which it is inſcribed *(Fig.* 2).

IV.

A rectilineal figure (ABCDE) is ſaid *to be deſcribed about a circle*, when each of the ſides AB, BC, CD, DE, EA) touches the circumference of the circle *(Fig.* 3).



DEFINITIONS.

V.

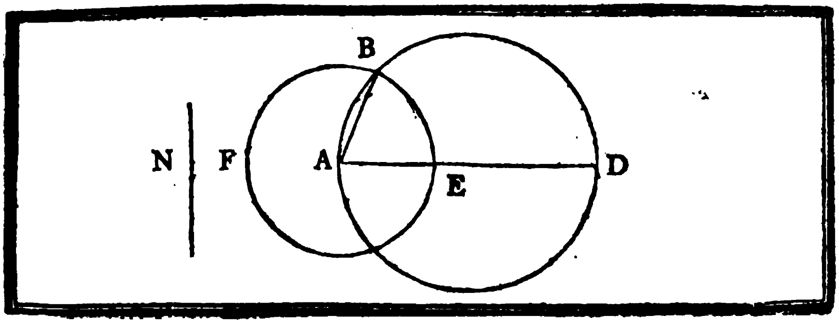
A *Circle* (ABCD) *is ſaid to be inſcribed in a rectilineal figure* (EFGH), when the circumference of the circle touches each of the ſides (EF, FG, GH, HE) of the figure in which it is inſcribed (*Fig.* 1).

VI.

A circle (ABCD) *is deſcribed about a rectilineal figure* (ABC), when the circumference of the circle paſſes thro' all the angular points (A, B, C) of the figure about which it is deſcribed (*Fig.* 2).

VII.

A ſtraight line (AB) *is ſaid to be placed in a circle* (ADBE), when the extremities of it (A & B) are in the circumference of the circle (*fig.* 3).



PROPOSITION I. *PROBLEM I.*

IN a given circle (ABD), to place a ſtraight line (AB) equal to a given ſtraight line (N), not greater than the diameter of the circle (ABD).

Given.	Sought.
A ⊙ ABD *together with the ſtraight line* N, *not* > *the diameter of this* ⊙.	*The ſtraight line* AB *placed in the* ⊙ ABD & = *to the given ſtraight line* N.

Reſolution.

Draw the diameter AD of the ⊙ ABD. *Poſ.* 1.

CASE I. If AD is = to N.

THERE has been placed in the given ○ ABD a ſtraight line = to the given N. *D.* 7. *B.* 4.

CASE II. If AD is > N.

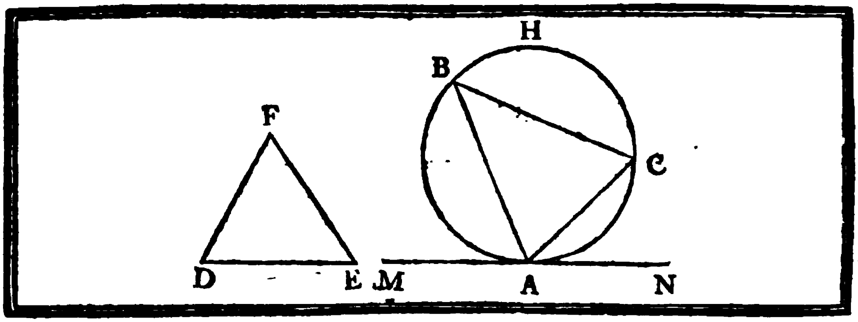
1. Make AE = to N. *P.* 3. *B.* 1.
2. From the center A at the diſtance AE deſcribe the ⊙ EBF, and draw AB. *Poſ.* 3.

DEMONSTRATION.

BECAUSE AB is = to AE *(D.* 15. *B.* 1*)*, and the ſtraight line N is = to AE *(Ref.* 1.*)*

1. The ſtraight line AB, placed in the ⊙ ABD, will be alſo = to N. { *Ax.* 1. *B.* 1. *D.* 7. *B.* 1.

Which was to be done.



PROPOSITION II. *PROBLEM II.*

IN a given circle (ABHC), to inscribe a triangle (ABC) equiangular to a given triangle (DFE).

Given.	Sought.
A ⊙ ABHC *together with the* △ DFE.	*The* △ ABC *inscribed in the* ⊙ ABHC, *equiangular to the* △ DFE.

Resolution.

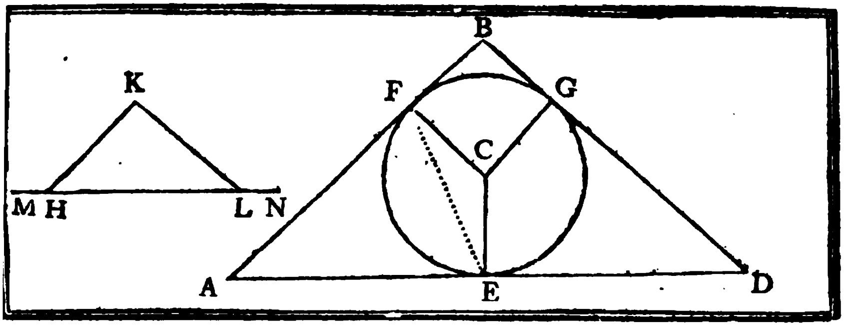
1. From the point M, to the ⊙ ABHC draw the tangent MN. *P.* 17. *B.* 3.
2. At the point of contact A in the line MN make the ∀ BAM = to the ∀ FED, and the ∀ CAN = to the ∀ FDE. *P.* 23. *B.* 1.
3. Draw BC. *Pos.* 1.

DEMONSTRATION.

BECAUSE the ∀ BCA is = to the ∀ BAM *(P.* 32. *B.* 3*)*, and the ∀ FED is = to the same ∀ BAM *(Ref.* 2*)*; also the ∀ CBA is = to the ∀ CAN *(P.* 32. *B.* 3.*)* and ∀ FDE is = to ∀ CAN *(Ref.* 2.

1. It follows that ∀ BCA is = to ∀ FED, and ∀ CBA = to ∀ FDE. *Ax.* 1. *B.* 1.
2. Consequently, the third ∀ BAC, of the △ ABC, is also = to the third ∀ DFE of the △ DFE, and this △ ABC is inscribed in the ⊙ ABHC. { *P.* 32. *B.* 1. *D.* 3. *B.* 4.

Which was to be done.



PROPOSITION III. *PROBLEM III.*

ABOUT a given circle (EFG) to describe a triangle (ABD), equiangular to a given triangle (HKL).

Given.	Sought.
The ⊙ EFG, *together with the* △ HKL.	*The* △ ABD *described about the* ⊙ EFG, *equiangular to the* △ HKL.

Resolution.

1. Produce the side HL, of the △ HKL, both ways. *Pos.* 2.
2. Find the center C of the ⊙ EFG, and draw the ray CE. *P.* 1. *B.* 3.
3. At the point C in CE, make the ∀ ECF = to the ∀ KHM, and ∀ ECG = to ∀ KLN. *P.* 23. *B.* 1.
4. Upon CE, CF, CG, erect the ⊥ AD, AB, DB produced. *P.* 11. *B.* 1.

Preparation.

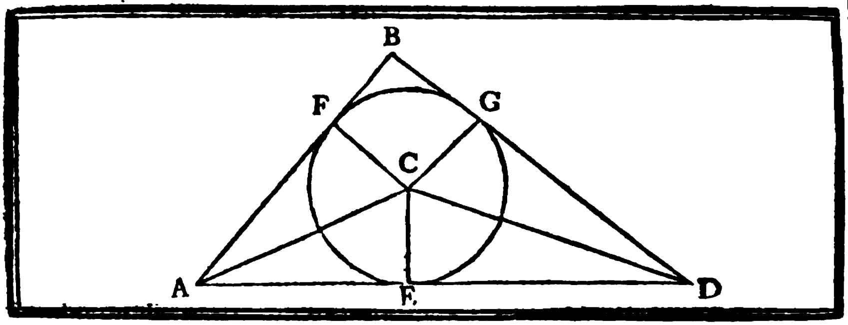
Draw the straight line FE. *Pos.* 1.

DEMONSTRATION.

BECAUSE the ∀ CEA, CFA are ∟ (*Ref.* 4.)

1. ∀ FEA + EFA are < 2 ∟, & AD, AB meet some where in A. { *Ax.* 8. *B.* 1. *Ax.* 11. *B.* 1.
It may be demonstrated after the same manner, that.
2. AD, DB & AB, DB meet somewhere in D & B.
And since AD, AB, DB are ⊥ at the extremities E, F, G of the rays EF, CF, CG (*Ref.* 4.)
3. These straight lines touch the ⊙ EFG; and the △ ABD formed by these straight lines is described about the ⊙ EFG. { *P.* 16. *B.* 3 *Cor.D.*4.*B.*4
Moreover, the 4 ∀ CEA + CFA + ECF + FAE of the quadrilateral figure AFCE being = to 4 ∟ (*P.* 32. *B.* 1), and the ∀ CEA + CFA = to 2 ∟ (*Ref.* 4).
4. The ∀ ECF + FAE are also = to 2 ∟. *Ax.* 3. *B.* 1.
5. Or = to ∀ KHM + KHL as being also = to 2 ∟. { *Ax.* 1. *B.* 1.
But ∀ ECF being = to ∀ KHM (*Ref.* 3). *P.* 13. *B.* 1.
6. The ∀ FAE is = to ∀ KHL, and ∀ GDE = to ∀ KLH. *Ax.* 3. *B.* 1.
7. Hence the third ∀ FBG of the △ ABD is = to the third ∀ HKL of △ HKL. *P.* 32. *B.* 1.
8. Therefore the △ ABD described about the ⊙ EFG is equiangular to the given △ HKL.

Which was to be done.



PROPOSITION IV. *PROBLEM IV.*

TO inscribe a circle (EFG) in a given triangle (ABD).

Given.	Sought.
The △ ABD.	*The* ⊙ EFG *inscribed in the* △ ABD.

Resolution.

1. Bisect the ∀ BAD, BDA by the straight lines AC, DC produced until they meet one another in C. *P.* 9. *B.* 1.
2. From the point C let fall upon AD the ⊥ CE. *P.* 12. *B.* 1.
3. And from the center C at the distance CE, describe the ⊙ EFG. *Pos.* 3.

Preparation.

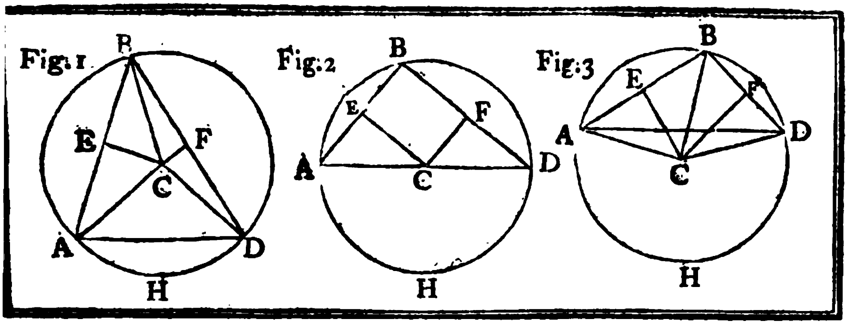
From the point C let fall upon AB & DB the ⊥ CF, CG. *P.* 12. *B.* 1.

Demonstration.

BECAUSE in the △ AFC, ACE, the ∀ FAC is = to the ∀ CAE *(Ref.* 1*)*, ∀ CFA = to ∀ CEA *(Prep. Ref.* 2 & *Ax.* 10. *B.* 1); & AC common to the two △.

1. The straight line CF is = to CE. *P.* 26. *B.* 1.
   In like manner it may be demonstrated, that
2. The straight line CG is = to CE.
3. Consequently, the straight lines CF, CE, CG are = to one another; and the ⊙ described from the center C at the distance CE will also pass thro' the points F & G. { *Ax.* 1. *B.* 1. *D.* 15. *B.* 1.
   And since the sides AD, AB, DB are ⊥ at the extremities E, F, G, of the rays CE, CF, CG *(Ref.* 2 *& Prep.)*.
4. These sides will touch the ⊙ in the points E, F, G. { *P.* 16. *B.* 3. *Cor.*
5. Therefore the ⊙ EFG is inscribed in the △ ABD. *D.* 5. *B.* 4.

Which was to be done.



PROPOSITION V. *PROBLEM V.*

TO describe a circle (ABDH), about a given triangle (ABD).

Given.	Sought.
The △ ABD.	*The* ⊙ ABDH *described about the* △ ABD.

Resolution.

1. Bisect the sides AB, DB in the points E & F. — *P.* 10. *B.* 1.
2. At the points E & F in AB, DB, erect the ⊥ EC, FC, produced until they meet in C. — *P.* 11. *B.* 1.
3. And whether the point C falls within (*fig.* 1.) or without (*fig.* 3.) or in one of the sides (*fig.* 2). of the △ ABD, from the center C at the distance CA describe the ⊙ ABDH. — *Pos.* 3.

Preparation.

Draw the straight lines CD, CB. — *Pos.* 1.

DEMONSTRATION.

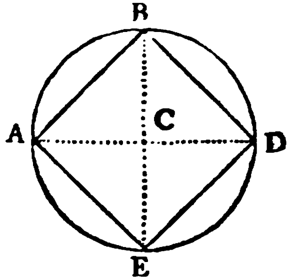
BECAUSE in the △ AEC, BEC, the side AE is = to the side EB (*Ref.* 1), EC common to the two △, & ∀AEC = to ∀BEC (*Ref.* 2 & *Ax.* 10. *B.* 1).

1. The straight line CB is = to CA. — *P.* 4. *B.* 1.
   It may be demonstrated after the same manner, that
2. The straight line CB is = to CD.
3. Consequently, the straight lines CA, CB, CD are = to one another; and the ⊙ ABDH described from the center C at the distance CA, will pass also thro' the points B & D. — { *Ax.* 1. *B.* 1. *D.* 15. *B.* 1.
4. Therefore this ⊙ ABDH is described about the △ ABD. — *D.* 6. *B.* 4.

Which was to be done.

COROLLARY

IF *the triangle* ABD *be acute angled, the point* C *falls within it* (fig. 1); *but if this triangle be obtuse angled, the point* C *falls without it* (fig. 3); *in fine if it be a right angled triangle, the point* C *is in one of the sides* (fig. 2).



PROPOSITION VI. *PROBLEM VI.*

TO inſcribe a Square (ABDE), in a given Circle (ABDE).

Given
The ⊙ ABDE.

Sought.
The □ ABDE *inſcribed in this* ⊙.

Reſolution.

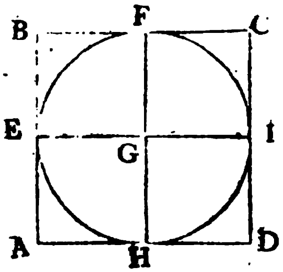
1. Draw the Diameters AD, BE, ſo as to cut each other at ∟. *P.* 11. *B.* 1.
2. Join their Extremities by the ſtraight Lines AB, BD, DE, EA. *Poſ.* 1.

DEMONSTRATION.

BECAUSE in the △ ABC, DBC the ſide AC is = to the ſide CD (*Reſ.*-1. & *D.* 15. *B.* 1.), BC common to the two △, & the ∀BCA = to ∀ BCD (*Reſ.* 1. & *Ax.* 10. *B.* 1).

1. The ſtraight Line AB is = to BD. *P.* 4. *B.* 1.
   It may be demonſtrated after the ſame manner, that
2. The ſtraight line BD is = to DE, DE = to EA & EA = to AB.
3. Conſequently, the ſtraight lines AB, BD, DE, EA are = to one another, or the quadrilateral figure ABDE is equilateral. *Ax.* 1. *B.* 1.
   And becauſe each of the ∀ ABD, BDE, DEA, EAB is placed in a ſemi ⊙.
4. Theſe ∀ are ∟, & the equilateral qadrilateral figure ABDE is rectangular. *P.* 31. *B.* 3.
5. Wherefore this quadrilateral figure is a ſquare inſcribed in the ⊙ ABDE. { *D.* 30. *B.* 1. *D.* 3. *B.* 4.

Which was to be done.



PROPOSITION VII. PROBLEM VII.

TO describe a Square (ABCD) about a given Circle (HEFI).

Given.	Sought.
The ⊙ HEFI.	*The* □ ABCD *described about the* ⊙ HEFI.

Resolution.

1. Draw the diameters EI, HF so as to cut each other at ∟. *P.*11. *B.*1.
2. At the Extremities H, E, F, I of those diameters erect the ⊥ AD, AB, BC, CD. *P.*11. *B.*1.

DEMONSTRATION.

1. THE lines DA, AB, BC, CD, are tangents of the ⊙ HEFI. { *P.*16. *B.*3. *Cor.*
2. And the straight line AD, is Plle. to EI, as also the straight line BC; because the ∀ HGE + GHA, & ∀ FGE + GFB are = to 2 ∟ (*Ref.* 1. & 2). *P.*28. *B.*1.
3. Consequently, AD is Plle. to BC, likewise AB, HF, DC are Plles. *P.*30. *B.*1.
4. Wherefore the quadrilateral figures AI, EC, AF, HC, AC are Pgmes. *D.*35. *B.*1.
5. From whence it follows, that the straight lines AD, EI, BC, also AB, HF, DC, are = to one another. *P.*34. *B.*1.
6. And since EI is = to HF (*D.* 15. *B.* 1.), the straight lines AD, BC, AB, DC are equal. *Ax.*1. *B.*1.

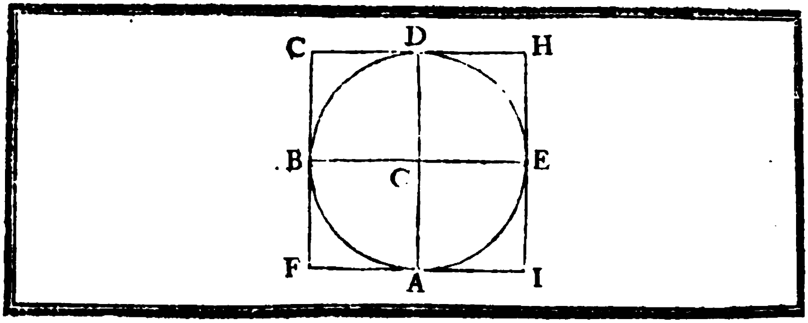
But ∀ EID of the Pgme. AI being a ∟ (*Ref.* 2).

7. The ∀ A, which is diagonally opposite to it, is also a ∟. *P.*34. *B.*1.

It may be proved after the same manner, that

8. The ∀ B, C, D are ∟.
9. Consequently, there has been described about the ⊙ HEFI a quadrilateral figure ABCD equilateral (*Arg.* 6.) & rectangular (*Arg.* 7. & 8); or a square. { *D.* 4. *B.*1. *D.*30. *B.*1.

Which was to be done.



PROPOSITION VIII. *PROBLEM VIII.*

TO inscribe a circle (ABDE) in a given square (FGHI).

Given.	Sought.
The □ FGHI.	*The* ⊙ ABDE *inscribed in the* □ (FGHI).

Resolution.

1. Bisect the sides FI, FG of the □ FGHI. *P.*10. *B.*1.
2. Thro' the points of section A & B, draw AD Plle. to FG or IH & BE Plle. to FI or GH. *P.*31. *B.*1.
3. From the center C, where AD, BE intersect each other, at the distance CA describe the ⊙ ABDE. *Pos.*3.

DEMONSTRATION.

BECAUSE the figures FE, BH, FD, AH, FC, AE, BD, CH are Pgmes. (*Ref.* 2. & *D.* 35. *B.* 1).

1. The straight line FA is = to BC & FB = to AC. *P.*34. *B.*1.

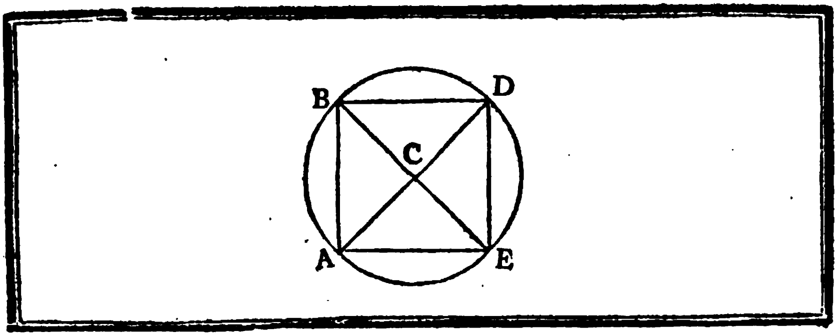
But the whole lines FI, FG being equal (*D.* 30. *B.* 1.) and FA, FB being the halves of those straight lines (*Ref.* 1).

2. The straight line FA is = to FB. *Ax.*7. *B.*1.
3. Consequently, BC is also = to AC; and likewise AC is = to CE & BC = to CD. *Ax.*1. *B.*1.
4. From whence it follows, that the straight lines AC, BC, CE, CD are = to one another, and the ⊙ described from the center C at the distance CA; passes also thro' the points B, D, E. { *Ax.*1. *B.*1. *D.*15. *B.*1.

But the ∀ DAF, EBG, ADH, BEI being ∟ (*P.* 34. *B.* 1.) as being interior opposite to the ∟ GFA, HGB, IHD, FIE (*D.* 30. *B.* 1).

5. The straight lines FI, FG, GH, HI are tangents of the ⊙ ABDE. { *P.*16. *B.*3. *Cor.*
6. Wherefore this ⊙ is inscribed in the square FGHI. *D.* 5. *B.*4

Which was to be done.



PROPOSITION IX. *PROBLEM IX.*

TO deſcribe a circle (ABDE), about a given ſquare (ABDE).

Given.	Sought.
The □ ABDE.	*The* ⊙ ABDE *deſcribed about the* □ ABDE.

Reſolution.

1. Draw the diagonals AD, BE. *Poſ.*1.
2. From the center C, where the diagonals interſect each other, and at the diſtance CA, deſcribe the ⊙ ABDE. *Poſ.*3.

DEMONSTRATION.

BECAUSE in △ ABE, EBD the ſide AB is = to the ſide BD AE = to ED (*D.* 30. *B.* 1.), & BE common to the two △.

1. The ∀ ABE is = to ∀ EBD, & the whole ∀ ABD is biſected by BE. *P.* 8. *B.*1.

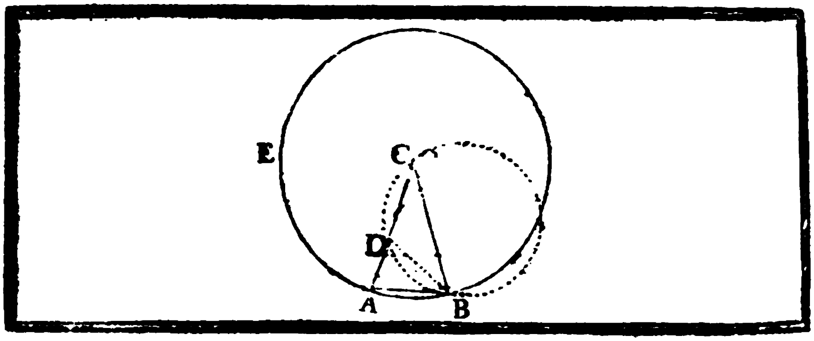
It may be demonſtrated after the ſame manner, that

2. The ∀ BAE, BDE, AED, are biſected by AD, BE.

But the whole ∀ ABD, BAE being = to one another (*D.* 30. *B.* 1).

3. Their halves, the ∀ CBA, CAB will be alſo equal. *Ax.*7. *B.*1.
4. Conſequently, CA is = to CB, likewiſe CA is = to CE, and CB = to CD. *P.* 6. *B.*1.
5. Hence CA, CB, CE, CD are = to one another, & the ⊙ deſcribed from the center C at the diſtance CA, will alſo paſs thro' the points B, D, E. { *Ax.*1. *B.*1. *D.*15. *B.*1. }
5. Wherefore the ⊙ ABDE is deſcribed about the □ ABDE. *D.* 6. *B.*4.

Which was to be done.



PROPOSITION X. PROBLEM X.

TO describe an Isosceles triangle (ACB), having each of the angles at the base (AB), double of the third angle (ACB).

Given.	Sought.
A line AC taken at will.	*The Isosceles △ ACB, having the ∀ CAB or CBA = to 2 ∀ ACB.*

Resolution.

1. Draw any straight line CA. *Pos.* 1.
2. Divide this line in the point C, so that the Rgle. of CA . AD be = to the □ of CD. *P.* 11. *B.* 2.
3. From the center C at the distance CA describe the ⊙ ABE. *Pos.* 3.
4. Place in this ⊙ the straight line AB = to CD & draw CB. *P.* 1. *B.* 4.

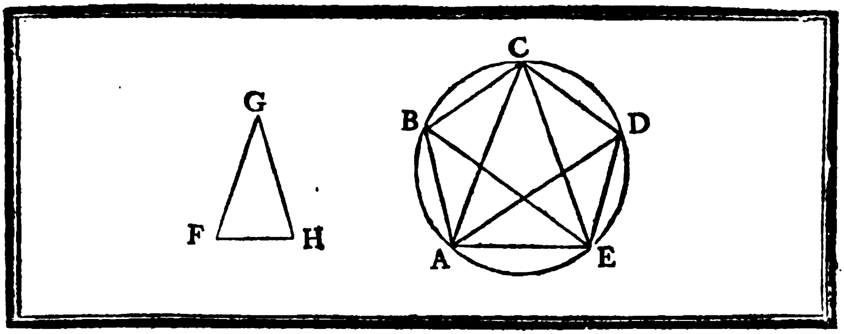
Preparation.

1. Draw the straight line DB. *Pos.* 1.
2. About the △ CDB describe a ⊙. *P.* 5. *B.* 4.

DEMONSTRATION.

BECAUSE the Rgle. CA . AD is = to the □ of CD (*Ref.* 2.) & the □ of AB is = to the □ of CD (*Ref.* 4. & *P.* 46. *Cor.* 3. *B.* 1).

1. The Rgle. CA . AD will be also = to the □ of AB. *Ax.* 1. *B.* 1.
2. Consequently, the straight line AB is a tangent of the ⊙ CDB. *P.* 37. *B.* 3.
3. From whence it follows that ∀ DBA is = to ∀ BCD. *P.* 32. *B.* 3.
   Therefore adding to both sides ∀ DBC.
4. The ∀ ABC will be = to the ∀ BCD + DBC. *Ax.* 2. *B.* 1.
   But ∀ BDA being also = to the ∀ BCD + DBC (*P.* 32. *B.* 1.
5. Therefore the ∀ BDA is = to ∀ ABC. *Ax.* 1. *B.* 1.
   Likewise, since CB is = to CA (*Ref.* 4. & *D.* 15. *B.* 1).
6. The ∀ BAC is = to the ∀ ABC. *P.* 5. *B.* 1.
7. Wherefore, ∀ BDA is = to ∀ BAC, & DB is = to AB. { *Ax.* 1. *B.* 1. *P.* 6. *B.* 1.
   And because CD is also = to AB (*Ref.* 4).
8. The straight line DB will be = to CD & ∀ CBD = to ∀ BCD. { *Ax.* 1. *B.* 1. *P.* 6. *B.* 1.
   Adding to both sides ∀ DBA or its equal ∀ BCD (*Arg.* 3).
9. The ∀ CBD + DBA or ∀ CAB is = to 2 ∀ BCD; and there has been described an Isosceles △ CAB having each of the ∀ at the base double of the ∀ at the vertex. *Ax.* 2. *B.* 1.
   Which was to be done.



PROPOSITION XI. *PROBLEM XI.*

TO inscribe an equilateral & equiangular pentagon (ABCDE) in a given circle (ACE).

Given.	Sought.
The ⊙ ACE.	*The equilateral & equiangular pentagone* ABCDE, *inscribed in the* ⊙ ACE.

Resolution.

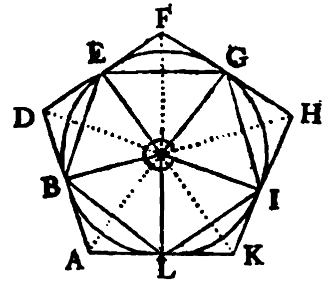
1. Describe the Isosceles △ FGH having each of the ∀ at the base FH double of the ∀ at the vertex G. *P.* 10. *B.* 4.
2. Inscribe in the ⊙ ACE a △ ACE equiangular to the △ FGH. *P.* 2. *B.* 4.
3. Bisect the ∀ CAE & CEA at the Base, by the straight lines AD, EB. *P.* 9. *B.* 1.
4. Draw the straight lines AB, BC, CD, DE. *Pos.* 1.

DEMONSTRATION.

BECAUSE each of the ∀ CAE, CEA is double of the ∀ ACE (*Ref.* 1. & 2.), & these ∀ are bisected (*Ref.* 3.).

1. The five ∀ACE, CAD, DAE, BEA, CEB are = to one another. *Ax.* 7. *B.* 1.
2. From whence it follows that the arches AE, ED, DC, CB, BA are = to one another, likewise the chords AE, ED, DC, CB, BA. { *P.* 26. *B.* 3. *P.* 29. *B.* 3.
But if to the = Arches AE, CD (*Arg.* 2.), be added the arch ABC.
3. The whole arch EABC is = to the whole arch ABCD, and ∀ CDE is = to the ∀ DEA. { *Ax.* 2. *B.* 1. *P.* 27 *B.* 3.
It may be demonstrated after the same manner, that
4. Each of the ∀ EAB, ABC, BCD is = to the ∀ CDE or DEA.
5. Wherefore there has been inscribed in the ⊙ ACE, an equilateral (*Arg.* 2.) & equiangular (*Arg.* 4.) pentagone. *D.* 3. *B.* 4.

Which was to be done.



PROPOSITION XII. *PROBLEM XII.*

TO describe an equilateral & equiangular pentagone (ADFHK) about a given circle.

Given.
The ⊙ LEG.

Sought.
The equilateral pentagone ADFHK *described about the* ⊙ LEG.

Resolution.

1. In the ⊙ LEG, inscribe an equilateral & equiangular pentagone. *P.* 11. *B.* 4.
2. To the points B, E, G, I, L draw the rays CB, CE, CG, CI, CL. *Pof.* 1.
3. At the extremities of these rays erect the ⊥ produced AD, DF, FH, HK, KA. *P.* 11. *B.* 1.

Preparation.

Draw the straight lines CA, CD, CF, CH, CK. *Pof.* 1.

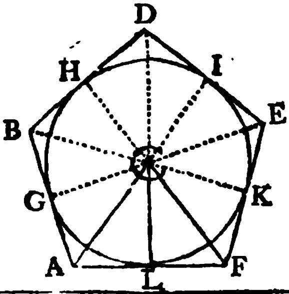
Demonstration.

BECAUSE the straight lines AD, DF, FH, HK, KA are ⊥ at the extremities of the rays CB, CE, CG, CI, CL. *(Ref. 3.)*

1. Those straight lines will touch the ⊙ in the points B, E, G, I, L. { *P.* 16. *B.* 3. *Cor.*
   And the ∀ DBE + DEB, FEG + FGE, HGI + HIG, KIL + KLI, ABL + ALB, taken two by two are < 2 ∟. *Ax.* 8. *B.* 1.
2. Therefore these straight lines AD, DF, FH, HK, KA will meet in the points D, F, H, K, A. *Lem.* 1. *B.* 1.
   But since in the △ CEF, CGF the side FE is = to the side FG *(P. 37. Cor. B. 3. & Ref. 3)*, CE = GC *(D. 15. B. 1.)* & CF common to the two △.

3. The ∠ CFE is = to the ∠ CFG & ∠ ECF = to ∠ GCF. *P.* 8. *B.* 1.
4. Consequently, ∠ EFG, is double of the ∠ CFG, & ∠ ECG double of the ∠ FCG; likewise ∠ GHI is double of the ∠ CHG & ∠ GCI double of ∠ GCH.
5. Moreover, ∠ ECG is = to ∠ GCI, on account of the equal arches EG, GI *(Ref.* 1.*)* *P.* 28. *B.* 3.
6. Consequently, ∠ FCG is = to ∠ GCH. *Ax.* 7. *B.* 1.
But the ∠ CGF, CGH, of the △ CFG, CHG being also equal *(Ref.* 3. & *Ax.* 10. *B* 1,) & CG common to the two △.
7. The straight line FG is = to GH & ∠ CFG is = to ∠ CHG. *P.* 26. *B.* 1.
8. Wherefore FH, is double of FG, & likewise DF is double of EF. *Ax.* 2. *B.* 1.
And because FG is = to EF *(P.* 37. *Cor. B.* 3*)*.
9. The straight line FH is also = to DF, *(Ax.* 6. *B.* 1*)*, & likewise the straight lines HK, KA, AD are = to FH, or DF.
Again ∠ EFG or DFH being double of the ∠ CFG, the ∠ GHI or FHK double of the ∠ CHG and also ∠ CFG = to ∠ CHG. *(Arg* 7*)*.
10. The ∠ DFH, FHK are = to one another, and likewise the ∠ HKA, KAD, ADF are = to DFH or FHK.
11. Consequently there has been described about the ⊙ LEG a pentagon ADFFHK *(Arg.* 1*)*. equilateral *(Arg.* 9*)*, and equiangular *(Arg.* 10*)*. *D.* 4. *B.* 4.

Which was to be done.



PROPOSITION XIII. *PROBLEM XIII.*

TO inscribe a circle (GHIKL), in a given equilateral and equiangular Pentagon (ABDEF).

Given	Sought
The equilateral & equiangular pentagon ABDEF.	*The* ⊙ GHIKL *inscribed in this pentagon.*

Resolution.

1. Bisect the two ∀ BAF, AFE of the pentagon ABDEF by the straight lines produced CA, CF. *P.* 9. *B.* 1.
2. From the point of concurse C let fall upon AF the ⊥ CL. *P.* 12. *B.* 1.
3. From the point C at the distance CL, describe the ⊙ GHIKL. *Pos.* 3.

Preparation.

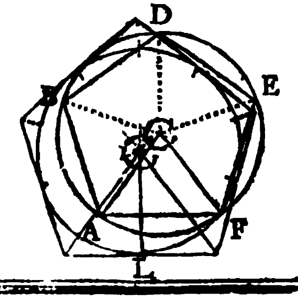
1. Draw the straight lines CB, CD, CE. *Pos.* 1.
2. From the point C let fall upon AB, BD, DE, EF the ⊥ CG, CH, CI, CK. *P.* 12. *B.* 1.

DEMONSTRATION.

BECAUSE in the △ ACF, ACB the side AF is = to the side AB, the side CA common to the two △ & ∀ CAF = to ∀ CAB. *(Ref. 1. & given).*

1. It follows that ∀ CFA is = to ∀ CBA. *P.* 4. *B.* 1.
But ∀ AFE being = to ∀ DBA and double of ∀ CFA *(Ref. 1).*
2. Hence, ∀ DBA is also double of the ∀ CBA, or ∀ CBD = to ∀ CBA. *Ax.* 6. *B.* 1.
It may be demonstrated after the same manner, that
3. The ∀ CDB is = to ∀ CDE & ∀ CED = to ∀ CEF.
Therefore in the △ CBG, CBH, ∀ CBG = to ∀ CBH *(Arg. 2),* ∀ CGB = to ∀ CHB *(Prep. 2 & Ax. 10. B. 1.),* & CB common to the two △. *P.* 26. *B.* 1.
4. Consequently, CG is = to CH; likewise CI, CK, CL are = to CH or to CG.
5. Therefore the ⊙ described from the center C at the distance CL will also pass thro' the points G, H, I, K. *D.* 15. *B.* 1.
And because the straight lines AB, BD, DE, EF, FA are ⊥ at the extremities of the rays CG, CH, CI, CK, CL *(Prep. 2 & Ref. 2).* *D.* 15. *B.* 1.
6. Those straight lines will touch the ⊙ GHIKL *(P. 16. Cor. B. 3)*; and this ⊙ is inscribed in the pentagon ABDEF. *D.* 5. *B.* 4.

Which was to be done.



PROPOSITION XIV. *PROBLEM XIV.*

TO defcribe a circle (ADF); about a given equilateral and equiangular pentagon (ABDEF).

Given	Sought
The equilateral & equiangular pentagon.	*The* ⊙ ADF, *defcribed about this pentagon.*

Refolution.

1. Bifect the ∀ BAF, AFE by the ftraight lines CA, CF produced. *P.* 9. *B.* 1.
2. From the point C, where thofe ftraight lines interfect each other, at the diftance CA defcribe the ⊙ ADF. *Pof.* 3.

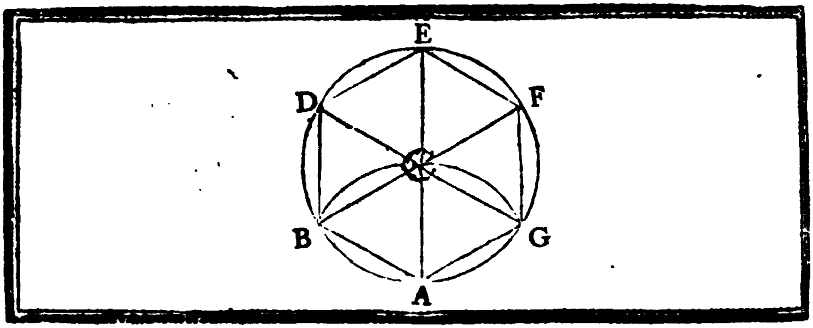
Preparation.

Draw the ftraight lines CB, CD, CE. *Pof.* 1.

DEMONSTRATION.

1. THE ftraight lines CB, CD, CE bifect the ∀ ABD, BDE, DEF. { *P.* 13. *B.* 4. *Cor.*
2. And becaufe the ∀ BAF is = to the ∀ AFE, the ∀ CAF will be alfo = to the ∀ CFA. *Ax.* 7. *B.* 1.
3. Wherefore CA is = to CF. *P.* 6. *B.* 1.
It may be demonftrated after the fame manner, that
4. Each of the ftraight lines CB, CD, CE is = to CA or to CF.
5. From whence it follows, that the ⊙ defcribed from the center C at the diftance CA will pafs thro' the points B, D, E, F. *D.* 15. *B.* 1.
6. Confequently the ⊙ ADF, is defcribed about the given pentagon ABDEF. *D.* 6. *B.* 4.

Which was to be done.



PROPOSITION XV. *PROBLEM XV.*

TO inscribe an equilateral and equiangular Hexagon (ABDEFG) in a given Circle (BEG).

Given	Sought
The ⊙ BEG.	*The equilateral & equiangular Hexagon* ABDEFG, *inscribed in the* ⊙ BEG.

Resolution.

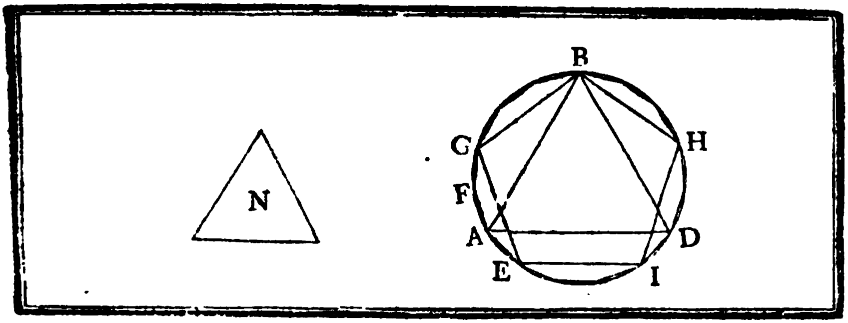
1. Find the center C of the ⊙ BEG, and draw any diameter AE. *P.* 1. *B.* 3.
2. From the center A, at the distance AC describe an arch of a ⊙ BCG. *Pos.* 3.
3. Draw the rays CG, CB produced to D & F. *Pos.* 1. & 2.
4. Draw the straight lines AB, BD, DE, EF, FG, GA. *Pos.* 1.

DEMONSTRATION.

BECAUSE in the △ BCA, the side BC is = to the side AC, & AB is also = to AC (*Ref.* 3. & *D.* 15. B. 1). *D.* 24. *B.* 1.

1. This △ is equilateral & equiangular. *P.* 5. *B.* 1.
2. Wherefore, ∀ BCA is = to the third part of 2 ∟. & likewise ∀ ACG is also = to the third part of 2 ∟. *P.* 32. *B.* 1.
   But the ∀ BCA + ACG + GCF being = to 2 ∟. (*P.* 13. *B.* 1),
3. The ∀ GCF is also = to the third part of 2 ∟; & the ∀ BCA, ACG, GCF are = to one another. *Ax.* 1. *B.* 1.
4. Consequently, the ∀ FCE, ECD, DCB which are = to them as being their vertical opposite ones, are also = to one another. *P.* 15. *B.* 1.
5. Hence, the arches BA, AG, GF, FE, ED, DB are = to one another, as likewise the chords BA, AG, GF, FE, ED, DB. *P.* 26. *B.* 3. *P.* 29. *B.* 3.
6. Therefore the Hexagon ABDEFG inscribed in the ⊙ BEG is equilateral.
   Moreover the arch BA being = to the arch ED (*Arg.* 5); if the common arch AGFE be added to both.
7. The arch BAGFE will be = to the arch AGFED. *Ax.* 2. *B.* 1.
8. From whence it follows, that ∀ EDB is = to ∀ DBA, and likewise each of the ∀ FED, GFE, AGF is = to the ∀ EDB, or to the ∀ DBA. *P.* 27. *B.* 3.
9. Therefore the equilateral Hexagon ABDEFG, inscribed in the ⊙ BEG is also equiangular. *D.* 3. *B.* 4.

Which was to be done.



PROPOSITION XVI. *PROBLEM XVI.*

TO inscribe an equilateral and equiangular quindecagon (EAFG &c.) in a given circle (EBI).

Given	Sought
The ⊙ EBI	*The equilateral & equiangular quindecagon* EAFG *&c.*

Resolution.

1. Describe an equilateral △ N. *P.* 1. *B.* 1.
2. Inscribe in the ⊙ EBI, a △ ABD, equiangular to the equilateral △ N. *P.* 2. *B.* 4.
3. And an equilateral & equiangular pentagon EGBHI. *P.* 11. *B.* 4.
4. Draw the chord EA & place it 15 times around in the ⊙ EBI. *P.* 1. *B.* 4.

DEMONSTRATION.

BECAUSE the △ ABD is equiangular to the equilateral △ N *(Ref. 2).*

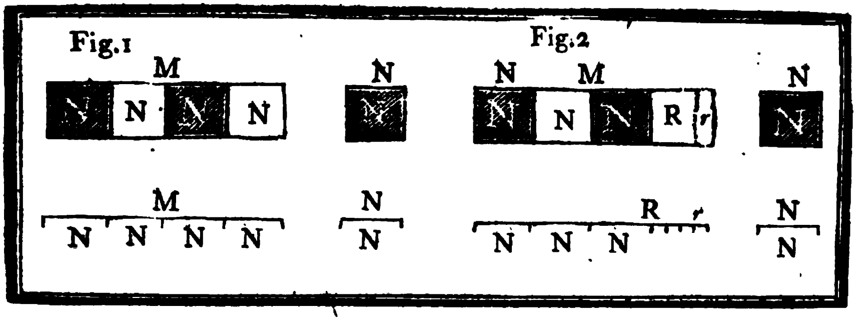
1. This △ is also equilateral, or AD is = to AB = to BD. *P.* 6. *B.* 1.
2. And the arches AD, AB, BD are = to one another, or each is the third part of the whole ○. *P.* 28. *B.* 3.

   Again, because the pentagon EGBHI is equilateral, *(Ref. 3).*
3. Each of the arches EG, GB, BH, HI, IE is the fifth part of the whole ○. *P.* 28. *B.* 3.

   But the arch AB being the third part *(Arg.* 2) and the arch EG or GB the fifth part of the ○ *(Arg. 3).*
4. There may be placed in the arch AB five sides of the quindecagon, and in each of the arches EG, GB three sides of the quindecagon, or in the arch EGB six sides of the quindecagon.
5. Consequently one of these sides may be placed in the arch AE, and the equilateral quindecagon EAFG &c. will be inscribed in the ⊙ EBI. *D.* 3. *B.* 4.

   Moreover, since each of its ∀ FAE is contained in an arch FHE which is = to thirteen parts of the fifteen, into which the circumference is divided,
6. These ∀ will be = to one another. *P.* 27. *B.* 3.
7. Therefore there has been inscribed in the ⊙ EBI, an equilateral & equiangular quindecagon EAFG.

Which was to be done.



DEFINITIONS.

I.

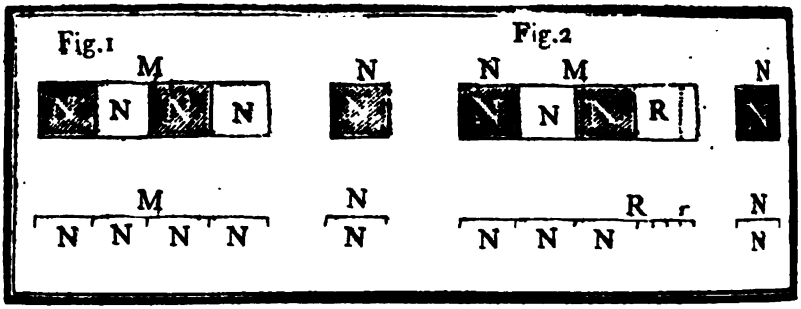
A Less magnitude is said to be a *part* of a greater magnitude, when the less measures the greater.

§. 1. *By the expression of measuring a magnitude* Euclid *means* to be contained in it a certain number of times without a remainder, *that is a less magnitude* N (fig. 1.) *measures a greater* M, *when the magnitude* N *is contained in* M *without a remainder twice, thrice, four times, and in general, any number of times whatsoever, or which comes to the same, when the less magnitude* N *repeated twice, thrice four times, and in general any number of times produces a whole, equal to the greater* M.

§. 2. *Those parts which measure a whole without a remainder, are called* aliquot parts, *and such as are not contained in a whole exactly, but are measured by some other determined quantity which measures also the whole, are called* aliquant parts.

Thus the numbers 2, 3, 4, 6 *are so many aliquot parts of the number* 12 *considered as a whole; as each of the numbers* 2, 3, 4, 6 *is found repeated in* 12 *a certain number of times without a remainder. But the numbers* 5, 7, 9 *&c. are aliquant parts of the same whole* 12; *as they do not measure* 12 *but with a remainder: although they are all measured by unity as well as* 12; *which often happens in other numbers different from unity, as in the number* 9 *which is commensurable to* 12 *by the nnmber* 3, *as also by unity.*

Likewise the magnitude N (fig. 2.) *is an aliquant part of the magnitude* M (= N + N + N + R *&c*), *if* N *measures* M *leaving a remainder* R, *and this remainder* R *be such, that it measures* N *or at least that one of its determined parts as r measures this remainder* R, *as also the magnitude* N *&* *consequently the whole* M.



DEFINITIONS.

§. 3. IN *general numbers are ſaid to be commenſurable to each other which may reſult from unity or one of its aliquot parts repeated a determined number of times: or what amounts to the ſame that which is meaſured by unity or one of its aliquot parts.*

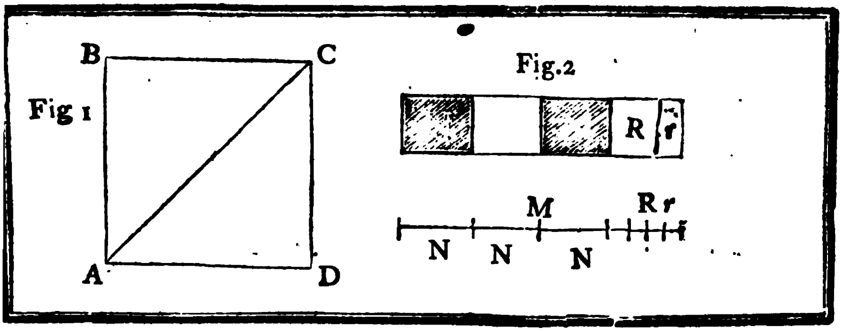
Thus the numbers 6, 9, 17, *and the fractions* $\frac{2}{3}$, $\frac{5}{7}$ *are commenſurable numbers; becauſe the firſt may be conceived to reſult from the determined and ſucceſſive addition of unity; and the laſt from that of the fractions* $\frac{1}{3}$ & $\frac{1}{7}$ *aliquot parts of unity.*

§. 4. *According to this definition,* a commenſurable quantity, *is that which reſults from the determined repetition of any determined quantity. A quantity is therefore commenſurable, when it contains one of its parts, as often as a determined number contains unity.*

§. 5. *Commenſurability is therefore ſomething relative. The magnitudes* M *and* N *are commenſurable, as having a common and determined meaſure* r *which can be taken for unity, and meaſure them both exactly; or, as thoſe two magnitudes may ariſe from the determined repetition of the ſame quantity* R, *be it what it will.*

§. 6. *It follows from this notion of commenſurable numbers, that they are all multiples of each other, or aliquot parts, or aliquant parts. For if the quantities* M *and* N, *are commenſurable,* N *meaſures* M, *or* M *meaſures* N, *or ſome other determined number* r *meaſures them both. In the firſt caſe, the number* M, *is a multiple of* N, *in the ſecond caſe* M, *is an aliquot part of* N, *and in the third, the leſſer of the two is an aliquant part of the leaſt. The ſame is true with reſpect to rational magnitudes in general.*

§. 7. *The number which cannot reſult from a determined repetition of unity or of one of its aliquot parts is called,* irrational or incommenſurable, *with reſpect to unity. And in general, magnitudes which cannot reſult from the determined repetition of the ſame determined quantity conſidered as unity, are* incommenſurable to one another, or irrational.



DEFINITIONS.

THUS *the side* (AD *or* DC) *of the square* (ABCD) *is incommensurable to ts diagonal* (AC), *or how much one contains of the other is inassignable* (Fig. 1).

. 8. *From whence it follows, that* if two magnitudes M and N, are incommensurable to each other, M cannot be a multiple of N; nor an aliquot part, or in fine an aliquant part of this same N, *for if it was, the magnitudes M and* N *could be measured by the same determined magnitude, which is repugnant to the notion of incommensurability* (Fig. 2)

II.

A greater magnitude is said to be a multiple of a less, when the greater is measured by the less.

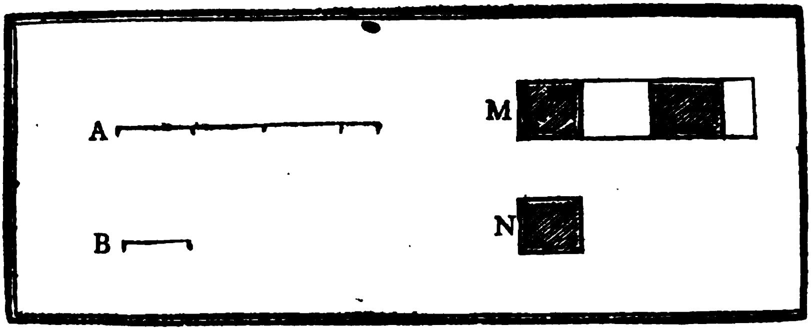
Thus, the number 12 *is said to be a multiple, of the number* 4, *because* 4 *measures* 12 *without a remainder.*

To the term of multiple corresponds that of submultiple, *which signifies, that a less magnitude is an aliquot part of a greater; thus* 4 *is a submultiple of* 12, *as* 12 *is a multiple of* 4.

III.

Ratio, is a mutual relation of two magnitudes of the same kind to one another n respect of quantity.

This definition is imperfect, and is commonly believed to be none of Euclid'*s, but the addition of some unskilful editor; for though the idea of ratio includes a certain relation of the quantities of two homogeneous magnitudes, yet this general character is not sufficient; because the quantities of two magnitudes are susceptible of several sorts of relations different from that of ratio. Thus, when in a circle the square of the perpendicular let fall from the circumference on the diameter, is represented as constantly equal to the difference of the squares of the ray, and of the portion of the ray intercepted between the center and the perpendicular, without doubt, this perpendicular is considered as bearing a certain relation to this portion of the ray, but it is manifest that this relation is not a ratio, since the quantities are compared only by the means of the ray which is a third homogeneous magnitude different from the quantities compared.*



DEFINITIONS.

IV.

MAGNITUDES are said to have a ratio to one another; when the less can be multiplied so as to exceed the other.

§. 1. The lines A *&* B *have a ratio to one another, because the line* B, *for example, taken three times and a half, is equal to the line* A, *and taken four times exceeds it. The Rgles* M *&* N *have also a ratio to one another, because the Rgle* N *taken three times and a half, is = to Rgle* M, *and repeated oftner exceeds it.*
But the line B, *and the Rgle* M *have no ratio to one another, because the line* B *repeated ever so often, can never produce a magnitude which would equal or exceed the Rgle* M. *Therefore, only magnitudes of the same kind can have a ratio to one another, as numbers to numbers, lines to lines, surfaces to surfaces, and solids to solids.*

§. 2. In consequence of this definition, a finite magnitude and an infinite one, have no ratio to one another, though they be supposed of the same kind. For a magnitude conceived infinite, is conceived without bounds, consequently a finite magnitude repeated ever so often (provided the number of repetions be determined) can never become equal or exceeds an infinite magnitude.

§. 3. *A* ratio *is* commensurable, *when the terms of the ratio* M & N *are commensurable to each other, & a ratio is said to be incommensurable when the terms of the ratio are* incommensurable.

§ 4. *The* antecedent *of the ratio of* M *to* N, *is the first of the two terms which are compared, and the other is called its consequent.*

V.

The first of four magnitudes is said to have the same ratio to the second, which the third has to the fourth, when any equimultiples whatsoever of the first and third being taken, and any equimultiples whatsoever of the second and fourth.

DEFINITIONS.

If the multiple of the first, be less than that of the second, the multiple of the third is also less than that of the fourth; or if the multiple of the first be equal to that of the second, the multiple of the third is also equal to that of the fourth, or if the multiple of the first be greater than that of the second, the multiple of the third is also greater than that of the fourth.

§. 1. *The ratio of the number 2 to the number 6, is the same as that of the number 8 to the number 24, for if the two antecedents 2 & 8 be multiplied by the same number M, and the two consequents 6 & 24 by another number N; the multiple 2 M of the first antecedent cannot be = or > or < the multiple 6 N of its consequent, unless the multiple of the second antecedent 8 M, be at the same time = or > or < the multiple 24 N of its consequent, for it is evident that*

If $2M$ *be* $=$ *to* $6N$,
$2M+2M+2M+2M$ is also $=6N+6N+6N+6N$, that is, $8M=24N$.
Likewise, *If* $2M$ *be* $>6N$, *then*
$2M+2M+2M+2M$ is also $>6N+6N+6N+6N$, that is, $8M>24N$.
And in fine, *If* $2M$ *be* $<6N$, *then*
$2M+2M+2M+2M$ is also $<6N+6N+6N+6N$, that is, $8M<24N$.

§. 2. *On the contrary, the numbers 2, 3 & 7, 8 are not in the same ratio; for if the antecedents be multiplied by 3, and the consequents by 2, there will result the four multiples 6, 6, 21, 16, where the multiple 6 of the Ist antecedent is equal to the multiple 6 of its consequent, whilst 21 multiple of the II. antecedent is greater than 16 multiple of its consequent.*

§. 3. *Incommensurable magnitudes can never have their equimultiples equal, otherwise they would be commensurable to one another, wherefore incommensurables are shewn to be proportional only from the joint excess or defect of their equimultiples; whereas commensurable magnitudes being capable of a joint equality, and inequality of their equimultiples, are shewn to be proportional from the joint equality or excess of their equimultiples, hence it is that the signs in this definition by which proportionality is discovered, are applicable to all kinds of magnitude whatsoever.*

§. 4. *What is true with respect to the correspondence of multiples, is also true with respect to that of submultiples. But it is probable that* Euclid *preferred the use of multiples to that of submultiples, because he could not prescribe to take submultiples without first shewing how to divide magnitude into equal parts, whilst the formation of multiples required no such principle. This Geometer had a right to assume for granted, that the double triple, or any multiple of a magnitude could be taken, bnt was under the necessity of shewing by the*

Resolution of a problem, how to take away an aliquot part from a given line, and the resolution of this problem supposing the doctrine of similitude, could not be given but in the IX. Proposition of the VI. Book.

VI.

Magnitudes which have the same ratio, are called proportionals.

When four magnitudes A,B,C,D *are proportional, it is usually exprest thus,* A : B = C : D *and in words, the first is to the second as the third to the fourth.*

VII.

When of the equimultiples of four magnitudes (taken as in the 5th definition) the multiple of the first is greater than that of the second, but the multiple of the third is not greater than the multiple of the fourth; then the first is said to have to the second a greater ratio than the third magnitude has to the fourth, and on the contrary, the third is said to have to the fourth a less ratio than the first has to the second.

§. 1. *Such are the ratios* 3 : 2 & 11 : 9 *for if the antecedents be multiplied by* 9, *and the consequents by* 13, *there will result* 27 : 26 ; 99 : 117.

$$\begin{array}{c} 3 : 2 ; 11 : 9 \\ 9 \quad 13 \quad 9 \quad 13 \\ \hline 27 : 26 ; 99 : 117 \end{array}$$

Where the correspondence of the multiples does not hold, the first antecedent 27 *being greater than its consequent* 26 *whilst the second antecedent* 99 *is less than its consequent* 117.

§. 2. *To discover by inspection the inequality of two ratios* A : B & C : D *by this character of the non correspondence of multiples, it suffices to chuse for multiples, the two terms of one of the two ratios, for Ex.* C : D, *and to multiply the antecedents* A & C *by the consequent* D *of this ratio; and the two consequents* B & D *by the antecedent* C *of the same ratio, in this manner.*

$$\begin{array}{c} A : B ; C : D \\ D \quad C ; D : C \\ \hline AD : BC ; CD : D.C \end{array} \qquad \begin{array}{c} 3 : 5 ; 7 : 9 \\ 9 \quad 7 \quad 9 \quad 7 \\ \hline 27 : 35 ; 63 ; 63 \end{array}$$

Which being done, the two products C.D & D.C *will be found equal, whilst the two others* A.D & B.C *are unequal, and in particular, if the multiple of one of the antecedents be greater than that of its consequent, whilst the multiple of the other is equal to its, then the terms of the lesser ratio have been chosen for multipliers. On the contrary, if the multiple of one of the antecedents be less than that of its consequent, whilst the multiple of the other is equal to its, then the terms of the greater ratio have been chosen for multipliers.*

VIII.

Analogy or proportion, is the similitude of ratios.

As a sign and character of proportionals has been already given (in Def. 5.) *this is a superfluous definition, a remark of some scholiast shufled into the text which interrupts the coherence of* Euclid's *genuine definitions.*

IX.

Proportion consists in three terms at least.

§. 1. *Proportion consisting in the equality of two ratios, and each ratio having two terms, in a proportion there are four terms, of which the first and fourth are called the* extreames, *and the second and third the* means, *those four terms are considered as only three, when the consequent of the first ratio at the same time holds the place of the antecedent of the second ratio : it is for this reason, that proportions are distinguished into* discrete, *and* continued. *A proportion is* discrete *when the two means are unequal, and it is called* continued *when these same terms are equal, thus this proportion* 2 : 4 = 5 : 10 *is* discrete *because the two mean terms* 4 & 5 *are unequal, on the contrary, the proportion* 2 : 4 = 4 : 8 *is a continued proportion on account of the equality of the mean terms* 4 & 4.

§. 2. *A series of magnitudes in continued proportion, forms* a geometrical progression, *such are the numbers* 1, 2, 4, 8, 16, 32, 64, &c.

X.

When three magnitudes are proportional the first is said to have to the third the duplicate ratio of that which it has to the second.

XI.

When four magnitudes are continual proportionals, the first is said to have to the fourth the triplicate ratio of that which it has to the second, and so on quadruplicate, &c. increasing the denomination still by unity in any number of proportionals.

XII.

In proportionals, the antecedent terms are called Homologous to one another, as also the consequents to one another.

XIII.

Proportion is said to be *alternate* when the antecedent of the first ratio is compared with the antecedent of the second, and the consequent of the first ratio with the consequent of the second.

$$\text{If } \left.\begin{matrix} A : B = C : D \\ 4 : 5 = 16 : 20 \end{matrix}\right\} \text{ then by alternation. } \left\{\begin{matrix} A : C = B : D \\ 4 : 16 = 5 : 20 \end{matrix}\right.$$

When the proportion is disposed after this manner, it is said to be done by permutation *or* alternately, permutando *or* alternando.

XIV.

But when the confequents are changed into antecedents, and the antecedents into confequents in the fame order, it is faid that the comparifon of the terms is made by *inverfion or invertendo.*

$$\left.\begin{array}{c} A : B = C : D \\ 3 : 9 = 4 : 12 \end{array}\right\} \textit{therefore invertendo.} \left\{\begin{array}{c} B : A = D : C \\ 9 : 3 = 12 : 4 \end{array}\right.$$

XV.

But the comparifon is made by *compofition* or *componendo*, when the fum of the confequents and antecedents is compared with their refpective confequents.

$$\begin{array}{c} A : B = C : D \\ 3 : 9 = 4 : 12 \end{array} \left\{\begin{array}{c} \textit{therefore} \\ \textit{componendo} \end{array}\right\} \begin{array}{c} A + B : B = C + D : D \\ 3 + 9 : 9 = 4 + 12 : 12 \end{array}$$

XVI.

The comparifon is made by *divifion* of ratio, or *dividendo* when the excefs of the antecedent above its confequent, is compared with its confequent.

$$\left.\begin{array}{c} \textit{If } A : B = C : D \\ 9 : 3 = 12 : 4 \end{array}\right\} \textit{dividendo.} \left\{\begin{array}{c} A - B : B = C - D : D \\ 9 - 3 : 3 = 12 - 4 : 4 \end{array}\right.$$

XVII.

The comparifon is made by *the converfion* of ratio, *or convertendo*, when the antecedent is compared to the excefs of the antecedent above its confequent.

$$\begin{array}{c} \textit{If } A : B = C : D \\ 9 : 3 = 12 : 4 \end{array} \left\{\begin{array}{c} \textit{therefore} \\ \textit{convertendo.} \end{array}\right\} \begin{array}{c} A : A - B = C : C - D \\ 9 : 9 - 3 = 12 : 12 - 4 \end{array}$$

XVIII.

A conclufion is drawn from *equality of ratio* or *ex æquo*, when comparing two feries of magnitudes of the fame number, fuch that the ratios of the firft be equal to the ratios of the fecond, each to each, (whether the comparifon be made in the fame order or in an inverted one), it is concluded that the extreames of the two feries are in proportion.

The fenfe of this definition is as follows, if A, B, C, D *be a feries of four magnitudes, and* a, b, c, d *a feries of four other magnitudes, fuch that*

$$\left.\begin{array}{c} A : B = a : b \\ B : C = b : c \\ C : D = c : d \end{array}\right\} \textit{or in an inverted order.} \left\{\begin{array}{c} A : B = c : d \\ B : C = b : c \\ C : D = a : b \end{array}\right.$$

In the one or the other case it is allowed to infer ex æquo, *when the ratio of the extreames* A : D *of the I. series is equal to the ratio of the extreames* a : d *of the II. series; or that* A : D = a : d.

I. A, B, C, D	15, 3, 45, 9
II. a, b, c, d	10, 2, 30, 6
A : D = a : d	15 : 9 = 10 : 6

XIX.

The equality of ratio is called ordinate ratio, when the ratio of the first series are equal to the ratios of the second series each to each in the same direct order.

For Example let A : B = a : b
B : C = b : c
C : D = c : d

Here the ratios are equal each to each in the same direct order, because the first magnitude is to the second of the first rank, as the first to the second of the other rank, and as the second is to the third of the first rank, so is the second to the third of the other, and so on in order. If therefore it is inferred that the extreames are proportional, or that A : D = a : d. *the inference is said to be made from direct equality,* or ex æquo ordinate.

XX.

On the contrary, *equality of ratio* is called *inverted* or *perturbate analogy*, in the second case, that is when the ratios of the first series are equa to those o the second series each to each, taking those last in an inverted order.

§. 1. *Let the two series of magnitudes be.*

$$\left.\begin{matrix} A, & B, & C, & D \\ a, & b, & c, & d \end{matrix}\right\} \text{ where it is supposed } \left\{\begin{matrix} A : B = c : d \\ B : C = b : c \\ C : D = a : b \end{matrix}\right.$$

Here the ratios of the I. series are equal to the ratios of the II. series each to each, but in an inverted order, that is the first magnitude is to the second of the first rank, as the last but one is to the last of the second rank, and as the second is to the third of the first rank, so is the last but two to the last but one of the second rank; and as the third is to the fourth of the first rank, so is the third from the last to the last but two of the second rank, and so in a cross order.

If therefore it be inferred that A : D = a : d.

This inference is said to be made ex æquo perturbate,

§. 2. *Beginners may easily distinguish the case of direct equality from that of perturbate equality, if they remember that when two terms are common to two proportions, and that they occupy indifferently either the first and third, or the second and fourth place, that it is always the case of direct equality; For Example.*

$$\begin{array}{lcl} A : B = a : b & & B : A = b : a & & A : B = a : b \\ B : C = b : c & or & B : C = b : c & or & C : B = c : b \\ \hline A : C = a : c & & A : C = a : c & & A : B = a : c \end{array}$$

Here are always two proportions which have in common the two terms B & b *occupying the first and third, or the second and fourth places; the two other terms* A & C *are proportional to the two others* a & c *taking them in the same order.*

§. 3. *On the contrary when the two terms which are common to the two proportions, are either the means or the extreames, it is the case of* perturbate *equality, for example*

$$\begin{array}{lcl} \text{If } A : B = b : c & & B : A = c : b & & A : B = b : c \\ B : C = a : b & or & B : C = a : b & or & C : B = b : a \\ \hline A : C = a : c & & A : C = a : c & & A : C = a : c \end{array}$$

In those three cases the terms B & b *which are common to the two proportions, are either the extremes or the means; consequently the other terms are in proportion, so that the two terms, which arise from the same proportion* A & C *or* a & c *remain extreams or means.*

These are the denominations given to the different ways of concluding by analogy, Euclid *now proceeds to demonstrate that they are just.*

POSTULATES.

I.

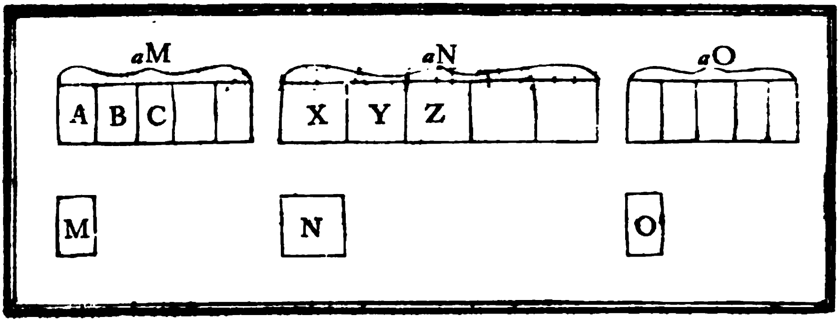
LET it be granted, that any magnitude may be doubled, tripled, quadrupled, or in general, that any multiple of it may be taken.

II.

That from a greater magnitude, there may be taken one or ſeveral parts equal to a leſs magnitude of the ſame kind.

ABREVIATIONS.

Mgn.	Magnitude.
Mult.	Multiple.
Equimult.	Equimultiple.



PROPOSITION I. THEOREM I.

IF any number of magnitudes (aM, aN, aO &c) be equimultiples of as many (M, N, O &c) each of each, the sum ($aM+aN+aO$ &c) of all the first is the same multiple of the sum ($M+N+O$ &c) of all the second, as any one of the first (aM) is of its part (M).

Hypothesis.

aM, aN, aO *are equimultiples of* M *each*, N *of*, O *each*.

Thesis.

$aM+aN+aO$ *is the same multiple of* $M+N+O$ *that* aM *is of* M, *or* aN *of* N *&c.*

Preparation.

The mgn. aM being the same multiple of M, that aN is of N *(Hyp.)*, as many magnitudes A, B, C, &c. as can be taken out of aM each equal to M, so many X, Y, Z, &c. can be taken out of aN, each equal to N.

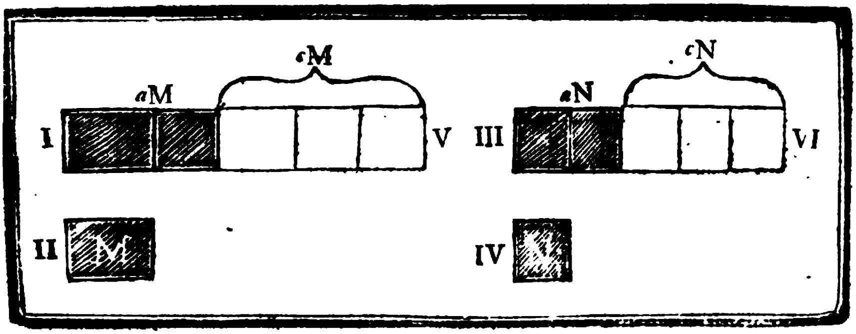
Let then A, B, C be each equal to M & X, Y, Z each equal to N. *Pos. 2. B. 5.*

DEMONSTRATION.

BECAUSE aM is the same multiple of M, that aN is of N *(Hyp)*,

1. As many magnitudes X, Y, Z, &c. as are in aN each equal to N, so many A, B, C, &c. are there in aM each equal to M.
But $A=M$ & $X=N$ *(Prep.)*,
2. Therefore $A+X=M+N$ *Ax. 2. B. 1.*
Likewise B being $=M$ & $Y=N$ *(Prep.)*,
3. It follows that $B+Y=M+N$ *Ax. 2. B. 1.*
Again, because $C=M$ & $Z=N$ *(Prep.)*,
4. It follows that $C+Z=M+N$ *Ax. 2. B. 1.*
Consequently there is in aM as many Magnitudes $=M$, as there are in $aM+aN=M+N$.
5. From whence it follows that $aM+aN$ is the same multiple of $M+N$, that aM is of M, or that aN is of N, & likewise $aM+aN+aO$ is the same multiple of $M+N+O$, that aM is of M or aN of N, &c.

Which was to be demonstrated



PROPOSITION II. THEOREM. II.

IF the first magnitude (*a* M) be the same multiple of the second (M), that the third (*a* N) is of the fourth (N); & the fifth (*c* M) the same multiple of the second (M), that the sixth (*c*N) is of the fourth (N); then shall the first together with the fifth (*a* M + *c* M) be the same multiple of the second (M), that the third together with the sixth (*a* N + *c* N) is of the fourth (N).

Hypothesis.

a M & *a* N } *likewise* { *c* M & *c* N *are equimultiples of* } M *each* & *of* N *each*

Thesis.

a M + *c* M *is the same multiple of* M, *that a* N + *c* N *is of* N.

Demonstration.

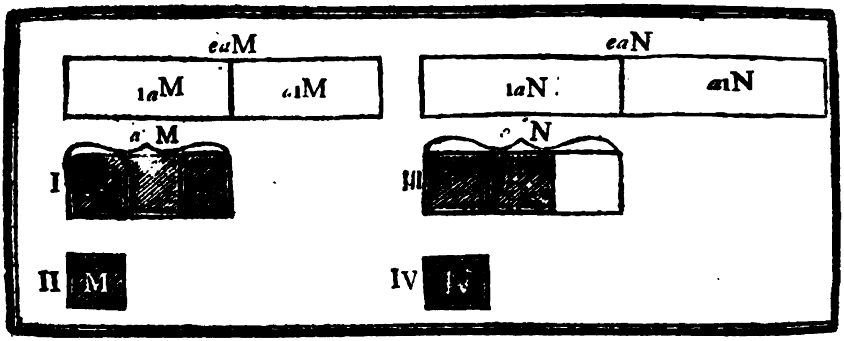
BECAUSE *a* M is the same multiple of M, that *a* N is of N (*Hyp.*),

1. There are as many magnitudes in *a* M = to M as there are in *a* N = to N.

   In like manner, because *c* M is the same multiple of M, that *c* N is of N (*Hyp.*),

2. There are as many magnitudes in *c* M = to M as there are in *c* N = to N.
3. Consequently, as many as are in the whole *a* M + *c* M equal to M, so many are there in the whole *a* N + *c* N = to N. *Ax.* 2. *B.* 1.
4. Therefore *a* M + *c* M is the same multiple of M that *a* N + *c* N is of N.

Which was to be demonstrated.



PROPOSITION III. THEOREM III.

IF the first magnitude (a M) be the same multiple of the second M, that the third (a N) is of the fourth (c N), and if of the first (a M) and third (a N) there be taken equimultiples (e a M, e a N); these (e a M, e a N) shall be equimultiples, the one of the second (M) and the other of the fourth (N).

Hypothesis.

I. a M & a N *are two equimultiples of* M & N *each of each*

II. e a M & e a N *are two equimultiples of* a M & a N *each of each*

Thesis.

e a M *is the same multiple of* M *that* e a N *is of* N.

Preparation.

Divide e a M into its parts 1 a M, a 1 M, &c. each = a M,
And e a N into its parts 1 a N, a 1 N, &c. each = a N.

DEMONSTRATION.

BECAUSE e a M is the same multiple of a M, that e a N is of a N (*Hyp.* 2.).

1. There are as many magnitudes in e a M = to a M as there are in e a N = to a N
2. Therefore the number of parts 1 a M, a 1 M, &c. in e a M, is = to the number of parts 1 a N, a 1 N, &c. in e a N.

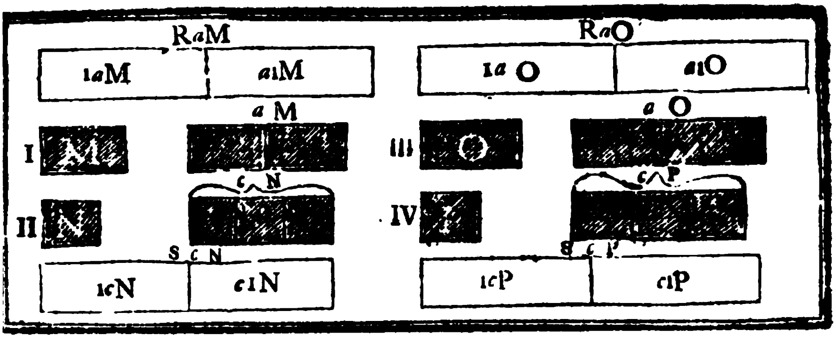
But because a M is the same multiple of M, that a N is of N, and that 1 a M = a M, 1 a N = a N.

3. The magnitude 1 a M is the same multiple of M, that 1 a N is of N.
4. In like manner a 1 M is the same multiple of M, that a 1 N is of N.

Since then I mgn. 1 a M is the same multiple of the II mgn. M.
that the III mgn. 1 a N is of the IV mgn. N
& that the V mgn. a 1 M is the same multiple of the II mgn. M
that the VI mgn. a 1 N is of the IV mgn. N.

5. It follows that the magnitude e a M, composed of the I & V mgn. 1 a M + a 1 M, is the same multiple of the II mgn. M, that the mgn. e a N, composed of the III & VI mgn. 1 a N + a 1 N is of the IV mgn. N. P. 2. B. 5.

Which was to be demonstrated



PROPOSITION IV. *THEOREM IV*,

IF four magnitudes (M, N, O, P,) are proportional : then any equimultiples (*a* M, *a* O) of the first (M) and third (O), shall have the same ratio to any equimultiples (*c* N, *c* P) of the second (N) and fourth (P).

Hypothesis. | Thesis.

I. M : N = O : P. | *a*M : *c* N = *a* O : *c* P.

II. {*a* M & *a* O} *are equimult. of* {M & O} *also* {*c* N & *c* P} *are equimult. of* {N & P}

Preparation.

1. Take of *a* M & of *a* O any equimult. R *a* M, R *a* O
2. Likewise of *c* N & of *c* P any equimult. S *c* N, S *c* P } *Pos.* 1. *B*. 5.

DEMONSTRATION.

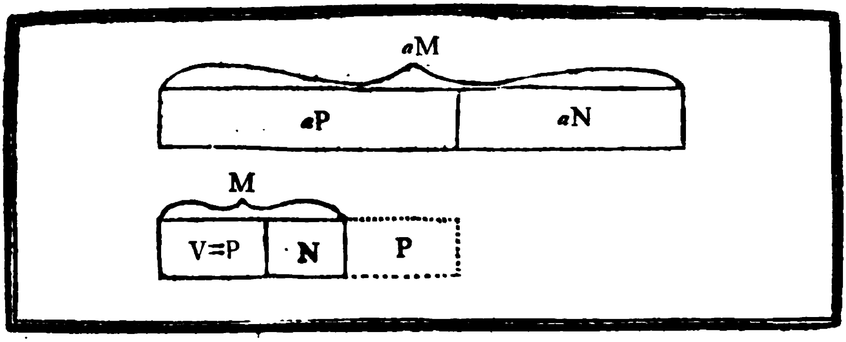
BECAUSE *a* M is the same mult. of M, that *a* O is of O (*Hyp.* 2), & the mgns. R *a* M, R *a* O are equimult. of the mgns. *a*M, *a*O (*Prep.* 1).

1. The magnitude R *a* M is the same multiple of M, that the magnitude R *a* O is of O. *P.* 3. *B.* 5.
2. In like manner, the magnitude S *c* N is the same multiple of N that S *c* P is of P.
   And as M : N = O : P (*Hyp.* 1.) & R *a* M, R *a* O are any equimultiples of the I term M and of the III O ; and S *c* N, S *c* P any equimultiples of the II term N and of the IV P (*Arg.* 1 & 2).
3. If R *a* M be >, = or < S *c* N, R *a* O will be >, = or < S *c* P. *D.* 5. *B.* 5.
   But the magnitudes R *a* M & R *a* O are any equimultiples of the magnitudes *a* M & *a* O, and the magnitudes S *c* N, S *c* P are any equimultiples of the magnitudes *c* N & *c* P (*Prep.* 1 & 2).
4. Consequently, the ratio, of *a* M to *c* N is = to the ratio of *a* O to *c* P ; or *a* M : *c* N = *a* O : *c* P *D.* 5. *B.* 5.

Which was to be demonstrated.

COROLLARY.

IT is manifest that if S *c* N *be* >, = *or* < R *a* M ; *likewise* S *c* P *will be* >, = *or* < R *a* O (*Arg.* 3.) ; *hence* *c* N : *a* M = *c* P : *a* O (*D.* 5. *B.* 5.). *Therefore, if four magnitudes be proportional, they are also by inversion or invertendo.*



PROPOSITION V. THEOREM V.

If a magnitude (a M) be the ſame multiple of another (M), which a magnitude (a N) taken from the firſt, is of a magnitude (N) taken from the other, the remainder (aP) ſhall be the ſame multiple of the remainder (V), that the whole (a M), is of the whole (M).

Hypotheſis.

I. *The mgns* a M & M *are two wholes*
The mgns a N & N *their parts taken away*
And the mgns a P & V *the remainders*

II. a M *is the ſame multiple of* M
that a N *is of* N.

Theſis.

a P *is the ſame multiple of* V, *that* a M *is of* M.

Preparation.

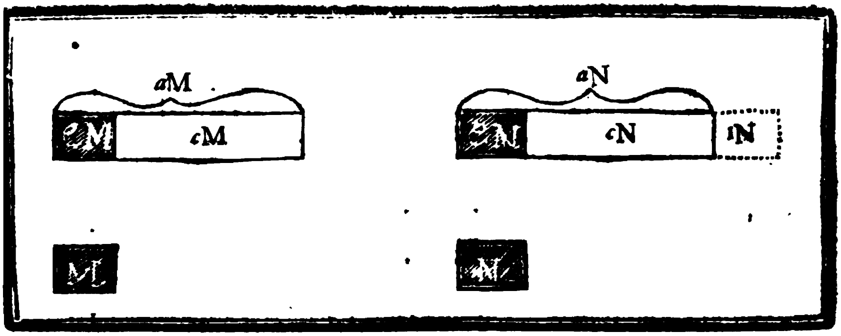
Take a magnitude P ſuch, that a P may be the ſame multiple of P, that a N is of N, or a M of M. *Pof.* 2. *B.* 5.

Demonstration.

Because a N is the ſame multiple of N, that a P is of P *(Prep.)*.

1. The ſum a N + a P, or a M, of the firſt, is the ſame multiple of the ſum N + P of the laſt, that a N is of N *P.* 1. *B.* 5.
   But a M is the ſame multiple of M, or of N + V, that a N is of N *(Hyp. 2)*.
2. Conſequently, the mgn. a M is equimultiple of the mgns. N + P, & N + V.
3. And of courſe N + P = N + V. *Ax.* 7. *B.* 1.
   Taking away the common mgn. N.
4. It follows that the mgn. P is = to the mgn. V. *Ax.* 3. *B.* 1.
5. Conſequently, a P being the ſame multiple of P, that a M is of M *(Prep.)*, a P is alſo the ſame multiple of V, that a M is of M,

Which was to be demonſtrated.



PROPOSITION VI. *THEOREM VI.*

IF two magnitudes (*a* M, *a* N) be equimultiples of two others (M & N) & if equimultiples (*c* M & *c* N) of these, be taken from the first two, the remainders (*e* M & *e* N) are either equal to these others (M & N), or equimultiples of them.

Hypothesis.

I. *a* M & *a* N *are two wholes*
c M & *c* N *their parts taken away*
e M & *e* N *the remainders*

II. *a* M & *a* N *also* *c* M & *c* N *are equimultiples of* M & N

Thesis.

I. *If* *e* M = M, *e* N *will be* = N.
II. *If* *e* M *be multiple of* M, *e* N *will be equimultiple of* N.

CASE I. If *e* M be = M.

Preparation.

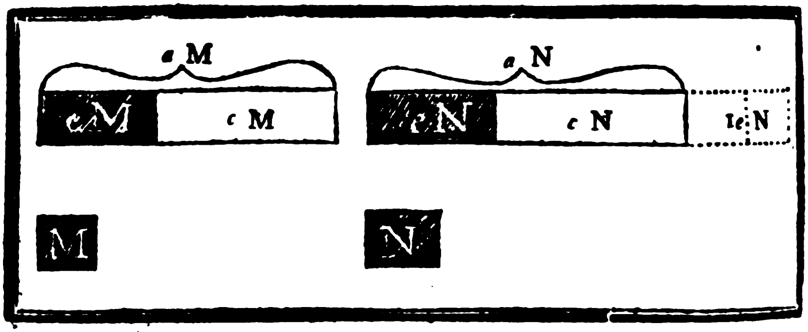
Let 1 N = N. *Pos.* 2. *B.* 5.

DEMONSTRATION.

BECAUSE *c* M is the same multiple of M, that *c* N is of N (*Hyp.* 2.), & that *e* M = M. (*Sup.* 1.), & 1 N = N (*Prep.*),

1. The mgn. *c* M + *e* M, or *a* M, will be the same multiple of M that *c* N + 1 N is of N.
But *a* M being the same multiple of M, that *a* N or *e* N + *c* N is of N (*Hyp.* 2.)
2. The two mgns. *c* N + 1 N & *e* N + *c* N are equimultiples of the same mgn. N.
3. Wherefore the mgn. *c* N + 1 N = *e* N + *c* N *Ax.* 6. *B.* 1.
Taking away the common mgn. *c* N,
4. It follows that 1 N is = *e* N *Ax.* 3. *B.* 1.
But 1 N is = N (*Prep.*);
5. Consequently, *e* N is = N *Ax.* 1. *B.* 1.
6. Therefore if *e* M be = M, *e* N is = N.

Which was to be demonstrated. 1.



CASE II. If *e* M be multiple of M.

Preparation.

Take 1 *e* N the same multiple of N, that *e* M is of M. *Pos.* 1. B. 5.

Demonstration.

BECAUSE *e* M is the same multiple of M, that 1 *e* N is of N *(Prep.)*, & that *c* M is the same multiple of M, that *c* N is of N *(Hyp. 2)*.

1. The magnitude *e* M + *c* M or *a* M, will be the same multiple of M, that 1 *e* N + *c* N is of N. *P.* 2. B. 5.

But *a* M being the same multiple of M that *a* N or *e* N + *c* N is of N *(Hyp. 2)*.

2. Therefore, the two mgns. 1 *e* N + *c* N & *e* N + *c* N are equimultiples of the same mgn. N. *Ax.* 6. B. 1.
3. Consequently, 1 *e* N + *c* N is = *e* N + *c* N

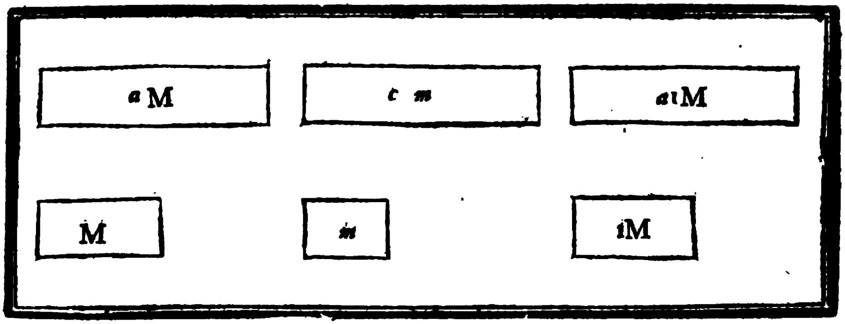
Taking away the common mgn. *c* N

4. It follows that the mgn. 1 *e* N is = *e* N *Ax.* 3. B. 1.

But 1 *e* N is the same multiple of N that *e* M is of M *(Prep.)*.

5. Therefore, if *e* M be an equimultiple of M, *e* N will be an equimultiple of N

Which was to be demonstrated. 11.



PROPOSITION VII. *THEOREM VII.*

EQUAL magnitudes (M & 1 M), have the same ratio to the same magnitude (*m*), and the same (*m*); has the same ratio to equal magnitudes (M & 1 M).

Hypothesis.	Thesis.
M & 1 M *are two equal mgns,*	*I.* M : *m* = 1 M : *m*
& *m is a third.*	*II.* *m* : M = *m* : 1 M

Preparation.

1. Take of M & of 1 M any equimultiples *a* M & *a* 1 M. } *Pof.* 1. *B.* 5
2. And of *m* any multiple whatever *c m*.

DEMONSTRATION.

BECAUSE *a* M & *a* 1 M are equimultiples of M & of 1 M (*Prep.* 1), & M = 1 M (*Hyp.*).

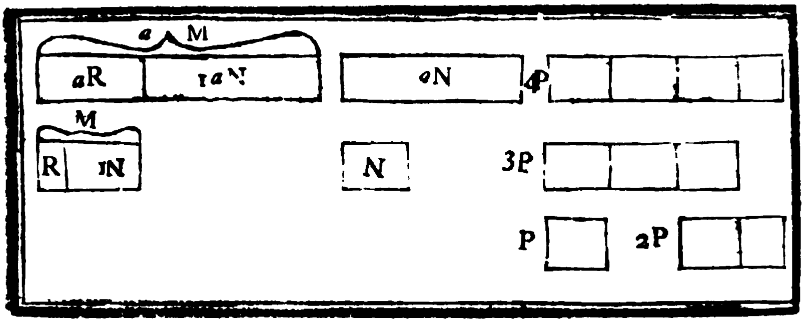
1. The mgn. *a* M is = *a* 1 M. *Ax.* 6. *B.* 1.
2. Therefore, if *a* M be >, =, or < *c m*; *a* 1 M will likewise be >, =, or < *c m*.
   But *a* M & *a* 1 M are equimultiples of the I term M. and of the III term 1 M, as *c m* and *c m* are of the II term *m* and of the IV term *m*,
3. Consequently M : *m* = 1 M : *m*. *D.* 5. *B.* 5.

Which was to be demonstrated. I.

And because *a* M = *a* 1 M (*Arg.* 1);

4. It also follows that, if *c m* be >, =, or < *a* M, likewise *c m* will be >, =, or < *a* 1 M.
5. Therefore *m* : M = *m* : 1 M. *D.* 5. *B.* 5.

Which was to be demonstrated. II.



PROPOSITION VIII. THEOREM VIII.

Of unequal magnitudes (M & N), the greater (M) has a greater ratio to the same (P), than the less (N) has; and the same magnitude (P) has a greater ratio to the less (N), than it has to the greater (M).

Hypothesis.	Thesis.
I. M > N.	*I.* M : P > N : P
II. P *is any magnitude.*	*II.* P : N > P : M

I. Preparation.

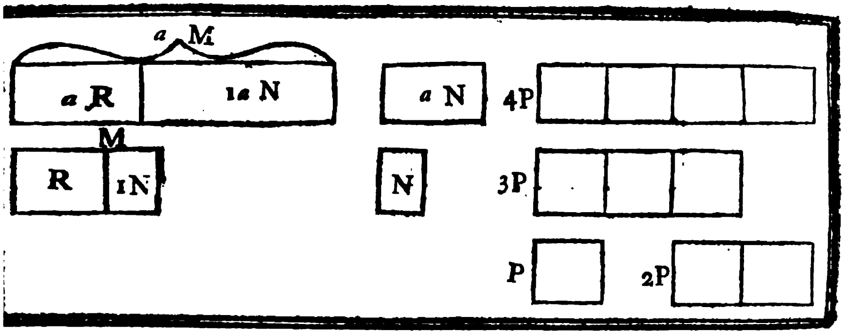
1. Take from the greater M a part 1 N = to the less N, and the remainder R will be either <, or > or infine = N ; Suppose first this remainder to be < N.
2. Take *a* R a multiple of this remainder > P ;
3. Take 1*a*N & *a*N the same mult. of 1 N & N that *a*R is of R. *Pos.* 1. B.1.
4. Take the mgn. 2 P double of P ; the mgn. 3 P triple of P and so on until the multiple of P be that which first becomes greater than *a* N, and let 4 P be that multiple.

Demonstration.

Because 4 P is the multiple of P which first becomes > *a*N (*Prep.* 4.)

1. The next preceding mult. 3 P is not > *a* N, or *a* N is not < 3 P.
Moreover *a* R and 1*a*N being equimultiples of R & of 1 N (*Prep.* 3),
2. The mgn. *a* R + 1 *a* N, or *a* M is the same multiple of R + 1 N or M, that *a* R is of R. *P.* 1. B. 5.
Or that *a* N is of N (*Prep.* 3).
3. Therefore *a* M and *a* N are equimultiples of M and of N.
Moreover, *a* N and 1*a*N being equimultiples of the = mgns. N and 1 N (*Prep.* 3 & 1).
4. The mgn. *a* N is = 1 *a* N *Ax.* 6. B. 1.
But *a* N is not < 3 P (*Arg.* 1).
5. Consequently, 1 *a* N is not < 3 P
But *a* R is > P (*Prep.* 2).
6. Therefore, by adding, *a* R + 1 *a* N or *a* M > 4 P.
Since then *a* M is > 4 P, and *a* N < 4 P (*Prep.* 4), and *a* M, *a* N are equimultiples of the antecedents M and N and 4 P, 4 P equimultiples of the consequents P and P (*Arg.* 3 & *Prep.* 4).
It follows that M : P > N : P *D.* 7. B. 5.

Which was to be demonstrated. 1.



Moreover, since aN is supposed < 4P *(Prep.4)*, & aM > 4P *(Arg.6)*.

8. It is evident that the mgn. 4 P is $>$ a N, & the same mgn. 4 P $<$ aM.
But 4 P and 4 P being equimultiples of the antecedents P and P, and a N, a M equimultiples of the consequents N and M,

9. It follows that P : N $>$ P : M. *D. 7. B. 5.*

Which was to be demonstrated. II.

II. Preparation.

If R be supposed $>$ 1 N or N.

5. Take 1 a N a multiple of 1 N $>$ P.
6. Take aR & aN the same multiples of R & of N that 1aN is of 1 N. *Pof. 1. B. 5.*
7. Let 4 P be the first multiple of P $>$ a R; consequently the next preceding multiple 3 P will not be $>$ a R, or a R will not be $<$ 3 P.

Demonstration.

It may be proved as before *(Arg. 1. 2 & 3)*, that

1. The mgns. a M and a N are equimultiples of the mgns. M & N.
Moreover, a R & a N being equimultiples of R & of N *(Prep. 6)*, and R being $>$ N *(Sup.)*,
2. It follows that a R is $>$ a N
But a R not being $<$ 3 P *(Prep. 7)*,
And the mgn. 1aN being $>$ P *(Prep. 5)*,
3. Then by adding, a R + 1 a N, or a M $>$ 4 P.
But aR being $<$ 4 P *(Prep.7)*, & this same aR being $>$ aN *(Arg.2)*,
4. Much more then a N is $<$ 4 P.
But aM & aN are equimultiples of the antecedents M & N *(Arg.1)* and 4 P, 4 P equimultiples of the consequents P & P, & moreover aM $>$ 4 P & a N $<$ 4 P *(Arg. 3 & 4)*.
5. Consequently M : P $>$ N : P. *D. 7. B. 5.*

Which was to be demonstrated. I.

Moreover, without changing the Preparation, it may be demonstrated as in the precedent case *(Arg. 8 & 9)*, that

6. The ratio of P : N is $>$ the ratio of P : M.

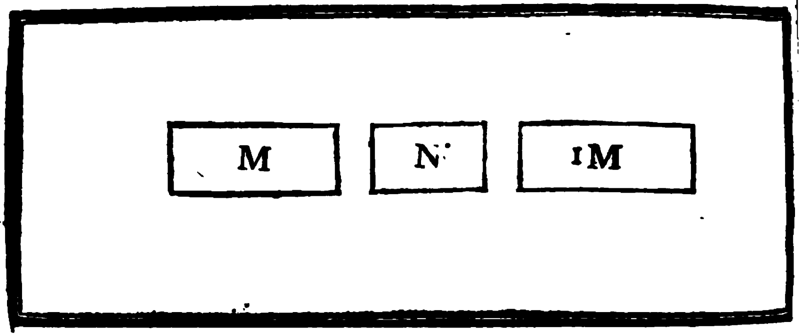
Which was to be demonstrated. II.

III.

And applying the same preparation and same reasoning to the last case when R = 1 N,

7. The demonstration will be completed as in the two precedent cases.

Which was to be demonstrated. I & II.



PROPOSITION IX. *THEOREM IX.*

MAGNITUDES (M & 1 M) which have the same ratio to the same magnitude (N) : are equal to one another. And those (M & 1 M) to which the same magnitude (N) has the same ratio, are equal to one another.

Hypothesis. Thesis.
M : N = 1 M : N. *The mgn.* M = 1 M.

DEMONSTRATION.

I.

If not, the two mgns. M & 1 M are unequal.

1. THEN the two mgns. M & 1 M have not the same ratio to the same mgn. N P. 8. B. 5.
But they have the same ratio to this same mgn. N *(Hyp.)* ;
2. Therefore the mgn. M is = to the mgn. 1 M.

Hypothesis. Thesis.
N : M = N : 1 M. *The mgn.* M = 1 M.

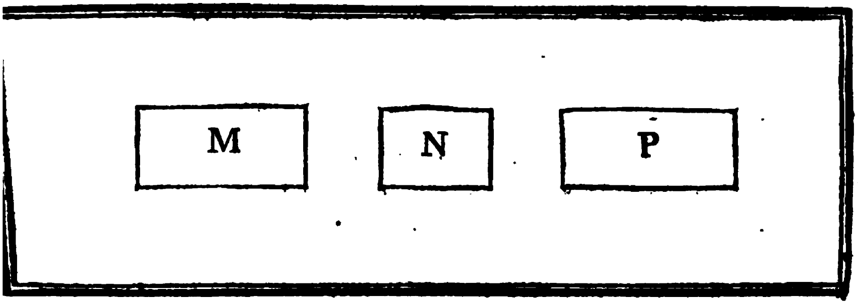
DEMONSTRATION.

II.

If not, the two mgns. M & 1 M are unequal.

1. THEN the same mgn. N has not the same ratio to the two mgns. M & 1 M. P. 8. B. 5.
But it has the same ratio to those two mgns. *(Hyp.)*.
2. Therefore the mgn. M is = to the mgn. 1 M.

Which was to be demonstrated.



PROPOSITION X. *THEOREM. X.*

THAT magnitude (M) which has a greater ratio than another (P) has unto the same magnitude (N) is the greater of the two, and that magnitude (P) to which the same (N) has a greater ratio than it has unto another magnitude (M) is the lesser of the two.

Hypothesis.	Thesis.
M . N *is* > P : N.	*The mgn.* M *is* > P.

DEMONSTRATION.

I.

If not; M is = P, or < P.

CASE I. If M be = P.

1. THEN the mgns. M & P have the same ratio to the same mgn. N. *P. 7. B. 5.*
But they have not the same ratio to the same mgn. N *(Hyp.)*;
2. Therefore the mgn. M is not = to the mgn. P.

CASE II. If M be < P.

3. THE ratio M : N would be < the ratio P : N *(Hyp.)*; *P. 8. B. 5.*
But the ratio M : N is not < the ratio P : N *(Hyp.)*;
4. Therefore the mgn. M is not < the mgn. P.
But neither is the mgn. M = P *(Arg. 2)*,
5. It remains then that M be > P.

Hypothesis.	Thesis.
N : P > N : M.	*The mgn.* P *is* < M.

DEMONSTRATION.

II.

If not, P is = or > M.

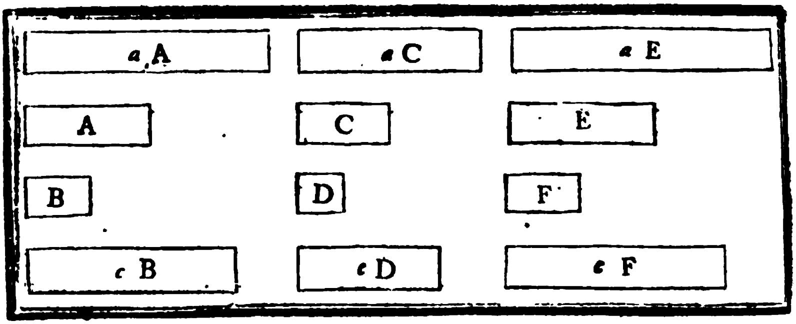
CASE I. If P be = M.

1. THE ratio N : M would be = to the ratio of N : P *P. 7. B. 5.*
2. Which being contrary to the Hypothesis, P cannot be = M.

CASE II. If P be > M.

3. THE ratio N : M would be > the ratio N : P. *P. 8. B. 5.*
4. Which being also contrary to the Hypothesis, P cannot be > M.
But neither is P = M. *(Arg. 2.)*;
5. Therefore P is < M.

Which was to be demonstrated.



PROPOSITION XI. *THEOREM XI.*

RATIOS (A : B & E : F) that are equal to a same third ratio (C : D), are equal to one another.

Hypothesis.	Thesis.
The ratios {A : B & E : F} *are* = *to the same ratio* C : D.	A : B = E : F.

Preparation.

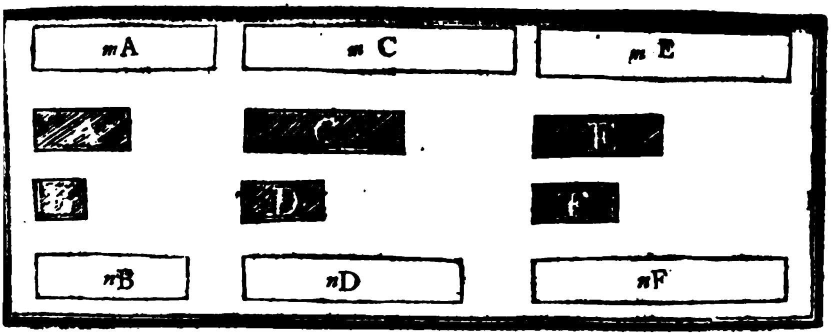
1. Take any equimultiples *a* A, *a* C, *a* E of the three antecedents A, C, E.
2. And any equimultiples *c* B, *c* D, *c* F of the three consequents B, D, F. } *Pos.* 1. *B.* 5.

DEMONSTRATION.

BECAUSE A : B = C : D *(Hyp.)*,

1. If the multiple *a* A be >, = or < the multiple *c* B, the equimultiple *a* C is likewise >, = or < the equimultiple C D *D.* 5. *B.* 5.
In like manner since C : D = E : F *(Hyp.)*
2. If the multiple *a* C be >, = or < the multiple *c* D, the equimultiple *a* E will be likewise >, = or < the equimultiple *c* F.
3. Consequently if the multiple *a* A be >, = or <, the multiple *c* B ; the equimultiple *a* E is likewise >, = or < the equimultiple *c* F.
4. Consequently, A : B = E : F. *D.* 5. *B.* 5.

Which was to be demonstrated.



PROPOSITION XII. *THEOREM XII.*

IF any number of magnitudes (A, B, C, D, E, F, &c) be proportionals. The sum of all the antecedents (A + C + E &c) is to the sum of all the consequents (B + D + F &c), as one of the antecedents is to its consequent.

Hypothesis.	Thesis.
The mgns. A, B, C, D, E, F *are proportionals* or A : B = C : D = E : F *&c.*	A+C+E : B+D+F = A : B.

Preparation.

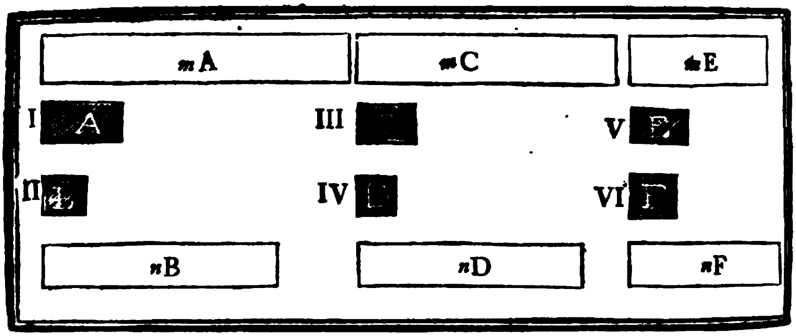
1. Take of the antecedents A, C, E the equimultiples *m* A, *m* C, *m* E
2. And of the consequents B, D, F the equimultiples *n* B, *n* D, *n* F } *Pos.* 1. *B.* 5.

DEMONSTRATION.

SINCE then A : B = C : D = E : F *(Hyp.)*;

1. If *m* A be >, = or < *n* B, likewise *m* C is >, = < *n* D; & *m* E is >, = or *n* F — *D.* 5. *B.* 5.
   Therefore adding on both sides the mgns. >, =, or <.
2. The mgns. *m* A + *m* C + *m* E will be constantly >, =, or < the mgns. *n* B + *n* D + *n* F according as *m* A is >, =, or < *n* B.
   But the mgns. *m* A + *m* C + *m* E & *m* A are equimultiples of the mgns. A + C + E & A *(Prep.* 1 *& P.* 1. *B.* 5.*)*; also the mgns. *n* B + *n* D + *n* F & *n* B are equimultiples of the mgns. B + D + F & B *(Prep.* 2 *& P.* 1. *B.* 5.*)*;
3. Consequently A + C + E : B + D + F = A : B — *D.* 5. *B.* 5.

Which was to be demonstrated.



PROPOSITION XIII. *THEOREM XIII.*

IF the first magnitude (A) has to the second (B), the same ratio, which the third (C) has to the fourth (D); but the third (C) to the fourth (D) a greater ratio than the fifth (E) to the sixth (F): the first (A) shall have to the second (B) a greater ratio than the fifth (E) has to the sixth (F).

Hypothesis.
I. A : B = C : D.
II. C : D > E : F.

Thesis.
A : B > E : F.

Preparation.

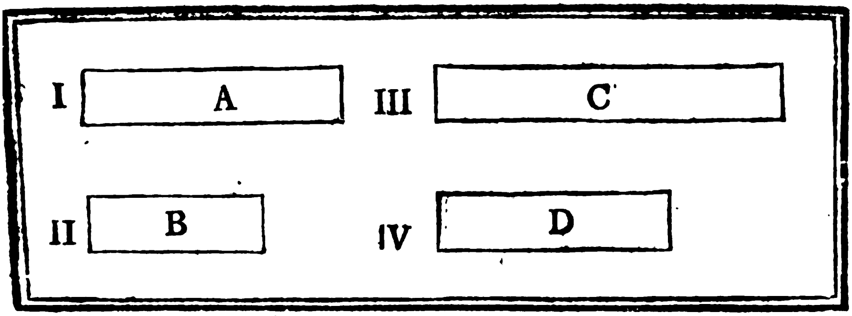
1. The ratio of C : D being > the ratio of E : F *(Hyp. 2)* there may be taken of the antecedents C & E, the equimult. *m* C & *m* E; and likewise of the consequents D & F the equimult. *n* D & *n* F, such, that *m* C is > *n* D, but *m* E is not > *n* F; *Pof. 1. B. 5. D. 7. B. 5.*
2. Take *m* A the same multiple of A that *m* C is of C,
3. And *n* B the same multiple of B that *n* D is of D. *Pof. 1. B. 5.*

DEMONSTRATION.

SINCE then A : B = C : D *(Hyp. 1.)*, and that *m* A, *m* C are equimultiples of the antecedents, & *n* B, *n* D equimultiples of the consequents *(Prep. 2 & 3)*.

1. The mgn. *m* A will be >, = or < *n* B; according as *m* C is >, = or < *n* D. *D. 5. B. 5.*
2. Therefore *m* A is also > *n* B.
But *m* E is not > *n* F *(Prep. 1)*, & the mgns. *m* A & *m* E are equimultiples of the antecedents A & E, & *n* B, *n* F equimultiples of the consequents B & F *(Prep. 1 & 2)*.
3. Consequently the ratio A : B, is > than the ratio E : F. *D. 7. B. 5.*

Which was to be demonstrated.



PROPOSITION XIV. *THEOREM XIV.*

IF four magnitudes (A, B, C, D) be proportionals, then if the first (A) be greater, equal, or less, than the third (C), the second (B) shall be greater, equal, or less, than the fourth (D).

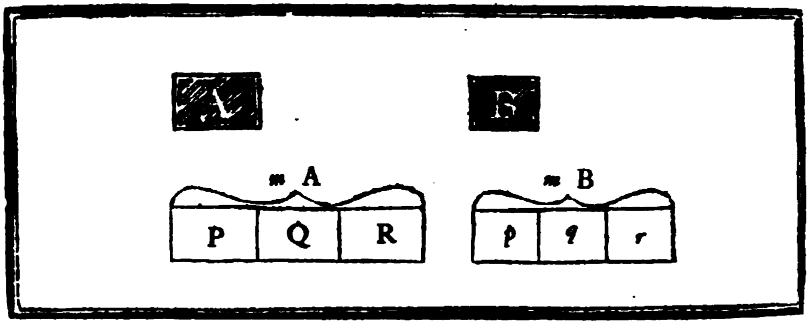
Hypothesis.	Thesis.
I. A : B = C : D	*According as* A *is* >, = *or* < C.
II. A *is* >, = *or* < C.	B *will be* >, = *or* < D.

CASE I. If A be > C.

Demonstration.

1. THEN the ratio of A : B is > the ratio C : B. *P.* 8. *B.* 5.
   But A : B = C : D *(Hyp. 1)*.
2. Therefore the ratio of C : D is > the ratio C : B. *P.* 13. *B.* 5.
3. From whence it follows, that D is < B or B > D. *P.* 10. *B.* 5.
   It may be demonstrated after the same manner, if A = C, that B will be = D; & if A be < C, that B will be < D.
4. Consequently, according as A is >, = or < C, B will be >, = or < D.

Which was to be demonstrated.



PROPOSITION XV. THEOREM XV.

MAGNITUDES (A & B) have the fame ratio to one another which their equimultiples (m A & m B) have.

Hypothefis.	Thefis.
The mgns. m A & m B *are equimult. of the mgns.* A & B.	A : B = m A : m B

Preparation.

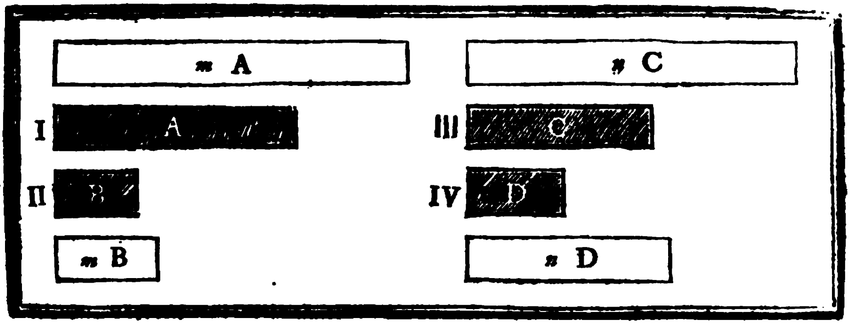
1. Divide m A into its parts P, Q, R each = A. } *Pof. 2. B. 5.*
2. And m B inro its parts p, q, r each = B.

DEMONSTRATION.

BECAUSE the mgns. m A, m B are equimultiples of the mgns. A & B *(Hyp.)*.

1. The number of parts P, Q, R &c. is = to the number of parts p, q, r &c.
   And P being = Q = R *(Prep. 1)*, & $p = q = r$ *(Prep. 2)*, *P. 7. B. 5.*
2. The mgn. P : p = Q : q = R : r &c. *P. 11. B. 5.*
3. Wherefore P + Q + R, or m A : $p + q + r$ or m B = P : p. *P. 12. B. 5.*
   But fince P = A & p = B *(Prep. 1 & 2)*,
4. The mgn. P : p = A : B. *P. 7. B. 5.*
5. Confequently A : B = m A : m B. *P. 11. B. 5.*

Which was to be demonftrated.



PROPOSITION XVI. *THEOREM XVI.*

If four magnitudes (A, B, C, D) of the same kind be proportionals, they shall also be proportionals when taken alternately.

Hypothesis. Thesis.

$A : B = C : D.$ $A : C = B : D.$

Preparation.

1. Take of the terms A & B of the first ratio, any equimult. mA, & mB. } *Pos.* 1. *B.* 5.
2. Take of the terms C & D of the second ratio any equimult. nC, nD. } *Pos.* 1. *B.* 5.

Demonstration.

Because mA & mB are equimult. of the mgns. A & B *(Prep. 1)*,

1. Then $A : B = mA : mB$. *P.* 15. *B.* 5.
   But $A : B = C : D$ *(Hyp.)*.
2. Therefore $C : D = mA : mB$. *P.* 11. *B.* 5.
3. Likewise $C : D = nC : nD$. *P.* 15. *B.* 5.
4. Consequently $mA : mB = nC : nD$. *P.* 11. *B.* 5.
5. Wherefore, if mA be $>, =$ or $< nC$, mB will be $>, =$ or $< nD$. *P.* 14. *B.* 5.
   But mA & mB being equimult. of the terms A & B considered as antecedents *(Prep. 1)*, & nC, nD equimult. of the terms C & D considered as consequents *(Prep. 2)*,
6. Consequently $A : C = B : D$. *D.* 5. *B.* 5.

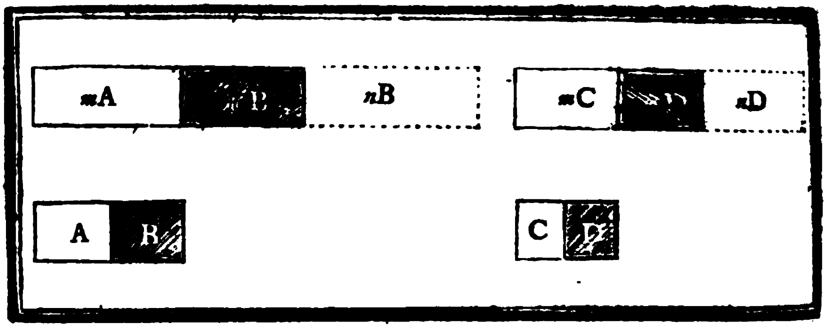
Which was to be demonstrated.

COROLLARY.

It follows from this proposition that if four mgns. are proportionals, according as the first is greater, equal or less than the second, the third is likewise greater, equal, or less than the fourth.

For since $A : B = C : D$ *(Hyp.)*,

1. *Then* $A : C = B : D$. *P.* 16. *B.* 5.
2. *Therefore, according as* A *is* $>, =$ *or* $<$ B, C *will be likewise* $>, =$ *or* $<$ D. *P.* 14. *B.* 5



PROPOSITION XVII. THEOREM XVII.

IF two magnitudes together (A + B) have to one of them (B), the same ratio which two others (C + D) have to one of these (D), the remaining one (A) of the first two (A + B) shall have to the other (B), the same ratio which the remaining one (C) of the last two (C + D) has to the other of these (D).

Hypothesis. Thesis.

A + B : B = C + D : D A : B = C : D

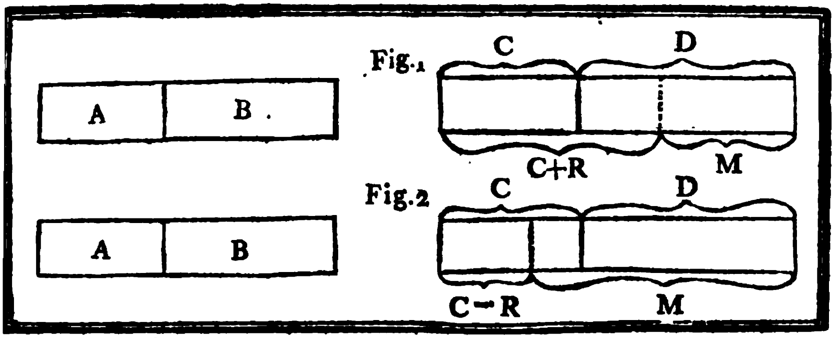
Preparation.

1. Take of the mgns. A,B,C,D any equimult. mA, mB, mC, mD.
2. And of the mgns. B & D any equimult. n B, n D. *Pos.* 1. *B.* 5

DEMONSTRATION.

1. THEN the whole mgn. m A + m B will be the same mult. of the mgn. A + B, that m A is of A, or m C of C. *P.* 1. *B.* 5
2. In like manner, the whole mgn. m C + m D is the same mult. of the mgn. C + D, that m C is of C. *P.* 1. *B.* 5
3. Consequently, m A + m B is the same mult. of A + B, that m C + m D is of C + D.
4. Also the mgns. mB+nB, mD+nD are equimult. of the mgns. B&D.
But A + B : B = C + D : D *(Hyp.)*, & m A + m B, m C + m D are equimult. of the antecedents A + B & C + D *(Arg. 3)*; also m B + n B, m D + n D are equimult. of the consequents B & D *(Arg. 4)*.
5. Consequently, if m A + m B be >, = or < m B + n B, m C + m D is also >, = or < m D + n D. *D.* 5. *B.* 5.
But if m A + m B be >, = or < m B + n B; taking away the common part m B.
6. The remainder m A will be >, = or < the remainder n B.
In like manner, if m C + m D be >, = or < m D + n D; taking away the common part m D.
7. The remainder m C will be >, = or < the remainder n D.
8. Wherefore, if m A be >, =, or < n B; m C will be likewise >, = or < n D.
But m A & m C are equimult. of A & of C considered as antecedents *(Prep. 1)*; & n B, n D equimult. of B & D considered as consequents *(Prep. 2)*.
9. Consequently, A : B = C : D. *D.* 5. *B.* 5.

Which was to be demonstrated.



PROPOSITION XVIII. *THEOREM XVIII.*

IF four magnitudes (A,B,C,D) be proportionals, the first and second together A+B) shall be to the second (B) as the third and fourth together (C+D) to he fourth (D).

Hypothesis.
A : B = C : D.

Thesis.
A + B : B = C + D : D.

Demonstration.

If not, A+B : B = C+D : another mgn. M < or > D.

CASE I. Let M < D, or M + R = D *(Fig. 1).*

SINCE then A + B : B = C + D : M, or A+B : B = C+M+R : M

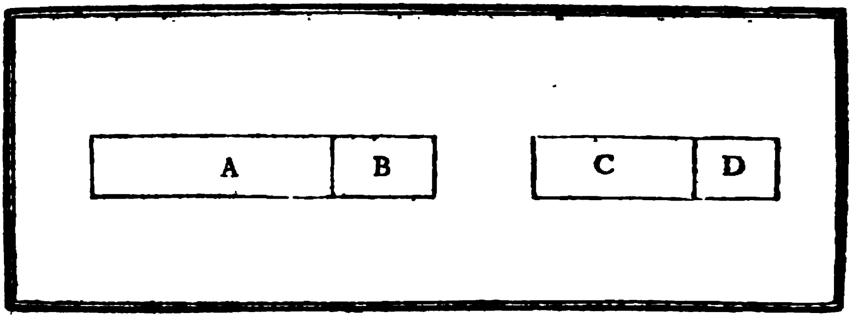
1. Dividendo A : B = C + R : M. *P.17. B.5.*
   But A : B = C : D *(Hyp.)* ;
2. Hence, C+R : M = C : D *P.11. B.5.*
   But C + R is > C *(Ax. 8. B. 1)* ;
3. Therefore M is > D, & the ſuppoſition of M < D, is impoſible *P.14. B.5.*

CASE II. Let M > D, or M = D + R *(Fig. 2).*

BECAUSE A + B : B = C + D : M, or A+B : B = C+D : D+R

4. Dividendo A : B = C − R : D + R *P.17. B.5.*
   But A : B = C : D. *(Hyp.).*
5. Hence, C−R : M = C : D. *P.11. B.5.*
   But C − R is < C *(Ax. 8. B. 1)* ;
6. Therefore M is < D, & the ſuppoſition of M > D, is impoſſible. *P.14. B.5.*
   Since then M is neither < D *(Arg. 3)* nor > D *(Arg. 7)*,
7. It follows that M is = D & A + B : B = C + D : D.

Which was to be demonſtrated.



PROPOSITION XIX. *THEOREM XIX.*

If a whole magnitude (A+B) be to a whole (C+D), as a magnitude (A) taken from the first is to a magnitude (C) taken from the other, the remainder (B) shall be to the remainder (D), as the whole (A + B) is to the whole (C + D).

Hypothesis.	Thesis.
$A + B : C + D = A : C$	$B : D = A + B : C + D.$

Demonstration.

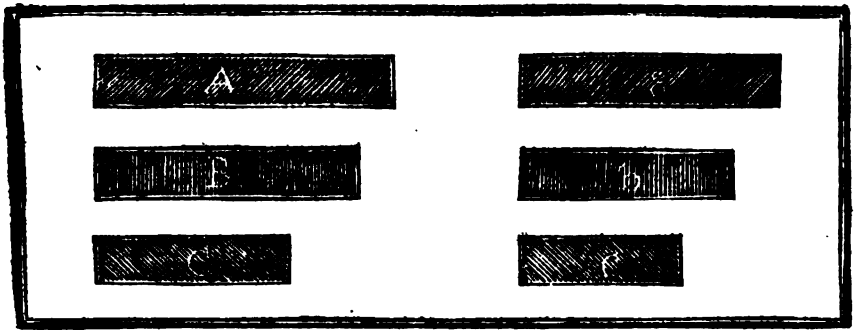
Because	$A + B : C + D =$	$A : C.$ (*Hyp.*).	
1. Therefore Alternando	$A + B : A = C + D : C$		*P.* 16. *B.* 5.
2. Then Dividendo	$B : A = D : C.$		*P.* 17. *B.* 5.
3. Alternando again	$B : D = A : C.$		*P.* 16. *B.* 5.
But since	$A + B : C + D =$	$A : C.$ (*Hyp.*).	
4. It follows that	$B : D = A + B : C + D.$		*P.* 11. *B.* 5.

Which was to be demonstrated.

COROLLARY.

If magnitudes taken jointly be proportionals, that is if $A + B : A = C + D : C$, *it may be inferred by conversion that* $A + B : B = C + D : D$ (*D.* 17. *B.* 5).

For $A + B : C + D = A : C$ (*Hyp.* & *P.* 16).
Wherefore $A + B : B + D = B : D$ (*P.* 19).
Consequently $A + B : B = C + D : D$ (*P.* 16).



PROPOSITION XX. *THEOREM XX.*

IF there be three magnitudes (A, B, C) and other three *(a, b, c)* which taken two and two in a direct order, have the same ratio; if the first (A) be greater than the third (C), the fourth *(a)* shall be greater than the sixth *(c)* and if equal, equal, and if less, less.

Hypothesis.	Thesis.
I. A : B = *a* : *b*	*According as* A *is* >, = *or* < C.
II. B : C = *b* : *c*	*a is also* >, = *or* < *c*.

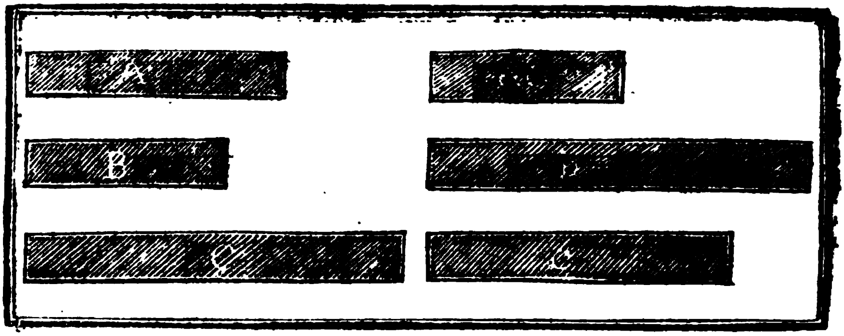
DEMONSTRATION.

CASE I. Let A be > C.

BECAUSE A is > C.

1. The ratio A : B is > C : B. *P.* 8. *B.* 5.
   But A : B = *a* : *b* *(Hyp.* 1*).*
   And C : B = *c* : *b* *(Hyp.* 2 & *P.* 4 *Cor. B.* 5*).*
2. Therefore, the ratio *a* : *b* is > *c* : *b*. *P.*13. *B.* 5.
3. Consequently, *a* is also > *c*. *P.*10. *B.* 5.
4. It may be proved after the same manner, that if A be = C, *a* shall be = *c*, & if A be < C, *a* shall be < *c*.
5. Consequently, according as A is >, = or < C, *a* will be also >, = or < *c*.

Which was to be demonstrated.



PROPOSITION XXI. *THEOREM XXI.*

IF there be three magnitudes (A, B, C), and other three *(a, b, c)*, which have the same ratio taken two and two, but in a cross order; if the first magnitude (A) be greater than the third (C), the fourth *(a)* shall be greater than the sixth *(c)*, and if equal, equal; and if less, less.

Hypothesis.	Thesis.
I. A : B = *b* : *c*	*According as* A *is* >, = *or* < C
II. B : C = *a* : *b*.	*a is also* >, = *or* < *c*.

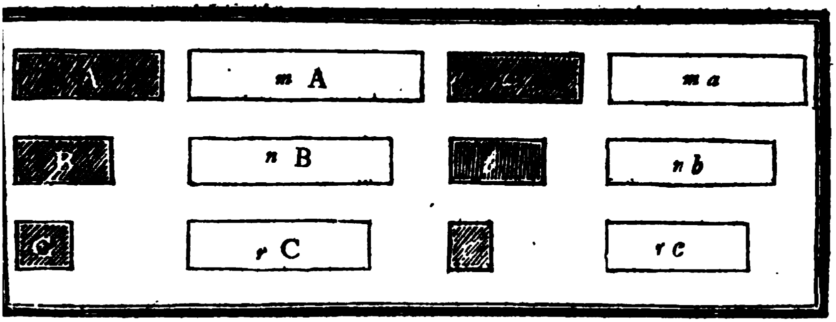
CASE I. Let A be > C.

DEMONSTRATION.

BECAUSE A is > C

1. The ratio of A : B > C : B — *P.* 8. *B.* 5.
   But A : B = *b* : *c* (*Hyp.* 1).
   & invertendo C : B = *b* : *a*. (*Hyp.* 2, & *P.* 4. *Cor. B.* 5.).
2. Consequently the ratio *b* : *c* > *b* : *a* — *P.* 13. *B.* 5.
3. Therefore *c* is < *a*, or *a* > *c* — *P.* 10. *B.* 5.
4. It may be demonstrated after the same manner, if A be = B, also *a* shall be = *c*; and if A be < C, *a* shall be < *c*
5. Consequently, according as A is >, = or < C, *a* shall be >, = or < *c*.

Which was to be demonstrated.



PROPOSITION XXII. *THEOREM XXII.*

IF there be any number of magnitudes (A, B, C, &c.) and as many others *a, b, c, &c.),* which taken two and two in order have the same ratio, the irst shall have to the last of the first magnitudes, the same ratio which the irst of the others has to the last, by equality of direct ratio, or *ex æquo or-inate.*

Hypothesis.	Thesis.
I. A : B = *a* : *b*	A : C = *a* : *c*.
II. B : C = *b* : *c*	

Preparation.

1. Take of A & *a* any equimult. *m* A & *m a*
2. And of B & *b* any equimult. *n* B & *n b*
3. And of C & *c* any equimult. *r* C & *r c*.

} *Pos.* 1. *B.* 5.

DEMONSTRATION.

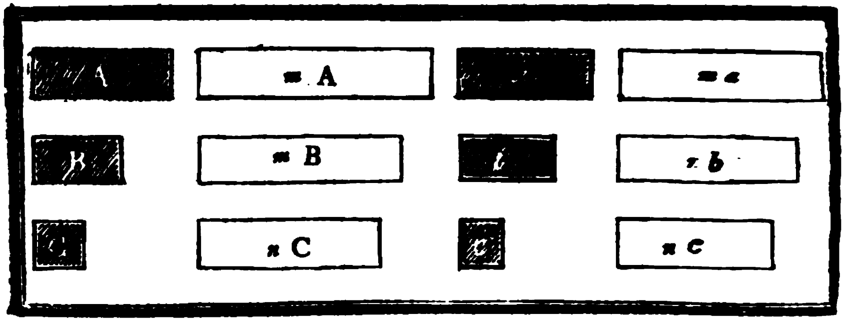
BECAUSE A : B = *a* : *b* (*Hyp.* 1).

1. It follows that *m* A : *n* B = *m a* : *n b* — *P.* 4. *B.* 5.

And because B : C = *b* : *c* (*Hyp.* 2).

2. It follows that *n* B : *r* C = *n b* = *r c* — *P.* 4. *B.* 5.
3. Therefore, *m* A, *n* B, *r* C & *m a*, *n b*, *r c* form two series of magnitudes which taken two by two in order have the same ratio.
4. Wherefore, by equality of ratio, according as the first *m* A of the first series is >, = or < the third *r* C, the first *m a* of the other series will be >, = or < the third *r c*. — *P.* 20. *B.* 5.
5. Consequently, A : C = *a* : *c*. — *D.* 5. *B.* 5.

Which was to be demonstrated.



PROPOSITION XXIII. *THEOREM XXIII.*

IF there be any number of magnitudes (A, B, C, &c.) and as many others *(a, b, c, &c.)* which taken two and two, in a cross order, have the same ratio; the first shall have to the last of the first magnitudes the same ratio which the first of the others has to the last, by equality of perturbate ratio or *ex æquo perturbate.*

Hypothesis.
I. A : B = *b* : *c*.
II. B : C = *a* : *b*.

Thesis.
A : C = *a* : *c*.

Preparation.

1. Take of A, B, *a*, any equimult. *m* A, *m* B, *m a*.
2. And of C, *b*, *c*, any equimult. *n* C, *n b*, *n c*.

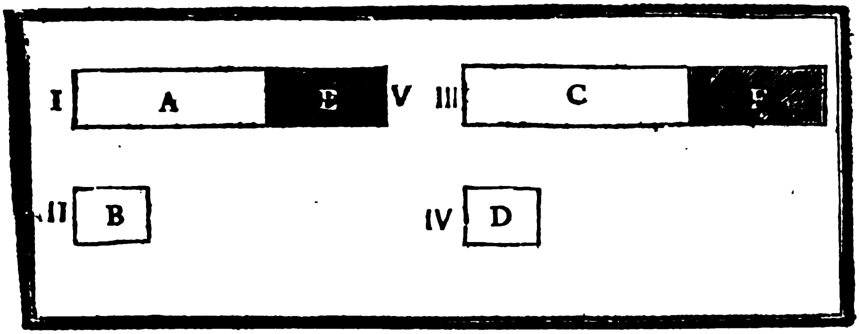
} *Pos.* 1. *B.* 5.

DEMONSTRATION.

BECAUSE *m* A & *m* B are equimult. of A & B *(Prep. 1)*.

1. It follows that A : B = *m* A : *m* B. *P.* 15. *B.* 5.
2. And *b* : *c* = *n b* : *n c*.
   But A : B = *b* : *c*. *(Hyp. 1)*.
3. Therefore, *m* A : *m* B = *n b* : *n c*. *P.* 11. *B.* 5.
   And because B : C = *a* : *b*. *(Hyp. 2)*.
4. It follows that *m* B : *n* C = *m a* : *n b*. *P.* 4. *B.* 5.
5. Wherefore, *m* A, *m* B, *n* C, & *m a*, *m b*, *n c* form two series of mgns. which taken two and two in a cross order have the same ratio.
6. Consequently, by equality of ratio, according as the first *m* A of the first series is >, = or < the third *n* C, the first *m a* of the other series will be >, = or < the third *n c*. *P.* 21. *B.* 5.
7. For which reason A : C = *a* : *c*. *D.* 5. *B.* 5.

Which was to be demonstrated.



PROPOSITION XXIV. *THEOREM XXIV.*

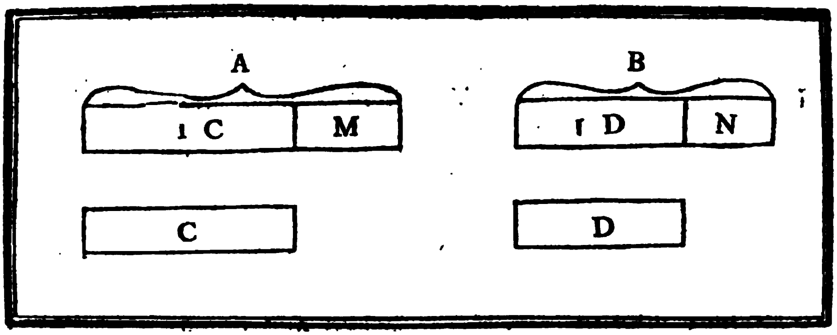
If four magnitudes (A, B, C, D) be proportionals and that a fifth (E) has to the second (B) the same ratio which a sixth (F) has to the fourth (D), the first and fifth together (A + E) shall have to the second (B), the same ratio which the third and sixth together (C + F) have to the fourth (D).

Hypothesis.	Thesis.
I. $A : B = C : D$	$A + E : B = C + F : D.$
II. $E : B = F : D.$	

Demonstration.

Because	$E : B = F : D$ *(Hyp. 2).*	
1. It follows invertendo	$B : E = D : F$	P. 4. B. 5. Cor.
And because	$A : B = C : D$ *(Hyp. 1).*	
2. Ex æquo ordinate	$A : E = C : F$	P. 22. B. 5.
3. Componendo	$A + E : E = C + F : F$	P. 18. B. 5.
But since	$E : B = F : D$ *(Hyp. 2).*	
4. It follows, Ex æquo ordinate	$A + E : B = C + F : D$	P. 22. B. 5.

Which was to be demonstrated.



PROPOSITION XXV. *THEOREM XXV.*

IF four magnitudes (A, B, C, D) are proportionals, the greateſt (A) and leaſt of them (D) together, are greater than the other two (B & C) together.

Hypotheſis.	Theſis.
I. A : B = C : D	A + D > B + C.
II. A *is the greateſt term,* & *Conſequently* (*) D *the leaſt.*	

Preparation.

Take 1 C = C & 1 D = D.

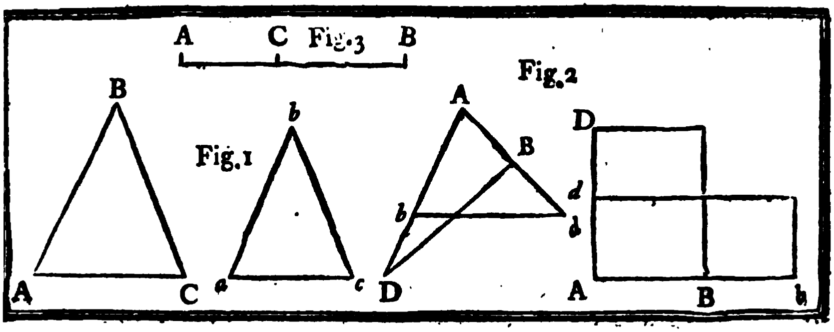
DEMONSTRATION.

BECAUSE A : B = C : D *(Hyp. 1)* & C = 1 C & D = 1 D *(Prep.)*.

1. It follows that	A : B = 1C : 1D	*P. 7. B 5.*
2. Wherefore	A : B = M : N	*P. 19. B. 5.*
But the mgn.	A being > B *(Hyp. 2)*.	
3. The mgn.	M is alſo > N	*P. 16. B. 5. Cor.*
Moreover, becauſe	C = 1C & D = 1D *(Prep. 1 & 2)*.	
4. It follows that	1C + D = 1D + C	
And ſince	M is > N *(Arg. 3)*.	
5. It follows that	1C + D + M > 1D + C + N, that is A + D is > B + C.	*Ax. 4. B. 1.*

Which was to be demonſtrated.

(*) EUCLID *ſuppoſes the conſequence of this Hypotheſis ſufficiently evident from the foregoing truths ; for ſince* A : B : C : D *(Hyp. 1.),* & A > C *(Hyp. 2.),* B *is* > D *(P. 14. B. 5.). Likewiſe* A *being* > B *(Hyp. 2.)* C *is* > D *(P. 16. Cor. B. 5.), Conſequently* D *is the leaſt of the IV terms.*



DEFINITIONS.

I.

SIMILAR *rectilineal figures (Fig. 1.)* are those (A B C, a b c), which have their several Angles (A, B, C, and a, b, c) equal, each to each, and the sides (A B, A C, B C, and a b, a c, b c,) about the equal angles, proportionals (that is A B : A C = a b : a c, also A B : B C = a b : b c, and A C : B C = a c : b c).

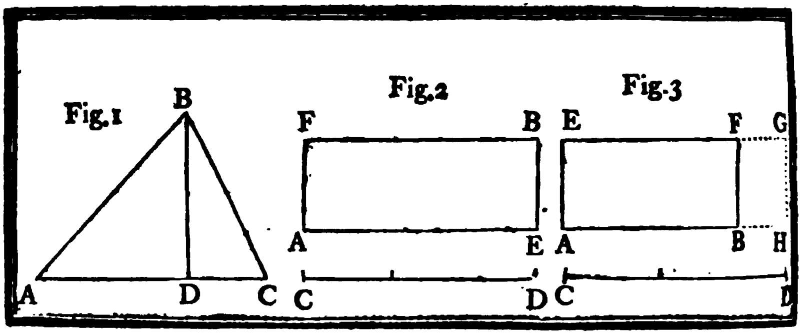
II.

THE Figures (D A B, d A b) are reciprocal *(Fig. 2.)*, when the antecedents (A D, A b) and the consequents (A d, A B) of the ratios, are in each of the figures, (that is A D : A d = A b : A B.

Or the figures (D A B, d A b) are reciprocal; *when the two sides* (A D, A B *and* A d, A b), *in each of those figures, about the same angle* (A), *or equal angles, are the extreams or means of the same proportion, that is, a side* (A D) *in the first figure is to a side* (A d) *of the other, as the remaining side* (A b) *of this other is to the remaining side* (A B) *of the first.*

III.

A Straight line (A B) is said to be cut in *mean and extream ratio, (Fig. 3.)* when the whole (A B), is to the greater segment (B C), as the greater segment, is to the less (A C).



DEFINITIONS.

IV.

THE *altitude of any figure* (A B C) *(Fig.1.)*, is the perpendicular (BD) let fall from the vertex (B) upon the baſe (A C).

IT follows from this Definition, that if two figures placed upon the ſame ſtraight line, have the ſame altitude, they are between the ſame parallels; becauſe from the nature of parallels the perpendiculars let fall from one to the other are always equal.

V.

A *Ratio* (AB. BC. CD : DE. EF. FG) *is compounded of ſeveral others* (A B : D E + B C : E F + C D : F G) when its terms reſult from the multiplication of the terms of thoſe compounding ratios.

VI.

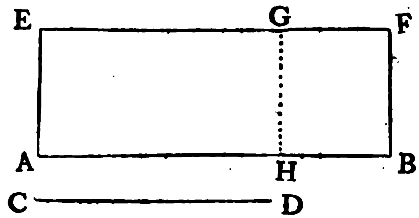
A *Parallelogram* (A B) *(Fig. 2)* is ſaid *to be applied to a ſtraight line* (CD), when it has for its baſe or for its ſide this propoſed ſtraight line (C D).

VII.

A *Deficient parallelogram* (A F), *(Fig.3)* is that whoſe baſe (A B) is leſs than the propoſed line (C D) to which it is ſaid to be applied.

VIII.

BUT the deficiency of a deficient parallelogram (AF), *(Fig.1)* is a parallelogram (BG) contained by the remainder of the propoſed ſtraight line (C D) and the other ſide (B F) of the deficient parallelogram.



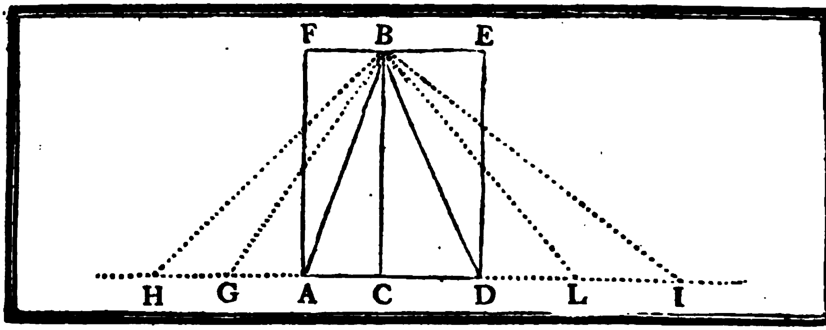
DEFINITIONS.

IX.

AN *exceeding parallelogram* (A F) is that, whoſe baſe (A B) is greater than the propoſed line (C D), to which it is ſaid to be applied.

X.

AND *the exceſs of an exceeding parallelogram* (AF) is a parallelogram (HF) contained by the exceſs of the baſe (A B) above the propoſed ſtraight line (C D) and the other ſide (B F) of the exceeding parallelogram.



PROPOSITION I. *THEOREM I.*

TRIANGLES (ABC, CBD), and parallelograms (CF, CE), of the same altitude, are one to another as their bases (AC, CD).

Hypothesis.	Thesis.
The △ ABC, CBD, *& pgms.* CF, CE, *have the same altitude*	*I. The* △ ABC : △ CBD = AC : CD. *II. The pgm.* CF : *pgm.* CE = AC : CD.

Preparation.

1. Produce AD indefinitely to H & I. *Pos.* 2. *B.* 1.
3. Take AG = AC = GH, also DL = CD = LI. *P.* 3. *B.* 1.
3. Draw BG, BH, BL, BI. *Pos.* 1. *B.* 1.

Demonstration.

BECAUSE the △ ABC, GBA, HBG, are upon equal bases AC, AG, GH, *(Prep. 2)*, & between the same plles. HI, FE, *(Hyp. & D. 35. B. 1. & Rem. D. 4. B. 6.)*.

1. Those △ are = to one another. *P.* 38. *B.* 1.
2. From whence it follows, that the △ HBC, & the base HC, are equimult of the △ ABC, & of the base AC.

It may be demonstrated after the same manner, that

3. The △ CBI, & the base CI, are equimult. of the △ CBD, & of the base CD.
4. Consequently, the mgns. HBC & HC, are equimult. of the mgns. ABC & AC *(Arg. 2)*, & the mgns. CBI & CI are equimult. of the mgns. CBD & CD, *(Arg. 3.)*.

But if the △ HBC, be >, = or < the △ CBI, the base HC is also >, = or < the base CI, *(P. 38. B. 1.)*.

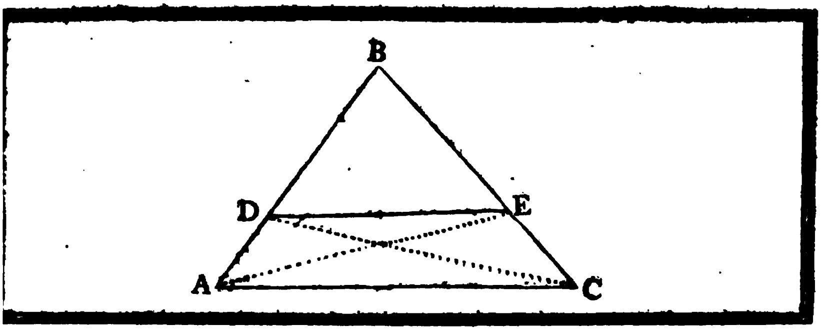
5. Consequently, the △ ABC : △ CBD = AC : CD. *D.* 5. *B.* 5.

Which was to be demonstrated. I.

But the △ ACB, CBD, being the halves of the pgms. CF, CE, *(P. 41. B. 1.)*

5. It follows, that △ ACB : △ CBD = pgm. CF : pgm. CE. *P.* 15. *B.* 5.
6. Wherefore the pgm. CF : pgm. CE = AC : CD. *P.* 11. *B.* 5.

Which was to be demonstrated. II.



PROPOSITION II. *THEOREM II.*

If a straight line (D E) be drawn parallel to one of the sides (A C) of a triangle (A B C): it shall cut the other sides (A B, B C) proportionally, that is A D : DB = C E : E B; and if the sides (A B, B C) be cut proportionally, the straight line (D E) which joins the points of section shall be parallel to the remaining side (A C) of the triangle.

Hypothesis.	Thesis.
The straight line D E *is plle. to* A C.	A D : D B = C E : E B.

Preparation.

Draw the straight lines A E, C D. *Pos.* 1. *B.* 1.

I. Demonstration.

BECAUSE	D E is plle. to A C *(Hyp.).*		
1. The	△ D A E is	= △ E C D.	*P.* 37. *B.* 1.
2. Consequently,	△ D A E : △ D B E =	△ E C D : △ D B E.	*P.* 7. *B.* 5.
3. But the	△ D A E : △ D B E =	A D : D B.	
& the	△ E C D : △ D B E =	C E : E B *(P.* 1. *B.* 6.)	
4. Therefore	A D : D B =	C E : E B.	*P.* 11. *B.* 5.

Which was to be demonstrated.

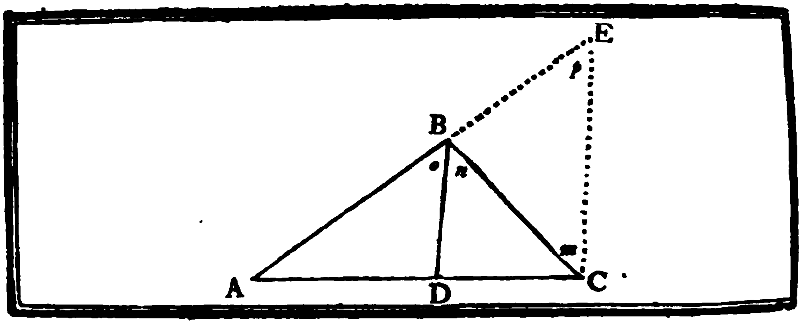
Hypothesis.	Thesis.
A D : D B = C E : E B.	*The straight line* D E *is plle. to* A C.

II. Demonstration.

BECAUSE the △ D A E, D B E are between the same plles, as also the △ E C D, D B E.

1. It follows that	△ D A E : △ D B E =	A D : D B.	} *P.* 1. *B.* 6.
& the	△ E C D : △ D B E =	C E : E B.	
But	A D : D B =	C E : E B. *(Hyp.)*,	
2. Therefore the	△ D A E : △ D B E =	△ E C D : △ D B E.	*P.* 11. *B.* 5.
3. Wherefore the	△ D A E is	= △ E C D.	*P.* 9. *B.* 5.
4. Consequently,	the straight line D E is plle. to A C.		*P.* 39. *B.* 1.

Which was to be demonstrated.



PROPOSITION III. THEOREM III.

If the angle (B) of a triangle (A B C) be divided into two equal angles by a straight line (B D) which cuts the base in (D), the segments of the base (A D, D C) shall have the same ratio which the other sides (A B, B C) of the triangle have to one another; and if the segments of the base (A D, D C) have the same ratio which the other sides (A B, B C) of the triangle have to one another, the straight line (B D) drawn from the vertex (B) to the point of section (D) divides the vertical angle (A B C) into two equal angles.

Hypothesis.	Thesis.
The straight line B D *divides the* ∀ A B C *into two equal parts, or* ∀ *o* = ∀ *n*.	A D : D C = A B : B C

Preparation.

1. Thro' the point C draw C E plle. to D B. — P. 31. B. 1.
2. Produce A B until it meets C E in E. — Pof. 2. B. 1.

I. Demonstration.

Because the straight lines D B, C E are plle. *(Prep. 1)*.

1. It follows that A D : D C = A B : B E. — P. 2. B. 6.
2. And that ∀ *n* = ∀ *m*, & ∀ *o* = ∀ *p*. — P. 29. B. 1.
   But, ∀ *o* being = to ∀ *n* *(Hyp.)*. — Ax. 1. B. 1.
3. The ∀ *m* is also = to ∀ *p*, & B C = to B E. — P. 6. B. 1.
4. Wherefore A D : D C = A B : B C. — P. 7. & 11. B. 5.

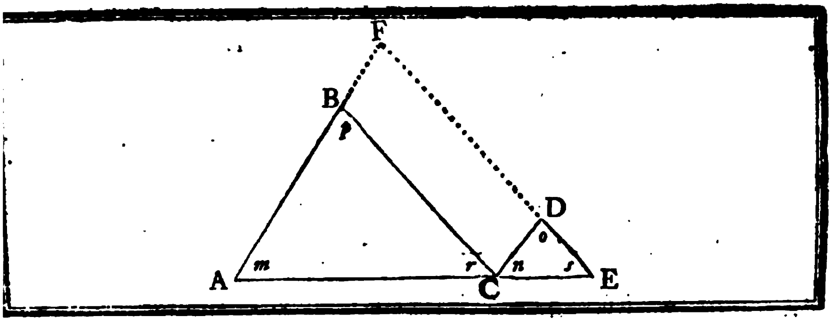
Which was to be demonstrated.

Hypothesis.	Thesis.
A D : D C = A B : B C.	B D *bisects* ∀ A B C *or* ∀ *o* = ∀ *n*

II. Demonstration.

Because the straight lines D B, C E are plle. *(Prep. 1)*.

1. It follows that A D : D C = A B : B E. — P. 2. B. 6.
   But A D : D C = A B : B C *(Hyp.)*
2. Wherefore A B : B E = A B : B C. — P. 11. B. 5.
3. Consequently, B E is = B C, & ∀ *m* = ∀ *p*. — P. 9. B. 5.
   But ∀ *m* is also = to ∀ *n*, & ∀ *p* = ∀ *o* *(P. 29. B. 1)*. — P. 5. B. 1.
4. Consequently, ∀ *n* is = to ∀ *o*, or B D bisects ∀ A B C. — Ax. 1. B. 1.



PROPOSITION IV. THEOREM IV.

THE fides (A C, A B & CE, C D, &c) about the equal angles (m & , &c) of equiangular triangles (ABC, CDE) are proportionals; and thofe fides AB, CD, &c) which are oppofite to the equal angles (r & s, &c) are homologous fides; that is, are the antecedents or confequents of the ratios.

Hypothefis.

The △ ABC, C D E *are equiangular,*
or ∀ m = ∀ n, ∀ r = ∀ s,
& ∀ p = ∀ o.

Thefis.

I. AB : AC = CD : CE.
AC : BC = CE : DE.
AB : BC = CD : DE.

II. The fides AB, CD; AC, CE; BC, DE *oppofite to the equal* ∀ *are homologous.*

Preparation.

1. Place the △ A B C, C D E, fo that the bafes A C, C E may be in the fame ftraight line.
2. Produce the fides A B, D E indefinitly to F. — *Pof. 2. B. 1.*

Demonstration.

BECAUSE the ∀ m + r of △ABC are < 2 ∟ (*P. 17. B. 1.*) & ∀ r = ∀ s. (*Hyp.*).

1. The ∀ m + s are alfo < 2 ∟, & AB, DE meet fomewhere in F. *Lem. B. 1.*
   But ∀ m being = to ∀ n & ∀ r = to ∀ s (*Hyp.*).
2. The ftraight lines A F, C D alfo B C, F E are plle. *P. 28. B. 1.*
3. And the quadrilateral figure C F is a pgrm. *D. 35. B. 1.*
4. Confequently, BC, F D; alfo C D, B F are = to one another. *P. 34. B. 1.*
   But B C being plle. to the fide F E of the △ F A E (*Arg. 2*).
5. Therefore AB : BF = AC : CE. *P. 2. B. 6.*
6. Or alternando AB : AC = BF : CE. *P. 16. B. 5.*
7. Or AB : AC = CD : CE, C D being = to B F. (*Arg. 4*). *P. 7. B. 5.*
   Likewife C D being plle. to the fide A F of the △ F E A.
8. It may be proved in the fame manner, that AC : BC = CE : DE.
9. Confequently, AB : BC = CD : DE. *P. 22. B. 1.*

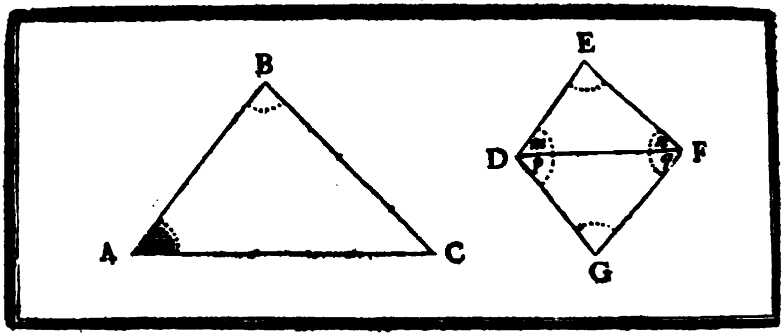
Which was to be demonftrated. 1.

But the fides A B, C D, alfo A C, C E & B C, D E are oppofite to the equal ∀ r & s, p & o, m & n.

10. Confequently, the fides A B, C D; A C, C E; B C, D E oppofite to the equal ∀ are homologous. *D. 12. B. 5.*

Which was to be demonftrated. 11.

Cor. *Therefore equiangular triangles are alfo fimilar* (*D. 1. B. 6.*)



PROPOSITION V. THEOREM V.

IF the sides of two triangles (A B C, D E F) be proportionals, those triangles shall be equiangular, and have their equal angles (A & *m*, C & *n*, &c) opposite to the homologous sides (B C, E F & A B, D E, &c).

Hypothesis.

The △ A B C, D E F *have their sides proportionals, that is,*

I. { A B : A C = D E : D F. A B : B C = D E : E F. A C : B C = D F : E F. }

II. *The sides* B C, E F, A B, D E, A C, D F. *are homologous.*

Thesis.

I. *The* △ A B C, D E F *are equiangular.*

II. *The* ∀ *opposite to the homologous sides are* =; *or* ∀ A = ∀ *m*, ∀ C = ∀ *n* & ∀ B = ∀ E.

Preparation.

1. At D in D F make ∀ *p* = ∀ A & at F, ∀ *q* = ∀ C. P. 23. B. 1.
2. Produce the sides D G, F G until they meet in G. Lem. B. 1.

DEMONSTRATION.

BECAUSE in the equiangular △ A B C, D G F (*Prep.* 1. & *P.* 32. *B.* 1), ∀ C = ∀ *q* & ∀ B = ∀ G.

1. A B : A C = D G : D F, & A B : A C = D E : D F. (*Hyp.* 1). P. 4. B. 6.
2. Therefore, D G : D F = D E : D F. & D G is = to D E. { P. 11. B. 5.
3. It may be proved after the same manner, that G F = E F. { P. 9. B. 5.

Since then in the two △ DEF, DGF, the sides DE, EF = the sides DG, GF (*Arg.* 2. & 3), & the base DF is common to the two △.

4. The ∀ *n* & *m* are = to to the ∀ *q* & *p* each to each. } P. 8. B. 1.
5. And the △ D E F, D G F are equiangular. }

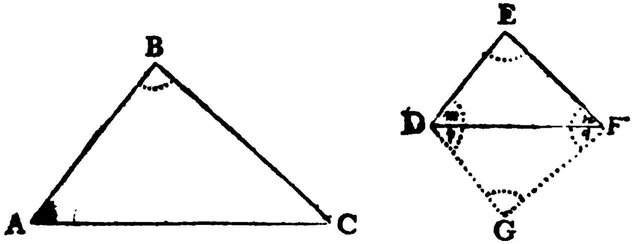
But the △ D G F, is equiangular to the △ABC (*Prep.* 1. & *P.* 32. *B.* 1),

6. From whence it follows that the △ A B C, D E F are equiangular. Ax. 1. B. 1.

Which was to be demonstrated. 1.

7. Moreover, the ∀ A, C & B opposite to the sides B C, A B, A C, being equal each to each, to the ∀ *m*, *n* & E opposite to the sides E F, D E, D F; homologous to the sides B C, A B, A C, each to each, because the one & the other of those ∀, are equal each to each to the ∀ *p*, *q*, G (*Prep.* 1. *P.* 32. *B.* 1. & *Arg.* 4).
8. It follows, that the ∀ A, *m*; also C, *n* & B, E opposite to the homologous sides are equal. Which was to be demonstrated. II.

Cor. *Therefore these triangles are also similar.* (*D.* 1. *B.* 6.)



PROPOSITION VI. THEOREM VI.

F two triangles (A B C, D E F) have one angle (A) of the one equal to e angle (*m*) of the other, and the sides (B A, A C, & E D, D F), about e equal angles proportionals, the triangles shall be equiangular, and shall ave these angles (C & *n*, also B & E) equal which are opposite to the homolo-ous sides (B A, E D & A C, D F).

Hypothesis.	Thesis.
I. ∀ A = *to* ∀ *m*.	*I. The* △ A B C, D E F *are equiangular.*
II. B A : A C = E D : D F.	*II. The* ∀ C *&* *n*, *also the* ∀ B *&* E *are*
III. BA, ED; AC, DF *are homologous.*	= *to one another.*

Preparation.

1. At the point D in the straight line D F make ∀ *p* = to ∀ A, or = to ∀ *m* & at the point F, ∀ *q* = to ∀ C. *P.* 23. *B.* 1.
2. Produce the sides D G, F G until they meet in G. *Lem. B.* 1.

DEMONSTRATION.

BECAUSE the △ A B C, D G F are equiangular (*Prep.* 1. & *P.* 32. *B.* 1), & particularly ∀ C = ∀ *q* & ∀ B = ∀ G.

. B A : A C = G D : D F *P.* 4. *B.* 5.

But . B A : A C = E D : D F. (*Hyp.* 2).

. Wherefore, G D : D F = E D : D F. *P.* 11. *B.* 5.

. Consequently, G D is = to E D. *P.* 9. *B.* 5.

Therefore the two △ D E F, D G F having the two sides E D, D F = to the two sides G D, D F (*Arg.* 3) & ∀ *m* = to ∀ *p* (*Prep.* 1).

. The ∀ *n*, *q* & E, G are =, & the △ DEF, DGF are equiangular. *P.* 4. *B.* 1.

But the △ABC, DGF being also equiangular (*Prep.* 1. & *P.* 32. *B.* 1),

. It follows, that the △ A B C, D E F are equiangular. *Ax.* 1. *B.* 1.

Which was to be demonstrated. 1.

Moreover, each of the angles C & *n* being = to ∀ *q* (*Prep.* 1. & *Arg.* 4).

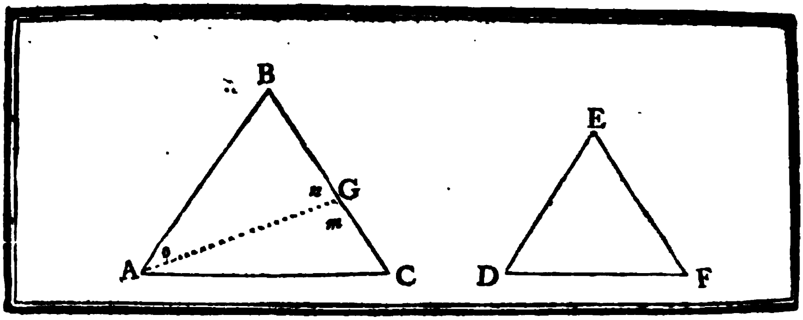
. The ∀ C is = to ∀ *n*. *Ax.* 1. *B.* 1.

. Consequently, ∀ A being = to ∀ *m* (*Hyp.* 1), ∀ B is also = to ∀ E. *P.* 32. *B.* 1.

And the sides B A, E D & A C, D F opposite to those angles being homologous (*Hyp.* 3. & *D.* 12. *B.* 5.).

. It follows, that the ∀ C & *n*, also B & E opposite to those homologous sides are = to one another. Which was to be demonstrated. 11.

Cor. *Therefore those triangles are also similar to each other.* (*P.* 4. *Cor. B.* 6).



PROPOSITION VII. *THEOREM VII.*

IF two triangles (A B C, D E F) have one angle of the one (B), equal to one angle of the other (E), and the sides (B A, A C & E D, D F) about two other angles (A & D), proportionals; then if each of the remaining angles (C & F) be either acute, or obtuse, the triangles shall be equiangular, and have those angles (A & D) equal, about which the sides are proportionals.

Hypothesis.	Thesis.
I. ∠ B *is* = *to* ∠ E.	*The* △ A B C, D E F *are equiangular,*
II. B A : A C = E D : D F	*& the* ∠ BAC *&* D *are* = *to one another.*
III. The ∠ C *&* F *are both either acute, or obtuse.*	

Demonstration.

If not, the ∠ B A C & D are unequal, and one as B A C is > the other D.

Preparation.

At the point A in the line A B, make ∠ *o* = ∠ D. P. 23. B. 1.

CASE I. If the ∠ C & F are both acute.

BECAUSE ∠ *o* is = to ∠ D *(Prep.)*, & ∠ B = to ∠ E *(Hyp. 1)*.

1. It follows, that ∠ *n* is = to ∠ F; & the △ A B G, D E F are equiangular. P. 32. B. 1.
2. Consequently, B A : A G = E D : D F. P. 4. B. 6.
   But B A : A C = E D : D F. *(Hyp. 2)*.
3. Consequently, B A : A G = B A : A C. P. 11. B. 5.
4. From whence it follows that A G is = to A C. P. 9. B. 5.
5. Wherefore, ∠ C is = to ∠ *m*. P. 5. B. 1
   And because in this case ∠ C is < ∟.
6. The ∠ *m* will be also < ∟; & ∠ *n* which is adjacent to it > ∟. P. 13. B. 1.
   But this ∠ *n* being = to ∠ F *(Arg. 1)*, which in this case is < ∟.
7. This same ∠ *n* will be also < ∟; which is impossible.
8. The ∠ B A C & D are therefore = to one another, & the third ∠ C is = to ∠ F, or the △ A B C, D E F are equiangular. P. 32. B. 1.

Which was to be demonstrated.

CASE II. If the ∀ C & F are both obtuse.

BY the same reasoning as in the first Case (*Arg.* 1. *to Arg.* 5.) it may be proved, that

. The ∀ C is = to ∀ *m*.

.. Therefore ∀ *m* is also > ∟, & the ∀ C + *m* will be > 2 ∟, which is impossible. *P.* 17. *B.* 1.

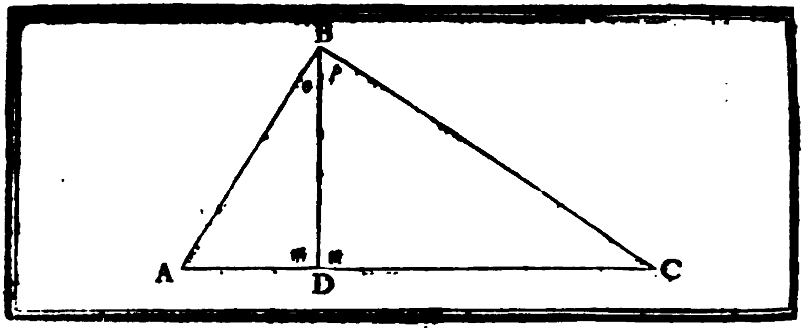
3. Consequently, the ∀ B A C & D are = to one another & the third ∀ C is = to ∀ F, or the △ A B C, D E F are equiangular. *P.* 32. *B.* 1.

Which was to be demonstrated.

REMARK.

IF the ∀ C *&* F *are both right angles the* △ A B C & D E F *are equiangular* (*Hyp.* 1. *& P.* 32. *B.* 2).

COR. *Therefore those triangles are similar to one another* (*P.* 4. *Cor. B.* 6).



PROPOSITION VIII. THEOREM VIII.

IN a right angled triangle (A B C), if a perpendicular (B D) be drawn from the right angle (A B C) to the base A C, the triangles (A D B, B D C) on each side of it are similar to the whole triangle (A B C) and to one another.

Hypothesis.	Thesis.
I. The △ A B C *is rgle. in* B.	*The* △ A D B, B D C *are similar to one another, & each is also similar to the whole* △ A B C.
II. B D *is* ⊥ *upon* A C.	

Demonstration.

BECAUSE in the two rgle. △ A D B, A B C, the ∀ *m* is = to ∀ A B C, (*Ax.* 10. *B.* 1.), & ∀ A common to the two △.

1. The ∀ *o* is = to ∀ C & the two △ A B C, A D B are equiangular. *P.* 32. *B.* 1.
2. Consequently, those two △ are also similar. { *P.* 4. *B.* 6. *Cor.*

It may be demonstrated after the same manner, that

3. The △ B D C is similar to the △ A B C.

Likewise in the two rgle. △ A D B, B D C, ∀ *m* being = to ∀ *n* (*Ax.* 10. *B.* 1.) & ∀ *o* = to ∀ C (*Arg.* 1.).

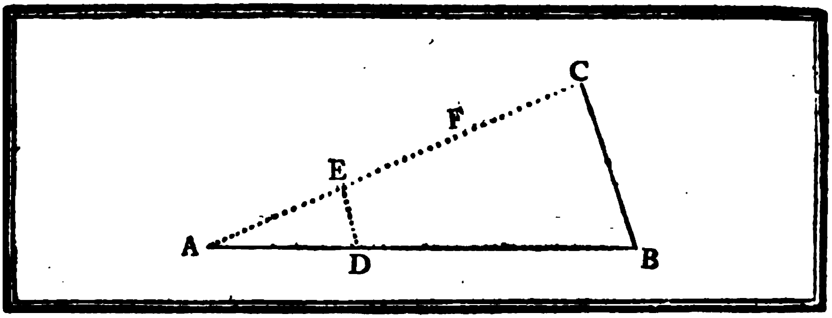
4. The ∀ A is = to ∀ *p*, & the two △ A D B, B D C, are equiangular. *P.* 32. *B.* 1.
5. From whence it follows that these △ are similar. { *P.* 4. *B.* 6. *Cor.*
6. Consequently, the ⊥ B D divides the △ A B C into two △ A D B, B D C similar to one another (*Arg.* 5.) & similar to the whole △ A B C (*Arg.* 2. & 3).

Which was to be demonstrated.

COROLLARY.

FROM this it is manifest that the perpendicular B D *drawn from the Vertex of a right angled triangle to the base, is a mean proportional between the segments* A D & D C *of the base; for the triangles* A D B, B D C *being equiangular,* A D : D B = D B : D C (*P.* 4. *B.* 6.).

Also, each of the sides A B *or* B C *of the triangle* A B C *is a mean proportional between the base & and the segment* A D *or* D C *adjacent to that side. for since each of the triangles* A D B, B D C *is equiangular with the whole* △ A B C, A C : A B = A B : A D, & A C : B C = B C : D C (*P.* 4. *B.* 6).



PROPOSITION IX. *PROBLEM I.*

FROM a given straight line (A B) to cut off any part required. (*For example the third part*).

Given.	Sought.
The straight line A B.	*The abscinded straight line* A D, *which may be the third part of* A B.

Resolution.

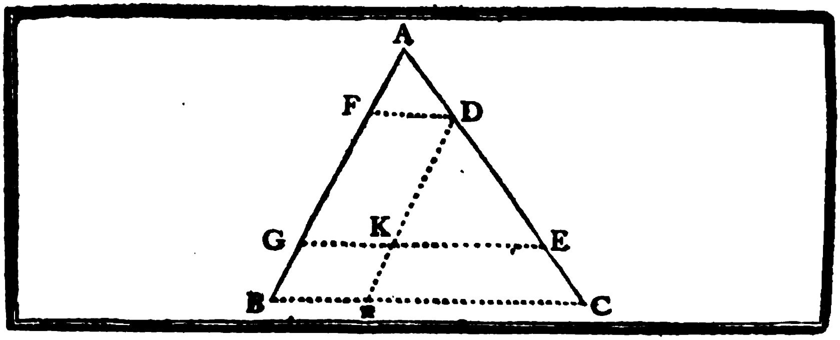
1. From the point A draw an indifinite straight line A C, making with A B any ∀ B A C.	*Pos.* 1. *B.* 1.
2. Take in AC three equal parts AE, EF, FC of any length.	*P.* 3. *B.* 1.
3. Join C B.	*Pos.* 1. *B.* 1.
4. And thro' E, draw E D plle. to C B, which will cut the straight line A B so that A D will be the third part.	*P.* 31. *B.* 1.

DEMONSTRATION.

BECAUSE ED is plle. to the side CB of the △ CAB (*Prep.* 4).

1. C E : E A = B D : D A.	*P.* 2. *B.* 6.
But C E : is double of E A (*Ref.* 2);	
2. Consequently, B D is also double of D A.	*D.* 8. *B.* 5.
3. Wherefore, A B is triple of A D.	
4. And the abscinded straight line A D is the third part of A B.	

Which was to be demonstrated.



PROPOSITION X. *PROBLEM II.*

TO divide a given ſtraight line (A B), ſimilarly to a given ſtraight line (A C) divided in the points (D, E &c)

Given.

I. *The ſtraight line* A B.
II. *The ſtraight line* A C *divided in the points* D, E *&c.*

Sought.

To divide A B *ſimilarly to* A C *in the points* F & G, *ſo that* A F : F G = A D : D E *& that* F G : G B = D E : E C.

Reſolution.

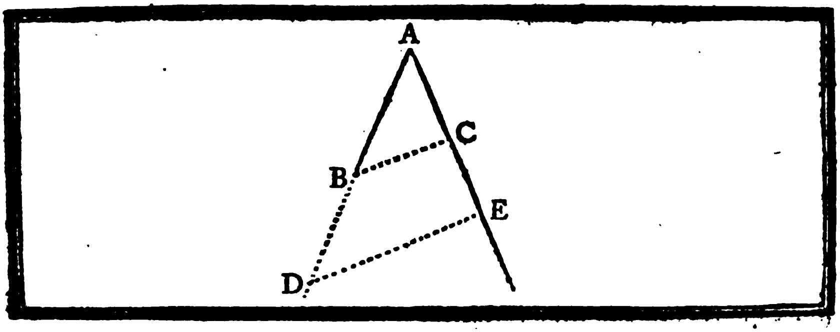
1. Join the given ſtraight lines A B, A C ſo as to contain any ∠ B A C. *Poſ.* 1. *B.* 1.
2. Draw C B, & from the points D & E, the ſtraight lines D F, E G plle. to C B, alſo D H plle. to A B. *P.* 31. *B.* 1.

DEMONSTRATION.

BECAUSE D F is plle. to the ſide E G of the △ A G E (*Ref.* 2. & *P.* 30. *B.* 1), and K E plle. to the ſide H C of the △ D H C (*Ref.* 2).

1. A F : F G = A D : D E.
And D K : K H = D E : E C. *P.* 7. *B.* 6.
But the figures K F, H G being pgrms. (*Ref.* 2. & *D.* 35. *B.* 1.).
2. It follows, that F G is = to D K & G D = K H. *P.* 34. *B.* 1.
3. Therefore, F G : G B = D E : E C. *P.* 7. & 11. *B.* 5.
4. Conſequently, the given ſtraight line A B is divided in the points F & G; ſo that A F : F G = A D : D E & F G : G B = D E : E C.

Which was to be done.



PROPOSITION XI. *PROBLEM III.*

TO find a third proportional (C E) to two given straight lines (A B, A C).

Given.
The two straight lines A B, A C.

Sought.
The straight line C E, *a third proportional to the two straight lines* A B, A C *that is such that* A B : A C = A C : C E.

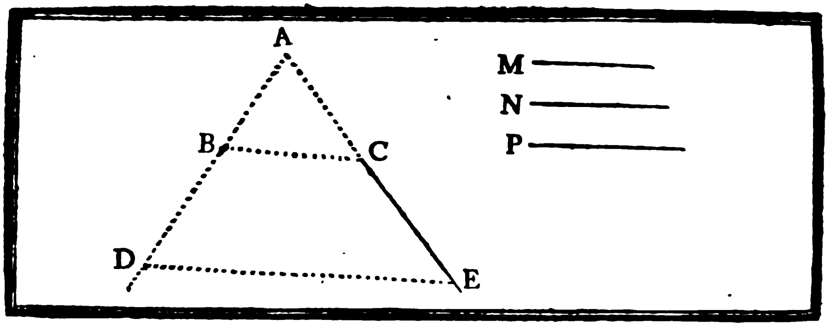
Resolution.

1. Join the two straight lines A B, A C so as to contain any ∀ B A C.
2. Produce them, & make B D = A C. *P. 3. B.1.*
3. Join B C. *Pos. 1. B. 1.*
4. And from the extremity D of the straight line A D draw D E plle. to B C. *P. 31. B. 1.*

Demonstration.

BECAUSE B C is plle. to D E *(Ref. 4)*.
1. A B : B D = A C : C E. *P. 2. B. 6.*
But B D is = to A C *(Ref. 2)*;
2. Consequently, A B : A C = A C : C E. *P. 7. & 11. B. 5.*

Which was to be done.



PROPOSITION XII. *PROBLEM IV.*

TO find a fourth proportional (CE) to three given ftraight lines (M, N, P).

Given.	Sought.
The ftraight lines M, N, P.	*The ftraight line* C E, *a fourth proportional to* M, N, P; *that is fuch, that* M : N = P : CE.

Refolution.

1. Draw the two ftraight lines A D, A E, containing any ∀ D A E. *P.* 3. *B.* 1.
2. Make A B = M; B D = N; A C = P. *Pof.* 1. *B.* 1.
3. Join B C.
4. From the extremity D of the ftraight line AD, draw DE, plle to B C. *P.* 31. *B.* 1.

DEMONSTRATION.

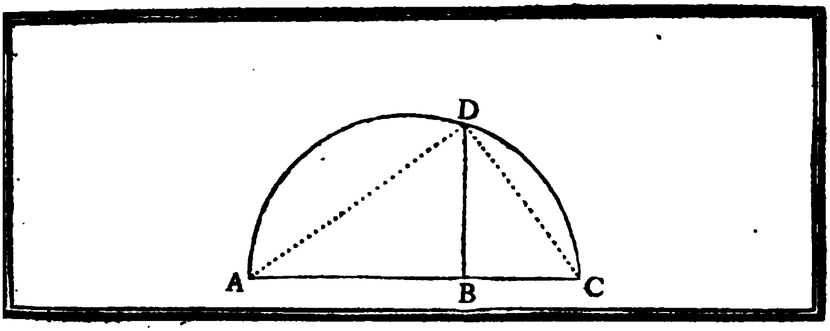
BECAUSE B C is plle. to D E (*Ref.* 4).

1. A B : B D = A C : C E. *P.* 2. *B.* 6.

But A B = M, B D = N, & A C = P (*Ref.* 2);

2. Confequently, M : N = P : C E. *P.* 7. & 11. *B.* 5.

Which was to be done.



PROPOSITION XIII. *PROBLEM V.*

TO find a mean proportional (B D); between two given straight lines A B, B C).

Given.	Sought.
The two straight lines A B, B C.	*The straight line* B D, *a mean proportional between* A B & B C, *that is such that* A B : B D = B D : B C.

Resolution.

1. Place A B, B C in a straight line A C.
2. Describe upon A C the semi ⊕ A D C. *Pos.*3. *B.*1.
3. At the point B, in A C, erect the ⊥ B D meeting the ○ in D. *P.*11. *B.*1.

Preparation.

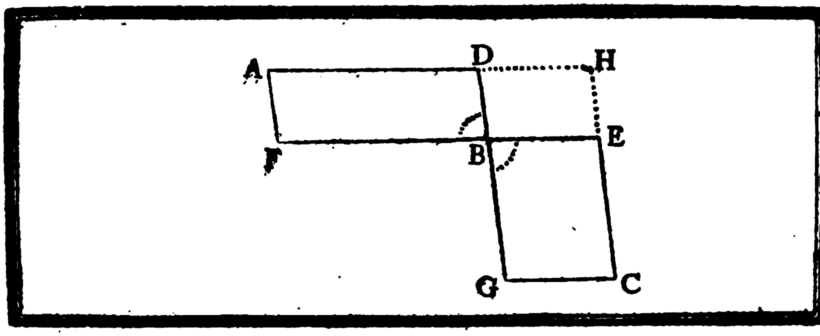
Join A D, & C D. *Pos.*1. *B.*1.

DEMONSTRATION.

BECAUSE the ∀ A D C is in a semi ⊙ (*Ref.* 2. & *Prep.*).

1. It is a right angle. *P.*31. *B.*3.
2. Wherefore, the △ A D C is right angled in D, & B D is a ⊥ let fall from the vertex D of the right angle, on the base AC (*Ref.* 3).
3. Consequently, A B : B D = B D : B C. { *P.* 8. *B.*6. *Cor.* }

Which was to be done.



PROPOSITION XIV. *THEOREM IX.*

EQUAL parallelograms (A B, B C), which have one angle of the one (F B D) equal to one angle of the other (G B E), have their sides (F B, B D & G B, B E), about the equal angles reciprocally proportional, (that is, F B : B E = G B : B D). And parallelograms that have one angle of the one (F B D) equal to one angle of the other (G B E) and the sides (F B, B D & G B, B E), about the equal angles reciprocally proportional, are equal.

Hypothesis.	Thesis.
I. The pgr. A B *is = to the pgr.* B C.	F B : B E = G B : B D.
II. ∀ F B D *is = to* ∀ G B E.	

Preparation.

1. Place the two pgrs. A B, B C so as the sides F B, B E may be in a straight line F E.
2. Complete the pgr. D E. *Pos. 2. B. 1.*

I. DEMONSTRATION.

BECAUSE the ∀ F B D, G B E are equal *(Hyp. 2)*; & F B, B E are in a straight line F E *(Prep. 1)*.

1. Therefore, G B, B D are in a straight line G D. *P. 14. B. 1.*

But the pgr. A B being = to the pgr. B C *(Hyp. 1)*.

2. The pgr. A B : pgr. D E = pgr. B C : pgr. D E. *P. 7. B. 5.*

But the pgrs. AB, DE also BC, DE have the same altitude *(D. 4. B. 6)*.

3. Hence pgr. AB : pgr. DE = FB : BE & pgr. BC : pgr. DE = GB : BD. *P. 1. B. 6.*
4. Consequently, F B : B E = G B : B D *(Arg. 2)*. *P. 11. B. 5.*

Which was to be demonstrated.

Hypothesis.	Thesis.
I. F B : B E = G B : B D.	*The pgr.* A B *is = to the pgr.* B C.
II. ∀ F B D *is = to* ∀ G B E.	

II. DEMONSTRATION.

1. IT may be demonstrated as before, that GB, BD are in the line GD.

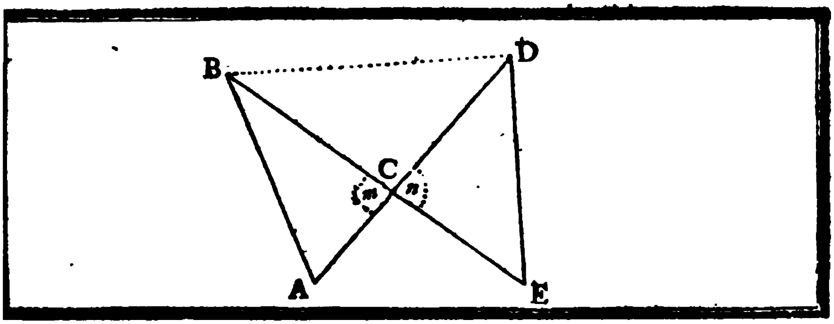
But the pgrs. AB, DE, & BC, DE, have the same altitude *(D. 4 B. 6)*.

2. Hence, pgr. AB : pgr. DE = FB : BE, & pgr. BC : pgr. DE = GB : BD. *P. 1. B. 6.*

But F B : B E = G B : BD *(Hyp.)*.

3. Wherefore, the pgr. A B : pgr. D E = pgr. B C : pgr. D E. *P. 11. B. 5.*
4. Consequently, the pgr. A B is = to the pgr. B C. *P. 9. B. 5.*

Which was to be demonstrated.



PROPOSITION XV. *THEOREM X.*

EQUAL triangles (ACB, ECD) which have one angle of the one *(m)* equal to one angle of the other *(n)*: have their sides (AC, CB, & EC, CD), about the equal angles, reciprocally proportional; & the triangles (ACB, ECD) which have one angle in the one *(m)* equal to one angle in the other *(n)*, and their sides (AC, CB, & EC, CD), about the equal angles reciprocally proportional, are equal to one another.

CASE I.

Hypothesis.	Thesis.
I. The △ ACB *is* = *to* △ ECD.	*The sides* AC, CB *&* EC, CD,
II. ∀ *m* *is* = *to* ∀ *n*.	*are reciprocally proportional, or* AC : CD = EC : CB.

Preparation.

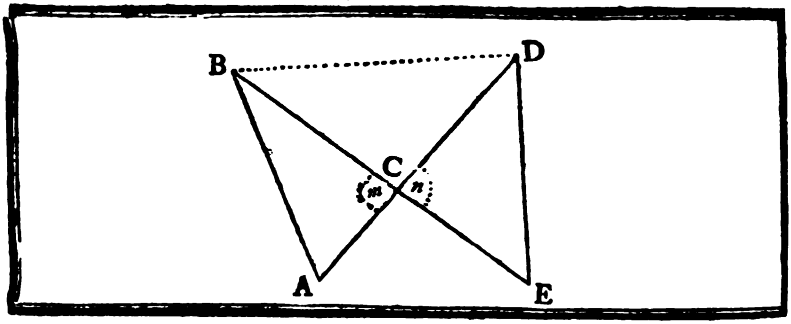
1. Place the △ ACB, ECD so that the sides AC, CD may be in the same straight line AD.
2. Draw the straight line BD. *Post.1. B.1.*

Demonstration.

BECAUSE ∀ *m* = ∀ *n* *(Hyp. 2.)*, & the straight lines AC, CD are in the same straight line AD *(Prep. 1)*.

1. The lines EC, CB are also in a straight line EB. *P.14. B.1.*
But the △ ACB being = to the △ ECD *(Hyp. 1)*.
2. The △ ACB : △ CBD = △ ECD : △ CBD. *P. 7. B.5.*
But the △ ACB, CBD also ECD, CBD have the same altitude *(Prep. 2. Arg. 1. & D. 4. Rem. B. 6)*.
3. Wherefore the △ ACB : △ CBD = AC : CD.
& the △ ECD : △ CBD = EC : CB. } *P. 1. B.6.*
4. Consequently, AC : CD = EC : CB *(Arg.2.&P.11.B.5)*.

Which was to be demonstrated.



CASE II.

Hypothesis.

I. AC : CD = EC : CB.

II. & ∀ *m* = ∀ *n*.

Thesis.

The △ ACB, *is* = *to the* △ ECD.

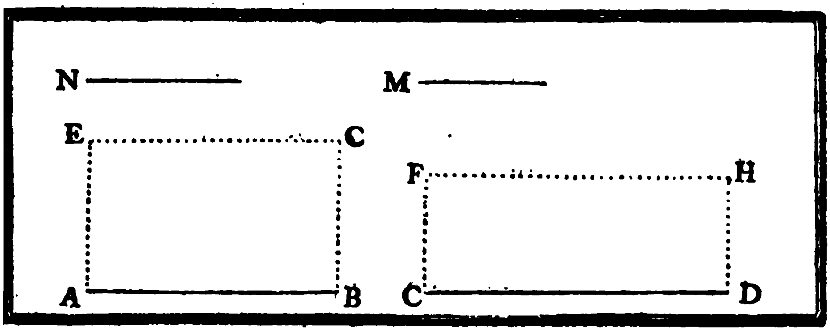
Preparation.

1. Place the two △ ACB, ECD ſo that the ſides AC, CD, may be in the ſame ſtraight line AD.
2. Draw the ſtraight line BD.

DEMONSTRATION.

1. IT may be demonſtrated, as in the firſt Caſe, that EC, CB are in the ſame ſtraight line EB.
   And becauſe the △ ACB, CBD, alſo the △ ECD, CBD have the ſame altitude (*Prep.* 2. *Arg.* 1. & *D.* 4 *Rem. B.* 6).
2. The △ ACB : △ CBD = AC : CD. } *P.* 1. *B.* 6.
   Likewiſe △ ECD : △ CBD = EC : CB.
   But AC : CD = EC : CB. (*Hyp.* 1).
3. Wherefore △ ABC : △ CBD = △ ECD : △ CBD. *P.* 11. *B.* 5.
4. Conſequently, the △ ABC is = to the △ ECD. *P.* 9. *B.* 5.

Which was to be demonſtrated.



PROPOSITION XVI. *THEOREM XI.*

If four straight lines (A B, C D, M, N) be proportionals, the rectangle contained by the extremes (A B. N) is equal to that of the means (C D. M). And if the rectangle contained by the extreames (A B. N) be equal to the rectangle contained by the means (C D. M), the four straight lines (AB, CD, M, N) are proportionals.

Hypothesis.	Thesis.
A B : C D = M : N.	*Rgle.* A B. N = *Rgle.* C D. M.

Preparation.

1. At the extremities A & C, of AB, CD, erect the ⊥ AE, CF. *P.*11. *B.*5.
2. Make A E = N, & C F = M. *P.* 3. *B.*1.
3. Complete the rgles. EB, F D. *P.*31. *B.*1.

I. Demonstration.

Because AB : CD = M : N (*Hyp.*) : & M = CF & N = AE (*Prep.*2).

1. AB : CD = CF : AE. *P.*7. & 11. *B.*5.
2. Therefore the sides of the rgles E B, F D about the equal ∀ A & C, (*Prep.* 1. & *Ax.* 10. *B.* 1.) are reciprocal. *D.* 2. *B.*6.
3. Consequently, the rgle. EB = rgle. FD, or the rgle under AB.AE = the rgle. under CD. CF. { *P.*14. *B.*6. *D.* 1. *B.*1.
 Consequently, A E being = N & C F = M (*Prep.* 2).
4. The rgle. under A B. N is also = to the rgle. under C D. M. *Ax.*2. *B.*2.

Which was to be demonstrated.

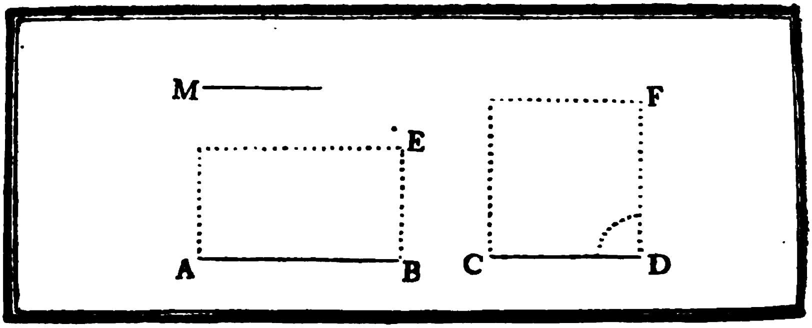
Hypothesis.	Thesis.
The rgle. AB. N *is* = *to the rgle.* CD. M.	A B : CD = M : N.

II. Demonstration.

Because the rgle. A B. N is = to the rgle C D. M (*Hyp.*) : & A E = N, & C F = M (*Prep.* 2).

1. The rgle. under A B. A E is = to the rgle under C D. C F. *Ax.*2. *B.*1.
 But these sides being about the equal ∀ E A B, F C D (*Prep.*1. & *Ax.*10. *B.*1).
2. A B : C D = C F : A E. *P.*14. *B.*6.
 And C F being = M & A E = N (*Prep.* 2).
3. A B : C D = M : N. *P.*7. & 11. *B.*5.

Which was to be demonstrated.



PROPOSITION XVII. *THEOREM XII.*

If three straight lines (AB, CD, M) be proportionals, the rectangle (AB.M) contained by the extremes is equal to the square of the mean (CD): And if the rectangle contained by the extreams (AB.M) be equal to the square of the mean (CD), the three straight lines (AB, CD, M) are proportionals.

Hypothesis. Thesis.
AB : CD = CD : M. *The rgle.* AB.M *is* = *to the* □ *of* CD.

Preparation.

1. At the extremities B & D of AB, CD erect the ⊥ BE, DF. P.11. B.1.
2. Make BE = M & DF = DC. P.3. B.1.
3. Complete the rgles. EA, FC. P.31. B.1.

I. Demonstration.

Because AB : CD = M (*Hyp.*), & CD = DF & M = BE (*Prep.* 2).

1. AB : CD = DF : BE. P.7. & 11. B.5.
Therefore the sides of the rgles. EA, FC about the equal ∠ B & D (*Prep.* 1. & *Ax.* 10. *B.* 1) are reciprocal. D. 2. B.6.
2. Consequently, the rgle. EA is = to the rgle. FC, or the rgle. under AB. BE = the rgle CD. DF. P.14. B.1.
3. Wherefore, BE being = M & DF = CD (*Prep.* 2), the rgle. AB. M is also = to the □ of CD. D. 1. B.6. Ax. 2. B.1.

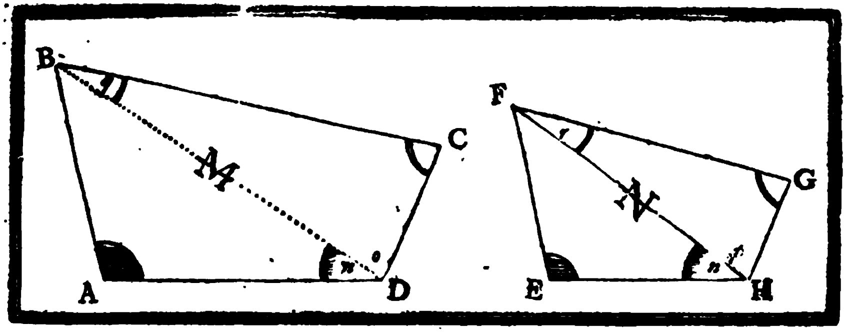
Hypothesis. Thesis.
The rgle. AB. M *is* = *to the* □ *of* CD. AB : CD = CD : M.

II. Demonstration.

Because the rgle. under AB.M is = to the □ of CD (*Hyp.*), & that BE is = M & DF = CD (*Prep.* 2).

1. The rgle. under AB. BE is = to the rgle. under CD. DF. Ax. 2. B.1.
But those sides are about the equal ∠ EBA, FDC (*Ax.* 10. *B.* 1. & *Prep.* 1).
2. Therefore, AB : CD = DF : BE. P.14. B.6.
And since DF = CD & BE = M (*Prep.* 2).
3. AB : CD = CD : M; P.7. & 11. B.5.

Which was to be demonstrated.



PROPOSITION XVIII. *PROBLEM VI.*

UPON a given ſtraight line (AD) to deſcribe a rectilineal figure (M) ſimilar, and ſimilarly ſituated to a given rectilineal figure (N).

Given.	Sought.
I. The ſtraight line AD. *II. The rectilineal figure* N,	*The rectilineal figure* M *ſimilar to a rectilineal figure* N & *ſimilarly ſituated.*

Reſolution.

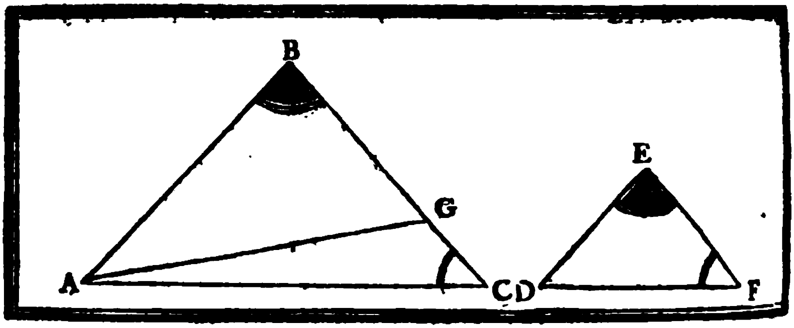
1. Join HF. *Poſ. 1. B. 1.*
2. At the points A & D in AD, make ∀A = ∀E & ∀*m* = ∀ *n*, wherefore the remaining ∀ ABD will be = to the remaining ∀ EFH.
3. At the points D & B in DB make ∀ *o* = ∀ *p* & ∀ *q* = ∀ *r*, conſequently the remaining ∀ C will be = to the remaining ∀ G. } *P. 23. 32. & Lem. B. 1.*

DEMONSTRATION.

BECAUSE the △ ABD is equiangular to the △ EFH, & the △ DBC equiangular to the △ HFG *(Ref. 2. & 3).*

1. BD : FH = BA : FE = AD : EH.
& BD : FH = DC : HG = CB : GF. } *P. 4. B. 6.*
2. Conſequently, BA : FE = AD : EH = DC : HG = CB : GF. *P. 11. B. 5.*
But ∀ *m* being = ∀ *n* *(Ref. 2)*, & ∀ *o* = ∀ *p* *(Ref. 3)*.
3. The whole ∀ *m* + *o* is = to the whole ∀ *n* + *p*. *Ax. 2. B. 1.*
4. Likewiſe ∀ ABC = ∀ EFG.
Moreover, ∀ A = ∀ E *(Ref. 2)*, & ∀ C = ∀ G *(Ref. 3)*.
4. Wherefore, the rectilineal figure M is equiangular to the rectilineal figure N, & their ſides about the equal ∀ are proportionals.
6. Therefore, the rectilineal figure M deſcribed upon the given line AD is ſimilar to the rectilineal figure EG, & is ſimilarly ſituated. *D. 1. B. 6.*

Which was to be done.



PROPOSITION XIX. THEOREM XIII.

SIMILAR triangles (A B C, D E F) are to one another in the duplicate ratio of their homologous sides (C B, F E or A C, D F, &c).

Hypothesis.	Thesis.
The triangles ABC, DEF *are similar. So that* ∀ C = ∀ F, *& the sides* A C, D F & C B, F E *are homologous.*	*The* △ ABC *is to the* △ DEF *in the duplicate ratio of* C B *to* F E *that is as* $CB^2 : FE^2$.*

Preparation.

Take C G a third proportional to C B, F E, & draw A G. { P.11. B.6. Pos.1. B.1.

DEMONSTRATION.

BECAUSE A C : C B = D F : F E (*Hyp.* & *D.* 1. *B.* 6.).

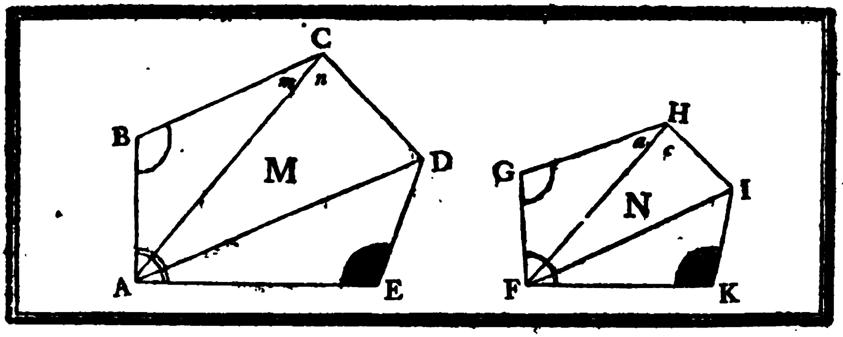
1. Alternando A C : D F = C B : F E. P.16. B.5.
   But C B : F E = F E : C G (*Prep.*).
2. Consequently, A C : D F = F E : C G. P.11. B.5.
3. Therefore, the sides of the △ AGC, DEF about the equal ∀ C & F (*Hyp.*) are reciprocal (*D.* 2. *B.* 6).
4. Hence the △ A G C is = to the △ D E F. P.15. B.6.
   But the △ A B C, A G C having the same altitude.
5. The △ A B C : △ A G C = C B : C G. P. 1. B.6.
6. Consequently, the △ A B C : △ D E F = C B : C G. P. 7. B.5.
   But since C B : F E = F E : C G. (*Prep.*).
7. C B : C G in the duplicate ratio of C B to F E, or as $CB^2 : FE^2$* D.10. B.5.
8. Wherefore, the △ A B C : △ D E F in the duplicate ratio of C B to F E, or as $CB^2 : FE^2$*. P.11. B.5.

Which was to be demonstrated.

COROLLARY.

FROM *this it is manifest, that if three lines* (C B, F E, C G) *be proportionals, as the first is to the third, so is any* △ *upon the first to a similar, & similarly described* △ *upon the second.*

* *See Cor.* 2. *of the following proposition.*



PROPOSITION XX. THEOREM XIV.

SIMILAR polygons (M & N) may be divided by the diagonals (AC, AD; FH, FI) into the same number of similar triangles (ABC, ACD, ADE, & FGH, FHI, FIK) having the same ratio to one another, that the polygons (M & N) have; and the polygons (M & N) have to one another the duplicate ratio of that which their homologous sides (AB, FG; or BC, GH &c.) have.

Hypothesis.

The polyg. M *is similar to the polyg.* N; *so that the* ∀ A, B, C, *&c. are* = *to the* ∀ F, G, H, *&c. each to each & the sides* AB, FG; *or* BC, GH, *&c. are homologous.*

Thesis.

I. *Those polygons may be divided into the same number of similar* △.

II. *Whereof, each to each has the same ratio which the polygons have.*

III. *And the polyg.* M : *polyg.* N *in the duplicate ratio of the homologous sides* AB, FG; *or as* $AB^2 : FG^2$.*

Preparation.

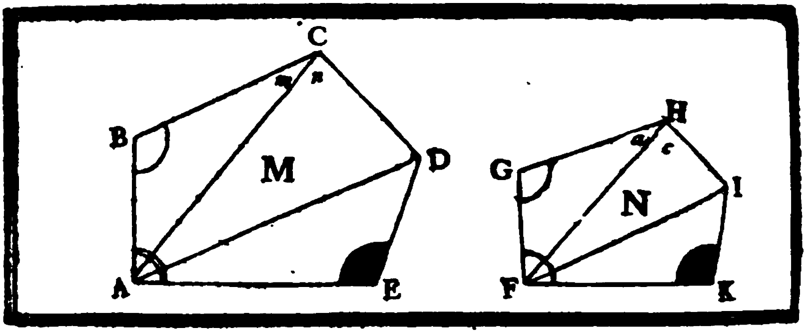
Draw AC, FH, likewise AD, FI. *Pos.* 1. *B.* 1.

DEMONSTRATION.

BECAUSE ∀B = ∀G & AB : BC = FG : GH (*Hyp.* & *D.* 1. *B.* 6).

1. The △ ABD is equiangular to the △ FGH. *P.* 6. *B.* 6.
2. Wherefore those △ are similar, & ∀ *m* = ∀ *a*. *P.* 4. *B.* 6. *Cor.*
   But the whole ∀ *m* + *n* is = to the whole ∀ *a* + *c* (*Hyp*).
3. Consequently, ∀ *n* is = to ∀ *c*. *Ax.* 3. *B.* 1.
   Since then by the simil. of the △ ABC & FGH (*Arg.* 2),
   AC : BC = FH : GH, *D.* 1. *B* 6.
   & by the simil. of the polyg. M & N, BC : CD = GH : HI.
4. It follows, Ex Æquo, that AC : CD = FH : HI. *P.* 22. *B.* 5.
   That is, the sides about the equal ∀ *n* & *c* are proportionals.
5. Therefore the △ ACD is equiangular to the △ FHI. *P.* 6. *B.* 6.
   And consequently is similar to it. *P.* 4. *B.* 6. *Cor.*
6. For the same reason, all the other △ ADE, FIK, &c. are similar.
7. Therefore, similar polygons may be divided into the same number of similar △. Which was to be demonstrated. 1.

* *See Cor.* 2. *of this proposition.*



Likewise, because the △ ABC, FGH are similar (*Arg.* 2).

6. The △ABC : △FGH = $AC^2 : FH^2$.* } P.19. B.6

And the △ACD : △FHI = $AC^2 : FH^2$.* }

7. Therefore, the △ABC : △FGH = △ACD : △FHI. P.11. B.5

It may be demonstrated after the same manner, that

8. The △ADE : △FIK = △ACD : △FHI.

9. Wherefore, △ABC : △FGH = △ACD : △FHI = △ADE : △FIK. P.11. B.5

10. Therefore, comparing the sum of the anteced. to that of the conseq.

△ABC + △ACD, &c. : △FGH + △FHI, &c. = △ABC : △FGH, &c. P.12. B.5

That is, the polyg. M : polyg. N = △ABC : △FGH = △ACD : △FHI, &c. P.7. B.5

Which was to be demonstrated. I I.

Since then the △ABC : △FGH = $AB^2 : FG^2$* (*P.*19. *B.* 6).

11. The polyg. M : polyg. N = $AB^2 : FG^2$*. P.11. B.5

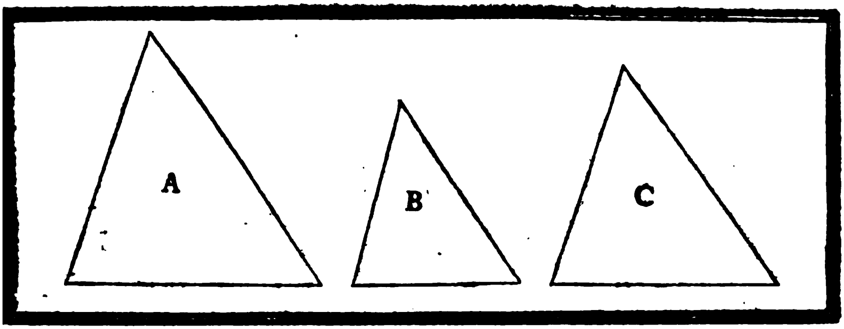
Which was to be demonstrated. I I I.

COROLLARY I.

*AS this Demonstration may be applied to quadrilateral figures, & the same truth has already been proved in triangles (P.*19*), it is evident universally, that similar* rectilineal figures are to one another in the duplicate ratio of their homologous sides. *Wherefore, if to* AB, FG *two of the homologous sides a third proportional* X *be taken; because* AB *is to* X *in the duplicate ratio of* AB : FG; *& that a rectilineal figure* M *is to another similar rectilineal figure* N, *in the duplicate ratio of the same sides* AB : FG; *it follows,* that if three straight lines be proportionals, as the first is to the third, so is any rectilineal figure described upon the first to a similar & similarly described rectilineal figure upon the second. (*P.*11. *B.*5).

* *COROLLARY II.*

ALL squares being similar figures (*D.* 30. *B.* 1. & *D.* 1. *B.* 6), *similar rectilineal figures* M & N, *are to one another as the squares of their homologous sides* AB, CD (*expressed thus* $AB^2 : CD^2$.) *for those figures are in the duplicate ratio of these same sides.*



PROPOSITION XXI. *THEOREM XV.*

RECTILINEAL figures (A, C) which are similar to the same rectilineal figure (B), are also similar to one another.

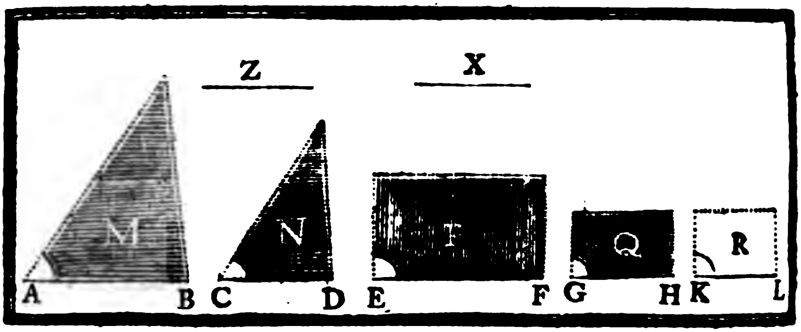
Hypothesis.	Thesis.
The rectilineal figures A *&* C *are similar to the figure* B.	*The rectilineal figure* A *is similar to the rectilineal figure* C.

DEMONSTRATION.

BECAUSE each of the figures A & C is similar to the figure B (*Hyp.*).

1. Each of those figures will be also equiangular to the figure B, & will have the sides about the equal ∀, proportional to the sides of the figure B. *D.* 1. *B.*6.
2. Consequently, those figures A & C will be also equiangular to one another, and their sides about the equal ∀, will be proportional. { *Ax.*1. *B.*1. *P.*11. *B.*5.
3. Consequently, the figures A & C are similar. *D.* 1. *B.*6.

Which was to be demonstrated.



PROPOSITION XXII. *THEOREM XVI.*

IF four ſtraight lines (AB, CD, EF, GH) be proportionals, the ſimilar rectilineal figures & ſimilarly deſcribed upon them (M, N, & P, Q) ſhall alſo be proportionals. And if the ſimilar rectilineal figures (M, N, & P, Q) ſimilarly deſcribed upon four ſtraight lines be proportionals, thoſe ſtraight lines ſhall be proportional.

I.

Hypotheſis. | Theſis.

I. AB : CD = EF : GH. | M : N = P : Q.

II. The figures M *&* N *deſcribed upon* AB, CD.
alſo the figures P *&* Q *deſcribed upon* EF, GH.
are ſimilar, & ſimilarly ſituated.

Preparation.

To the lines AB, CD take a III proportional Z.
To the lines EF, GH take a III proportional X. } *P.11. B6.*

DEMONSTRATION.

BECAUSE AB : CD = EF : GH. { *(Hyp. 1).*
& CD : Z = GH : X. { *(Hyp. 1. Prep. & P. 11. B. 5).*
1. AB : Z = EF : X. *P. 22. B5.*

But the figures M, N, & P, Q being ſimilar & ſimilarly deſcribed upon the ſtraight lines AB, CD, & EF, GH *(Hyp. 2).*

2. AB : Z = M : N { *P. 20. B6.*
& EF : X = P : Q. { *Cor. 1.*
3. Wherefore, M : N = P : Q *(Arg. 1).* *P. 11. B5.*

II.

Hypothesis.	Thesis.
I. M : N = P : Q.	AB : CD = EF : GH.
II. Those figures are similar & similarly described upon the straight lines AB, CD & EF, GH.	

Preparation.

1. To AB, CD, EF take a IVth proportional KL. — *P.*12. *B.*6.
2. Upon KL describe the rectil. figure R, similar to the rectil. figures P or Q, & similarly situated. — *P* 18. *B.*6.

DEMONSTRATION.

BECAUSE AB : CD = EF : KL *(Prep. 1)*, & upon those straight lines have been similarly described the figures M, N, & P, R, similar each to each *(Hyp. 2. & Prep. 2)*.

1. M : N = P : R *(Ist. part of this proposition.)*

But M : N = P : Q *(Hyp. 1)*.

2. Consequently, P : R = P : Q — *P.*11. *B.*5.
3. Wherefore, R = Q — *P.* 9. *B.*5.

Moreover, those figures being similar & similarly described upon the straight lines GH, KL *(Prep. 2)*.

4. Q : R = □ of GH : □ of KL. — *P.*20. *B.*6. *Cor.* 2.

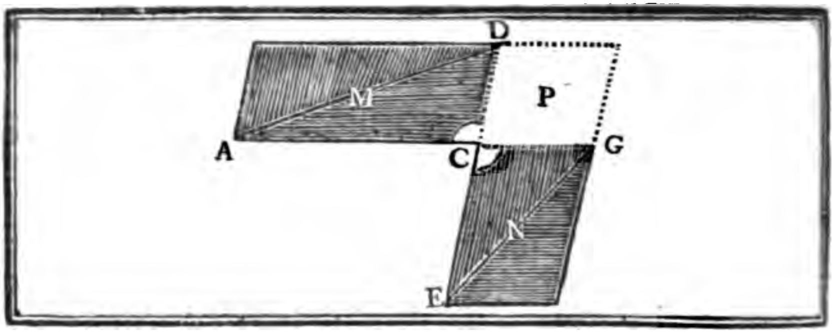
And Q being = R *(Arg. 3)*.

4. The □ of GH is = to the □ of KL. — *P.*16. *B.*5. *Cor.*
5. Consequently, GH = KL. — *P.*46. *B.*1. *Cor.*

Since then AB : CD = EF : KL *(Prep. 1)*, & GH = KL *(Arg. 5)*.

6. AB : CD = EF : GH. — *P.* 7. *B.*5.

Which was to be demonstrated.



PROPOSITION XXIII. THEOREM XVII.

EQUIANGULAR parallelograms (M & N) have to one another the ratio which is compounded of the ratios of their sides (AC, CD & E C, CG) about the equal angles.

Hypothesis.	Thesis.
The pgrs. M & N *are equiangular, so that* ∀ ACD = ∀ ECG.	*Pgr.* M : *Pgr.* N = AC. CD : EC. CG

Preparation.

1. Place A C & C G in the same straight line A G; therefore E C & C D are also in a straight line E D. *P.14. B.1.*
2. Complete the pgr. P. *Pos.1. B.1.*

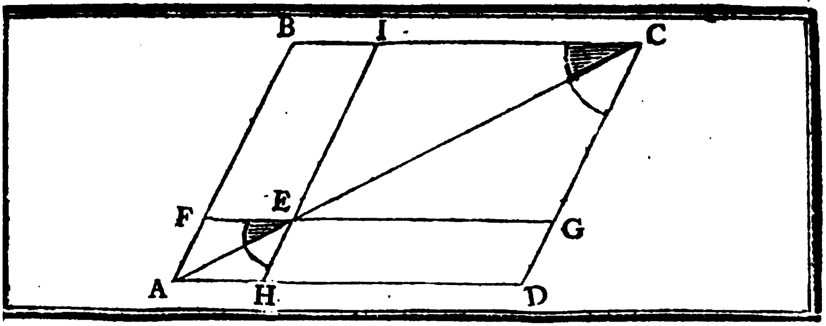
DEMONSTRATION.

BECAUSE the pgrs. M, P, N form a series of three magnitudes

1. M : MP = N : N.P. *D. 5. B.5.*
2. And alternando M : N = M.P : N.P. *P.16. B.5.*
3. Consequently the ratio of the first M to the last N, is compounded of the ratios M : P & P : N. *D. 5. B.6.*
   But since A C : C G = M : P
   & D C : C E = P : N. } *P. 1. B.6.*
4. The ratio of the sides A C : C G is the same as that of the pgrs. M : P; & the ratio of the sides D C : C E, the same as that of the pgrs. P : N.
   Since then the ratio of M : N is compounded of the ratios M : P, & P : N *(Arg. 1)*.
5. This same ratio is compounded of their equals; the ratios A C : C G & C D : E C, of the sides about the equal ∀ ACD, ECG.
6. Consequently, M : N = A C . C D : E C . C G. *D. 5. B.6.*

Which was to be demonstrated.

Cor. *The same truth is applicable to the triangles* (A C D, E C G) *having an angle* (A C D) *equal to an angle* (E C G). *for the diagonals* (A D, E G) *divide the pgrs. into two equal parts* (*P. 34. B. 1*).



PROPOSITION XXIV. *THEOREM XVIII.*

THE parallelograms (FH, IG) about the diagonal (AC) of any parallelogram (BD), are similar to the whole, and to one another.

Hypothesis.	Thesis.
I. B D *is a pgr.*	I. *The pgrs.* AFEH, EICG *are similar to the pgr.* ABCD.
II. FH,IG *are pgrs about the diagonal* AC.	II. *And similar to one another.*

DEMONSTRATION.

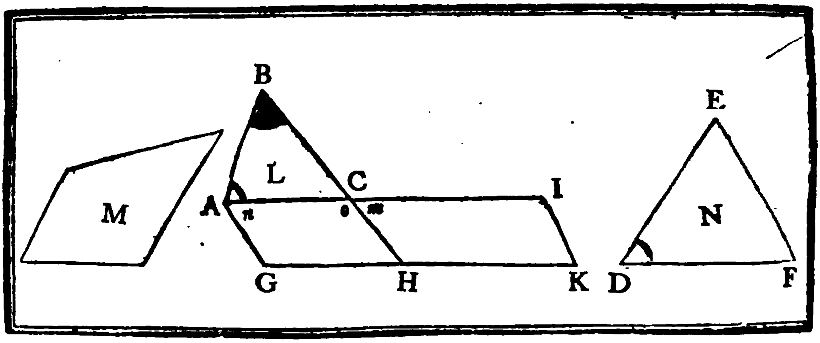
BECAUSE FE is plle. to BC (*Hyp.* 1. & 2. & *P.* 30. *B.* 1).

1. The △AFE is equiang. to the △ABC in the order of the letters. In like manner, because H E is plle. to D C. } *P.*29. *B.*1.
2. The △AHE is equiang. to the △ADC, in the order of the letters.
3. Therefore the pgr. AFEH is also equiangular to the pgr. ABCD, in the order of the letters.
   And because in the △AHE,ADC, the ∀AHE & D are equal (*Arg.*2), as also in the △ AFE, ABC, the ∀ AFE & B (*Arg.* 1).
4. AH : HE = AD : DC & AF : EF = AB : CB. *P.* 4. *B.*6.
   Moreover, because the ∀AEH,ACD; also FEA,BCA are equal (*Arg.*1.&2).
5. HE : AE = DC : AC & AE : EF = AC : CB. *P.* 4. *B.*6.
6. Therefore, ex æquo, HE : EF = DC : CB. *P.*22. *B.*5.
   And because the ∀ E A H, E F A are common to the two △ A H E, A D C & A F E, A B C.
7. HA : EA = DA : CA & EA : AF = CA : AB. *P.* 4. *B.*6.
8. Therefore, ex æquo, H A : A F = D A : A B. *P.*22. *B.*5.
9. Wherefore the pgrs. AFEH, ABCD have their angles equal, each to each in the order of the letters (*Arg.* 3); & the sides about the equal angles, proportionals (*Arg.* 4. 6. 8.).
10. Consequently, those pgrs. are similar. *D.* 1. *B.*6.
11. It may be demonstrated after the same manner that the pgrs. E I C G, A B C D are similar.

Which was to be demonstrated. 1.

12. Consequently, the pgrs. AFEH, EICG are also similar to one another. *P.*21. *B.*1.

Which was to be demonstrated. 11.



PROPOSITION XXV. *PROBLEM VII.*

TO describe a rectilineal figure (N), which shall be similar to a given rectilineal figure (L), and equal to another (M).

Given.	Sought.
I. The rectilineal figure L.	*The rectil. figure* N, *similar to the rectil.*
II. The rectilineal figure M.	*figure* L, *&* = *to the rectil. figure* M.

Resolution.

1. Upon the straight line AC, describe the pgr. AH = to the given rectilineal figure L. *P.*44. *B.*1.
2. And on the straight line CH a pgr. CK = to the given rectilineal figure M, having an ∀ *m* = to the ∀ *n*. *P.*45. *B.*1.
3. Consequently, the sides A C, C I, & G H, H K will be in a straight line. *P.*14. 29. & 34. *B.*1.
4. Between A C, & C I find a mean proportional D F. *P.*13. *B.*6.
5. Upon this straight line D F, describe the rectil. figure N, similarly & similar to the rectilineal figure L. *P.*18. *B.*6.

DEMONSTRATION.

BECAUSE the pgrs. AH, CK have the same altitude *(Ref.* 2. *&* 3*)*.

1. pgr. AH : pgr. CK = AC : CI. *P.* 1. *B.*6.

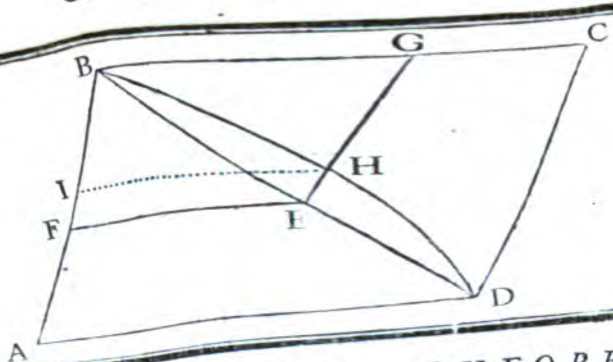
But the pgr. AH = rectil. L, & the pgr. CK = rectil. M *(Ref.* 1. *&* 2*)*.

2. Consequently, L : M = AC : CI. *P.*11. *B.*5.

But AC : DF = DF : CI *(Ref.* 4.*)*, & upon the straight lines AC, DF have been similarly described the similar figures L & N, *(Ref.* 5*)*.

3. Consequently, L : N = AC : CI. *P.*20. *B.*6. *Cor.*
4. Hence, L : N = L : M *(Arg.* 2*)*.
5. Wherefore, N = M *P.*11. *B.*5.
6. Therefore, there has been described a rectilineal figure N, similar to the rectilineal figure L *(Ref.* 5*)*, & equal to the rectilineal figure M *(Arg.* 5*)*. *P.*14. *B.*5.

Which was to be done.



PROPOSITION XXVI. THEOREM X

IF two similar parallelograms (A C, F G) have a common angle and be similarly situated, they are about the same diagonal (B D).

Hypothesis.

I. A C is a pgr. & B D its diagonal.
II. F G is a pgr. similar to AC & having an ∀ F B G common with A C.

Thesis.

The pgr. F G is placed the diagonal BD of th

DEMONSTRATION.

If not, let another line BHD different from BED be the diagonal of the pgr. A C, cutting the side G E in the point H.

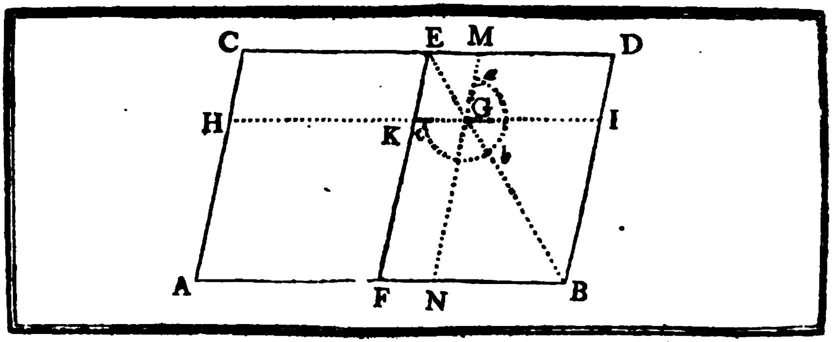
Preparation.

Thro' the point H draw H I p||le. to C B or D A.

THE pgrs. A C, I G being about the same diagonal B H D, ∀ F B G being common to the two pgrs. *(Sup. & Prep.)*,

1. The pgr. A C is similar to the pgr. I G.
2. Consequently, C B : B A = G B : B I.
   But the pgrs. A C & F G being also similar, & ∀ B common to t two pgrs. *(Hyp. 2)*.
3. It follows, that C B : B A = G B : B F.
4. Consequently, G B : B I = G B : B F,
5. Therefore, B I = B F.
6. Which is impossible.
7. Hence, a line B H D, different from the line B E D is not diagonal of the pgr. A C.
8. Consequently, the line B E D is the diagonal, & the pgr. F is placed about it.

Which was to be demonstr



PROPOSITION XXVII. *THEOREM XX.*

OF all parallelograms (A G) applied to the same straight line (A B), and deficient by parallelograms (N I) similar and similarly situated to that (F D) which is described upon the half (FB) of the line (A B); that (A E) which is applied to the other half (AF), and is similar to its defect (FD), is the greatest.

Hypothesis.	Thesis.
I. A E *is a pgr. applied to the half* A F *of the straight line* A B. *II. Which is similar & similarly situated to its defect the pgr.* FD, *described on the other half* FB.	A E *is the greatest of all the pgrs. such as* A G, *applied to* A B, *that have their defects such as* N I, *similar & similarly situated to the pgr.* F D, *defect of* A E, *described upon* F B *the half of* A B.

Preparation.

1. Draw the diagonal B E, *Pos.1. B.1.*
2. Thro' any point G, taken in B E, draw I H, M N plle. to B A, A C. *P.31. B.1.*
   In order to have a pgr. A G, applied to A B, deficient by a pgr. N I, similar to the pgr. F D & similarly situated. *P.26. B.6.*

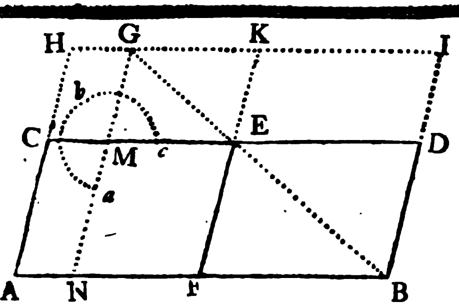
DEMONSTRATION.

C A S E I. When the point N falls in the half F B.

BECAUSE the pgr. G D is = to the pgr. G F (*P.43. B.1*); adding the common pgr. N I.

1. The pgr. N D will be = to the pgr. F I. *Ax.2. B.1.*
   But because the pgr. A K is also = to the pgr. F I. (*P. 36. B. 1*),
2. The pgr. N D is = to the pgr. A K. *Ax.1. B.1.*
   And adding to both sides the pgr. F G.
3. The gnomon *a b c* is = to the pgr. A G. *Ax.2. B.1*
4. Consequently, the whole pgr. FD, or its equal the pgr. AE (*Hyp.2*), is > pgr. A G. *Ax.8. B.1.*

Which was to be demonstrated.



CASE II. When the point N falls in the half A F.

The pgr. N E being = to the pgr. I E (*P*. 43. *B*. 1), if the common pgr. F D be added to both sides.

The pgr. N D will be = to the pgr. F I. *Ax*.2. *B*.1.

But because the pgr. A K is also = to the pgr. F I (*P*. 36. *B*. 1).

The pgr. N D will be = to the pgr. A K *Ax*.1. *B*.1.

Therefore the common pgr. F M being taken away from both sides.

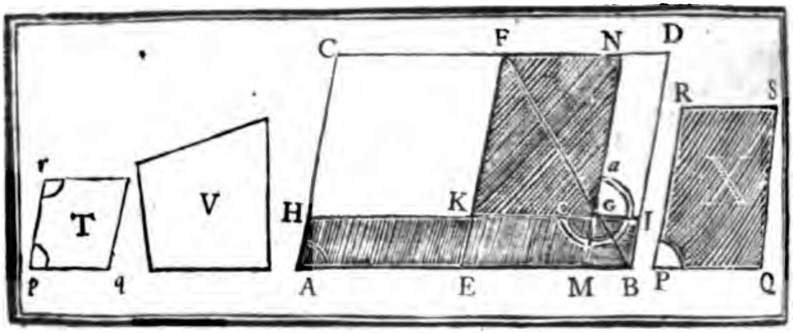
The remaining pgr. F D is = to the gnomon *a b c*. *Ax*.3. *B*.1.

But the pgr. F D is = to the pgr. A E. *P*.36. *B*.1.

Wherefore the pgr. A E is = to the gnomon *a b c*. *Ax*.1. *B*.1.

Consequently the pgr. A E is > the pgr. A G. *Ax*.8. *B*.1.

Which was to be demonstrated.



PROPOSITION XXVIII. *PROBLEM VIII.*

To a given straight line (A B) to apply a parallelogram (A G) equal to a given rectilineal figure (V), and deficient by a parallelogram (M I), similar to a given parallelogram (T); but the given rectilineal figure (V) must not be greater than the parallelogram (A F) applied to half of the given line, having its defect (E D) similar to the given parallelogram (T).

Given.	Sought.
I. The straight line A B, *& the pgr* T. *II. The rectilineal figure* V, *not* > *pgr*. ED, *similar to* T, *applied to* AE, *half of* AB.	*The construction of a pgr*. AG, *applied to* AB, *which may be* = *to* V, *& deficient by a pgr*. M I *similar to* T.

Resolution.

1. Divide A B into two equal parts in E. *P.10. B.1.*
2. Upon E B describe a pgr. E D, similar to the pgr. T, & similarly situated. *P.18. B.6.*
3. Complete the pgr. A D. *P.21. B.1.*
   The pgr. A F will be either = or > V; since it cannot be < V, by the determination.

CASE I. If A F be = V.

There has been applied to AB, a pgr. AF = to the rectilineal V, & deficient by a pgr. E D similar to the pgr. T.

CASE II. If A F be > V, & consequently E D > V, A F being = E D. *P.36. B.1.*

4. Describe a pgr. X similar to the pgr. T (or to the pgr. ED) (*Ref.* 2), & similarly situated, & equal to the excess of E D, or its equal AF, above V (i. e. make X = ED—V), & let R S, F D & R P, F E be the homologous sides. *P.45. B.1.*
   And because X is simil. to ED & < ED; (E D being = V + X).
   The sides R S, R P are < their homologous sides F D, F E.
5. Make then F N = R S, & F K = R P. *P. 3. B.1.*
6. And complete the pgr. N K. *P.31. B.1.*

DEMONSTRATION.

THE pgr. K N, being equal & similar to the pgr. X *(Ref.4,5.&6)*; which is itself similar to the pgr. E D *(Ref. 4)*.

1. The pgr. K N is similar to the pgr E D. *P.21. B.6.*
2. Wherefore those two pgrs. K N, E D, are about the same diagonal. *P.26. B.6.*
Draw this diagonal F G B, & complete the description of the figure. Since then the pgr. M I, is also about the same diagonal F B.
3. It is similar to the pgr. E D. *P.24. B.6.*
4. Consequently similar to the pgr. T *(Ref. 2)*. *P.21. B.6.*
But the pgr. D G being = to the pgr. E G *(P. 43. B. 1)*, if the common pgr. M I be added on both sides.
5. The pgr M D will be = to the pgr. E I. *Ax.2. B.1.*
But the pgr. A K being also = to the pgr. E I (*P.* 36. *B.* 1).
6. The pgr. M D is = to the pgr. A K. *Ax.1. B.1.*
And adding to both sides the common pgr. E G.
7. The gnomon *a b c* will be = to the pgr. A G. *Ax.2. B.1.*
But the pgr. E D being = to the figures V & X taken together *(Ref. 4.)*, or to V & K N, since X is = K N *(Ref. 5. & 6)*; if K N be taken away from both sides.
8. The remaining gnomon *a b c* = V. *Ax.3. B.1.*
9. Consequently, the pgr. A G is = to V *(Arg. 7)*.
But pgr. A G has for defect pgr. M I, similar to pgr. T (*Arg.* 4).
10. Therefore, there has been applied to A B a pgr. A G = V, deficient by a pgr. M I, similar to the pgr. T. *D. 8. B.6.*

Which was to be done.

REMARK.

SEVERAL *Editors of* New Elements of Euclid *have left out this proposition & the following, as useless; because they were ignorant of their use. They are notwithstanding absolutely necessary for the analysis of the ancients, corresponding to the analitic resolution of equations of the second degree.*

This XXVIIIth proposition corresponds to the case, where the last term of the equation is positive.

For reducing the given space V *to an equiangular pgr.* T; *let* $V = nl$; *the ratio of the sides* Q P, P R *of the pgr.* X (*or* T), $m : n$; $AB = a$, $AM = x$ & $MB = a - x$. *Consequently, since the defect* M I, *should be similar to the pgr.* T *or to the pgr.* X.

$$QP : PR = BM : MG \quad (D. 1. B. 6).$$

$$m : \quad n = a - x : \frac{n}{m}(a - x).$$

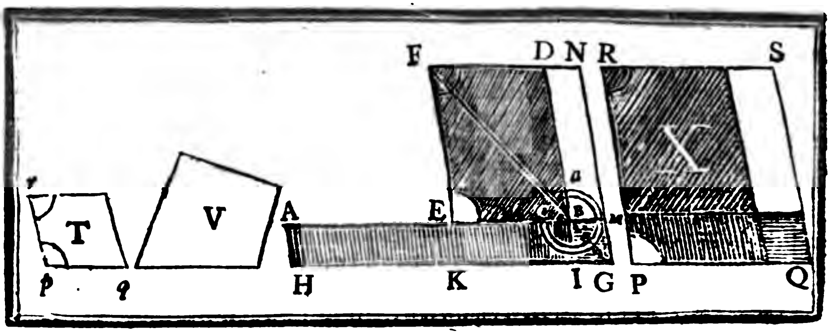
And because the pgr. G A ($= MA.\ MG$) *should be equal to the given space* V ($= nl$), *there results the following equation* (*P.*23. *B.*6).

$$\frac{n}{m}(a - x)\, x = V \text{ or } n\, l.$$

Which is reduced to $\frac{n}{m} xx - \frac{n}{m} a x + V = 0.$

Or substituting for V *its value, & multiplying by m & dividing by n.*

$$xx - ax + ml = 0.$$



PROPOSITION XXIX. *PROBLEM IX.*

TO a given ſtraight line (A B), to apply a parallelogram (A G), equal to a given rectilineal figure (V), exceeding by a parallelogram (M I), ſimilar to another given (T).

Given.	Sought.
I. The ſtraight line A B, *& the pgr.* T.	*The conſtruction of a pgr.* A G, *applied to* A B, *equal to the rectilineal figure* V, *&*
II. The rectilineal figure V.	*having for exceſs a pgr.* MI, *ſimilar to* T.

Reſolution.

1. Divide A B into two equal parts in E. *P.10. B.1.*
2. Upon E B, deſcribe a pgr. E D, ſimilar to the pgr. T, & ſimilarly ſituated. } *P.18. B.1.*
3. Deſcribe a pgr. X (or P S) = V + E D, ſimilar & ſimilarly ſituated to the pgr. T; & conſequently ſimilar to the pgr. ED (*Ref.2.P.21.B.6*); & let the ſides RS, FD; RP, FE be homologous.
4. Since X, (as = V + E D), is > E D; the ſide R S is > FD, & the ſide R P > F E; wherefore, having produced F D & F E, make F N = R S & F K = R P; & complete the pgr. F K G N, which will be equal & ſimilar to the pgr. X. *P.31. B.1.*

DEMONSTRATION.

THE pgr. K N being equal and ſimilar to the pgr. X, which is itſelf ſimilar to the pgr. E D (*Ref. 3*).

1. The pgr. K N is ſimilar to the pgr. E D. *P.21. B.6.*
2. Wherefore thoſe two pgrs K N, E D are about the ſame diagonal. *P.26. B.6.*
   Draw this diagonal F B G, & complete the deſcription of the figure.
   Since X is = to V + E D; & X = pgr. K N (*Ref. 3. & 4*).
3. The pgr. K N = V + E D. *Ax.1. B.1.*
   Therefore taking away from both ſides the common pgr. E D.
4. The remaining gnomon *a b c* is = to the rectilineal figure V. *Ax.3. B.1.*
   But becauſe A E = E B (*Ref. 1*).
5. The pgr. A K = the pgr. E I. *P.36. B.1.*
6. Conſequently, this pgr. A K is = to the pgr. N B. *P.43. B.1.*

Therefore adding to both sides the common pgr. M K.
7. There will result the pgr. A G = to the gnomon *a b c*. *Ax.2. B.1.*
But the gnomon *a b c* is = to the rectilineal figure V (*Arg.* 4).
8. Consequently, the pgr. A G is = to the rectilineal figure V. *Ax.1. B.1.*
Since then this pgr. A G has for excess the pgr. M I, similar to the pgr. E D (*P.* 24. *B.* 6.); & consequently similar to the pgr. T (*Ref.* 2. *P.* 21. *B.* 6).
9. There has been applied to A B, a pgr. A G = to the rectilineal figure V, having for excess a pgr. M I, similar to the pgr. T.

Which was to be done.

REMARK.

If as in the foregoing case A B *be made* = *a*, *the given square* V *(reduced to a pgr. equiangular to the pgr.* T*)* = *n l*; *the ratio of the sides* Q P, P R *of the pgr.* X *(which is the same as that of the sides of the pgr.* T*)* $m : n$; & A M = *x*, *consequently,* M B = $x - a$. *there will result an equation of the same kind.*

For since the defect M I *should be similar to the pgr.* T *or* X, *we will have as before the following proportion.*

$$QP : PR = MB : MG \; (D. 1. B. 6).$$

$$m : n = x - a : \frac{n}{m}(x - a).$$

And because the pgr. A G (= A M. M G) *should be equal to the given space* V (= *n l*), *there results the following equation,*

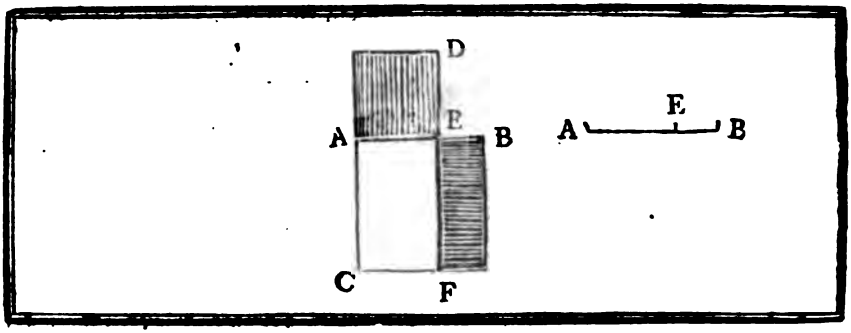
$$\frac{n}{m}(x - a)\,x = V \; (P. 23. B. 6).$$

which is reduced to $\frac{n}{m}xx - \frac{n}{m}ax - V = 0.$

And substituting for V *its value n l, then multiplying by m & dividing by n.*

$$xx - ax - ml = 0.$$

From whence it appears that the XXIX*th Prop. corresponds to the Case, in which the last term of the equation is negative.*



PROPOSITION XXX. PROBLEM X.

TO cut a given ſtraight line (A B) in extreme and mean ratio (in E).

Given. *The ſtraight line* A B.

Sought. *The point* E, *ſuch that* BA : AE = AE : BE

Reſolution.

1. Upon the ſtraight line A B deſcribe a ſquare B C. *P.*46. *B.*1.
2. Apply to the ſide C A, a pgr. C D = to the ſquare B C, whoſe exceſs A D is ſimilar to B C, which will conſequently be a ſquare. *P.*29. *B.*6.

DEMONSTRATION.

BECAUSE BC = CD *(Ref.* 2*)*; by taking away the common rgle. C E from each.

1. The remainder B F = A D. *Ax.*3. *B.*1.

But B F is alſo equiangular with A D *(P.* 15. *B.* 1*)*.

2. Therefore their ſides F E, E B, E D, A E about the equal angles, are reciprocally proportional, that is F E : ED = A E : E B. *P.*14. *B.*6.

But F E is = C A *(P.* 34. *B.* 1*)*, or = to B A, & E D = A E. *D.*30. *B.*1.

3. Wherefore, B A : A E = A E : E B. *P.*7. & 11. *B.*5.

But becauſe B A is > A E *(Ax.* 8. *B.* 1*)*.

4. The ſtraight line A E is > E B. *P.*14. *B.*5.
5. Conſequently, the ſtraight line A B is cut in extreme & mean ratio in E.

Which was to be done.

Otherwiſe.

Divide B A in E, ſo that the rect. A B. B E be = to the □ of A E. *P.*11. *B.*2.

DEMONSTRATION.

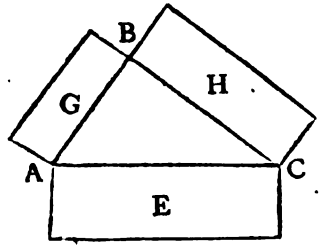
BECAUSE B A . B E is = to the □ of A E *(Ref.)*.

1. B A : A E = A E : B E. *P.*17. *B.*6.

And becauſe B A is > A E *(Ax.* 8 *B.* 1*)*.

2. The ſtraight line A E is > B E. *P.*14. *B.*5.
3. Conſequently, the ſtraight line AB is cut in extreme & mean ratio in E. *D.* 3. *B.*6.

Which was to be done.



PROPOSITION XXXI. THEOREM XXI.

IN every right angled triangle (A B C), the rectilineal figure (E) described upon the hypothenuse (A C) is equal to the sum of the similar and similarly described figures (G & H), upon the sides (A B, B C) containing the right angle.

Hypothesis.

I. A B C *is a rgle.* △ *in* B.
II. *The fig.* E *is described upon the hypoth.* A C *of this* △.
III. *And the figures* G & H *are similar to* E, & *similarly described upon the two other sides* A B, B C.

Thesis.

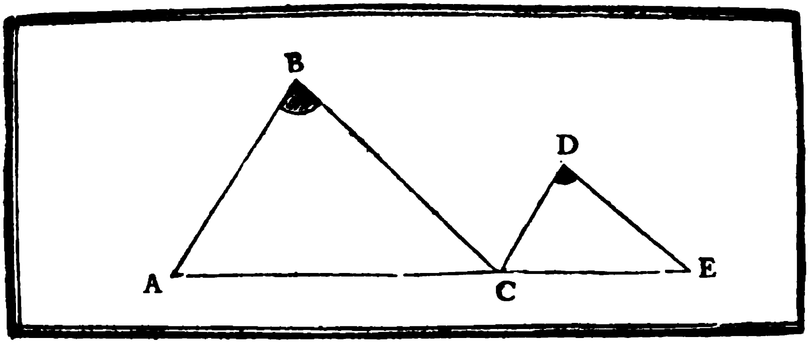
fig. E = *fig.* G + H.

Demonstration.

BECAUSE the figures E, G, H are similar, & similarly described upon the homologous sides A C, A B, B C *(Hyp. 3)*.

.	G : E = □ of A B : □ of A C.	P. 20. B. 6. Cor. 2.
And	H : E = □ of B C : □ of A C.	
.	Consequently, G + H : E = □ of A B + □ of B C : □ of A C.	P. 24. B. 5.
	But because the △ A B C is rgle. in B *(Hyp. 1.)*.	
.	The □ of A B + □ of B C is = to the □ of A C.	P. 47. B. 1.
.	Therefore, the figure E is = to the figures G + H.	P. 16. B. 5. Cor.

Which was to be demonstrated.



PROPOSITION XXXII. *THEOREM XXII.*

IF two triangles (A B C, C D E), which have two sides (A B, B C) of the one, proportional to two sides (C D, D E) of the other, be joined at one angle (C), so as to have their homologous sides (A B, C D, B C, D E) parallel to one another, the remaining sides (A C, C E) shall be in a straight line.

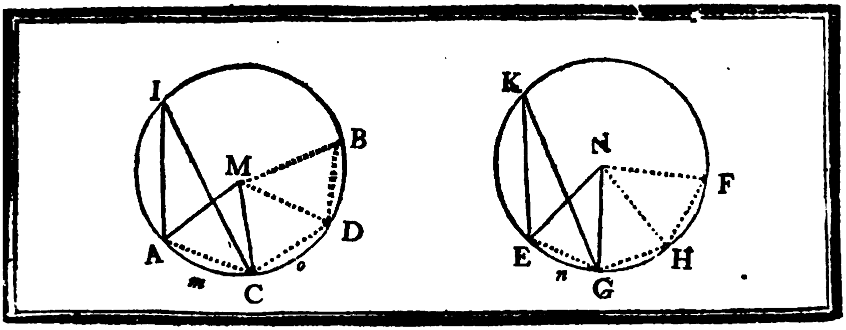
Hypothesis.	Thesis.
I. A B : B C = C D : D E.	*The remaining sides* A C, C E *of these* △
II. The △ A B C, C D E, *are joined in* C.	*are in a straight line* A E.
III. So that AB *is plle. to* CD, *&* BC *plle. to* D E.	

Demonstration.

BECAUSE the plles. A B, C D are cut by the straight line B C, & the plles. B C, D E by the straight line D C *(Hyp. 2)*.

1. The ∀ B is = to ∀ B C D & ∀ D is = to ∀ BCD. *P.29. B.1.*
2. Consequently, ∀ B is = to ∀ D. *Ax.1. B.1.*
   And besides A B : B C = C D : D E *(Hyp. 1)*.
3. The △ A B C, C D E are equiangular. *P. 6. B.6.*
4. Therefore, ∀ A is = to ∀ D C E, being opposite to the homologous sides B C, D E.
   Adding then to both sides ∀ B, or its = ∀ B C D *(Arg. 1)*, together with the common ∀ B C A.
5. The ∀ A + B + B C A will be = to the ∀ DCE + BCD + BCA. *Ax.2. B.1.*
   But the ∀ A + B + B C A are = to 2 ∟ *(P. 32. B. 1)*.
6. Consequently the ∀ D C E + B C D + B C A are also = to 2 ∟. *Ax.1. B.1.*
7. Wherefore the straight lines A C, C E are in the same straight line A E. *P.14. B.1.*

Which was to be demonstrated.



PROPOSITION XXXIII. THEOREM XXIII.

IN equal circles (A I B C, E K F G), angles, wether at the centres or circumferences (A M C, E N G or A I C, E K G), as alſo the ſectors (A M C *m*, E N G *n*) have the ſame ratio with the arches (A *m* C, E *n* G) on which they ſtand, have to one another.

Hypotheſis.

I. The ⊙AIBC,EKFG *are*=*to one another.*

II. The ∠ *at the centers* AMC,ENG *& the* ∠ *at the* ○AIC, EKG *ſtand upon the arches* A *m* C, E *n* G.

Theſis.

I. ∠AMC : ∠ENG = A*m*C : E*n*G.

II. ∠A I C : ∠EKG = A*m*C : E*n*G.

*III. Sect.*AMC*m*:*Sect.*ENG*n*=A*m*C:E*n*G.

Preparation.

1. Join the chords A C, E G. *Poſ.*1. *B.*1.
2. In the ○ A I B C, draw the chords C D, D B &c, each = to A C, & in the ○ E K F G a pareil number of cords G H, H F &c, each = to E G. *P.* 1. *B.*4
3. Draw M D, M B &c, alſo N H, N F &c. *Poſ.*1. *B.*1.

DEMONSTRATION.

BECAUSE on one ſide the cords A C, C D, D B, & on the other the cords E G, G H, H F are = to one another (*Prep.* 2).

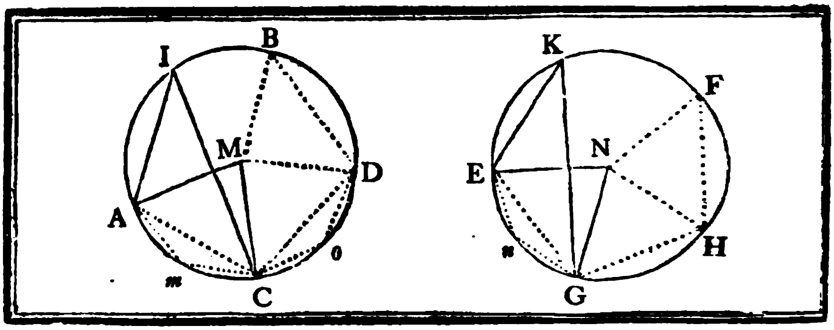
1. The arches A *m* C, C *o* D, D B are all equal on the one ſide, as the arches E *n* G, G H, H F are on the other. *P.*28. *B.*3.
2. Conſequently, the ∠ A M C, C M D, D M B &c, & E N G, G N H, H N F &c, are alſo = to one another, on one ſide & the other.
3. Wherefore, the ∠ A M B & the arch A C D B, are equimult. of the ∠ A M C & of the arch A *m* C.
4. Likewiſe, ∠ E N F & the arch EGHF are equimult. of ∠ E N G, & of the arch E *n* G.
   But becauſe the ⊙ A I B C, E K F G are equal (*Hyp.* 1).
   According as the arch A C D B is >, = or < the arch E G H F; ∠ A M B is alſo >, = or < ∠ E N F. *P.*27. *B.*3.
5. Wherefore, ∠ A M C : ∠ E N G = A *m* C : E *n* G. *D.* 5. *B.*5.

Which was to be demonſtrated. I.

Moreover, ∠ A M C being double of ∠ A I C, & ∠ E N G double of ∠ E K G (*P.* 20. *B.* 3).

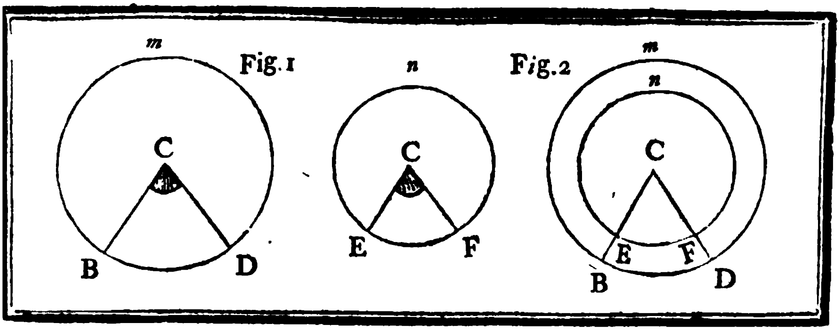
6. It follows that ∠ A M C : ∠ E N G = ∠ A I C : ∠ E K G. *P.*15. *B.*5.
7. Conſequently, ∠ A I C : ∠ E K G = A *m* C : E *n* G. *P.*11. *B.*5.

Which was to be demonſtrated. II.



PREP. 4. In the arches A C, C D, take the points *m* & *o*, & join A *m*, C *m*; C *o*, D *o* &c. *Pos.* 1. *B.* 1.
Since then the two sides A M, M C are = to the two sides C M, M D (*D.* 15. *B.* 1), & the ∀ A M C, C M D are equal (*Arg.* 2).
8. The base AC is = to the base CD, & the △AMC = to the △CMD. *P.* 4. *B.* 1.
Moreover, because the arch A *m* C is = to the arch C *o* D (*Arg.* 1).
9. The complement A I B D C of the first is = to the complement C A I B D of the second. *Ax* 3. *B* 1.
10. Wherefore ∀ A *m* C is = to ∀ C *o* D. *P.* 27. *B.* 3.
11. Therefore the segment A *m* C is similar to the segment C *o* D. *Ax.* 2. *B.* 3.
Besides they are subtended by equal cords (*Arg.* 8).
12. Consequently, the segment A *m* C is = to the segment C *o* D. *P.* 24. *B.* 3.
But since the △ A M C is also = to the △ C M D (*Arg.* 8).
13. The sector A M C *m* is = to the sector C M D *o*. *Ax.* 2. *B.* 1.
Likewise, the sector D M B is equal to each of the two foregoing A M C *m*, C M D *o*.
14. Therefore the sectors A M C, C M D, D M B are = to one another.
15. It is demonstrated after the same manner, that the sectors E N G, G N H, H N F are = to one another.
16. Wherefore, the sect. A M B D C, & the arch A C D B are equimult. of the sect. A M C *m*, & of the arch A *m* C, the sect. E N F H G, & the arch E G H F are equimult. of the sect. E N G *n*, & of the arch E *n* G.
But because the ⊙ A I B C, E K F G are equal (*Hyp.* 1).
If the arch A C D B be = to the arch E G H F, the sect. A M B D C is also = to the sect. E N F H G, as is proved by the reasoning employed in this third part of the demonstration to arg. 12 inclusively.
And, if the arch A C D B be > the arch E G H F, the sect. A M B D C is also > the sect. E N F H G, & if less, less.
Since then there are four magnitudes, the two arches A *m* C, E *n* G, & the two sect. A M C *m*, E N G *n*. And of the arch A *m* C, & sect. A M C *m*, the arch A C D B & sect. A M B D C are any equimult. whatever; & of the arch E *n* G, & sector E N G *n*, the arch E G H F, & the sect. E N F H G are any equimult. whatever.
And it has been proved that, if arch A C D B be >, = or < the arch E G H F, sect. A M B D C is also >, = or < the sect. E N F H G.
17. It follows, that sect. A M C : sect. E N G = A *m* C : E *n* G. *D.* 5. *B.* 5.

Which was to be demonstrated. III.



COROLLARY I.

THE angle at the center, is to four right angles, as the arch upon which it stands, is to the circumference.

For *(Fig. 1)*, $\angle BCD : \llcorner = BD$: to a quadrant of the $\odot$.

Wherefore, quadrupling the consequents.

$\angle BCD : 4\llcorner = BD : \odot$. *P.15. B.5.*

CORALLARY II.

THE arches E F, B D of unequal circles, are similar, if they subtend equal angles C & C, either at their centers, or at their $\odot$ *(Fig. 2)*.

For $EF : \odot EnF = \angle ECF : 4\llcorner$. } *(Cor. 1.)*

But $\angle BCD$ or $\angle ECF : 4\llcorner = BD : \odot BmD$. }

Consequently, $EF : \odot EnF = BD : \odot BmD$. *P.11. B.5.*

Therefore, the arches E F, B D are similar.

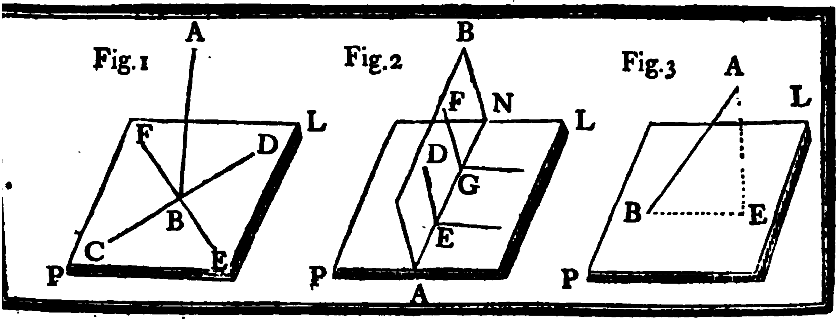
CORALLARY III.

TWO rays C B, C D cut off from concentric circumferences similar arches E F, B D *(Fig. 2)*.

REMARK.

I*T is in consequence of the proportionality established in* Cor. 1. *that an arch of a* ircle (B D) *is called the* MEASURE *of its correspondent angle* (B C D)*; that is of the* ngle at the center, subtended by this arch; the circumference of a circle being the nly curve, whose arches, increase or diminish in the ratio of the correspondent an*les, about the same point.*

The whole circle is conceived to be divided into 360 *equal parts, which are* alled DEGREES; *and each of these degrees into* 60 *equal parts, called* INUTES; *and each minute into* 60 *equal parts, called* SECONDS &c. *in conse*uence of this hypothesis, & the correspondence established between the arches, & the ngles, we are obliged to conceive all the angles about a point in the same plane* (that s the sum of 4 $\llcorner$), *as divided into* 360 *equal parts, in such a manner, that the* ngle of a degree is no more than the 360th part of 4 $\llcorner$, or the 90th of a $\llcorner$, & con*quently, of an amplitude to be subtended by the* 360*th part of the circumference.*



DEFINITIONS.

I.

A *SOLID* is that which hath length, breadth and thickness.

II.

That which bounds a Solid is a superficies.

III.

A straight line (A B) *is perpendicular to a plane* (P L) (*Fig.* 1), if it be perpendicular to all the lines (C D & F E), meeting it in this plane; *that is, the line* (A B) *will be perpendicular to the plane* (P L), *if it be perpendicular to the lines* (C D & F E) *which being drawn in the plane* (PL) *pass through the point* (B), *so that the angles* (A B C, A B D, A B E & A B F) *are right angles.*

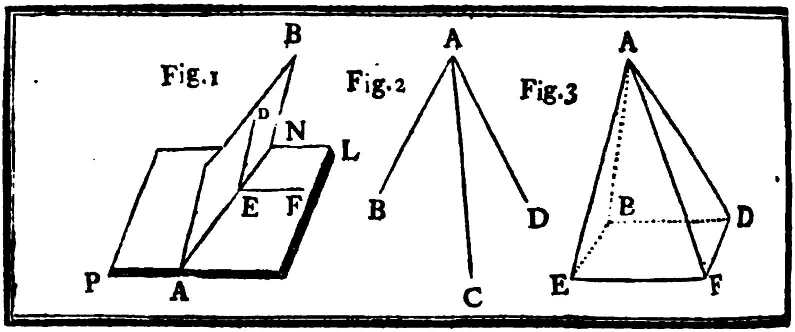
IV.

A plane (A B) (*Fig.* 2.) *is perpendicular to a plane* (P L), if the lines (D E & F G) drawn in one of the planes (as in A B) perpendicularly to the common section (A N) of the planes, are also perpendicular to the other plane (P L).

The common section of two planes is the line which is in the two planes: as the line (A N), *which is not only in the plane* (A B), *but also in the plane* (P L); *therefore if the lines* D E & F G *drawn perpendicular to* A N *in the plane* A B *are also perpendicular to the plane* P L; *the plane* A B *will be perpendicular to the plane* P L.

V.

The inclination of a straight line (A B) *to a plane,* (*Fig.* 3.) is the acute angle (A B E), contained by the straight line (A B), and another (B E) drawn from the point (B), in which A B meets the plane (P L), to the point (E) in which a perpendicular (A E) to the plane (P L) drawn from any point (A) of the line (A B) above the plane, meets the same plane.



DEFINITIONS.

VI.

THE *inclination of a plane* (A B) (*Fig.* 1) *to a plane* (P L); is the acute angle (D E F) contained by two ſtraight lines (E D & E F) drawn in each of the planes, (that is D E in the plane A B & E F in the plane P L) from a ſame point (E), perpendicular to their common ſection (A N).

VII.

Two planes are ſaid to have the ſame or a like inclination to one another, which two other planes have, when their angles of inclination are equal.

VIII.

Parallel planes are ſuch which do not meet one another tho' produced.

IX.

Similar ſolid figures are thoſe which are contained by the ſame number of ſurfaces, ſimilar and homologous.

X.

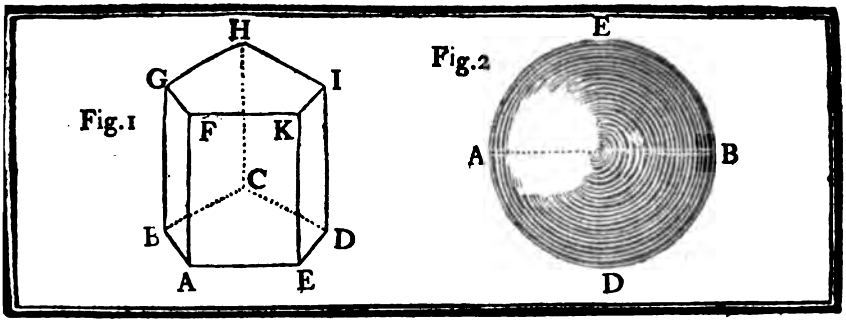
Equal & ſimilar Solids are thoſe which are contained by the ſame number of equal, ſimilar and homologous ſurfaces.

XI.

A ſolid Angle (A) is that which is made by the meeting of more than two plane angles (B A C, C A D & B A D), which are not in the ſame plane, in one point (A).

XII.

A Pyramid (E B A D F) (*Fig.* 3) is a ſolid contained by more than two triangular planes (B A D, B A E &c.) having the ſame vertex (A), and whoſe baſes (viz. the lines E B, B D &c.) are in the ſame plane (E B D F).



DEFINITIONS.

XIII.

A *Prism* is a solid figure (AHE) *(Fig.* 1.*)* contained by plane figures, of which two that are opposite (viz. GHIKF & BCDA*)* are equal similar, and parallel to one another; and the other sides (as GA, AK, KD, &c.) are parallelograms.

If the opposite parallel planes be triangles, the prism is called a triangular one, (and it is only of those prisms that Euclid treats in the XIth and XIIth Book), *if the opposite planes are polygons, they are called polygon prisms.*

XIV.

A Sphere is a solid figure (AEBD) *(Fig.* 2.*)* whose surface is described by the revolution of a semicircle (AEB) about its diameter, which remains unmoved.

XV.

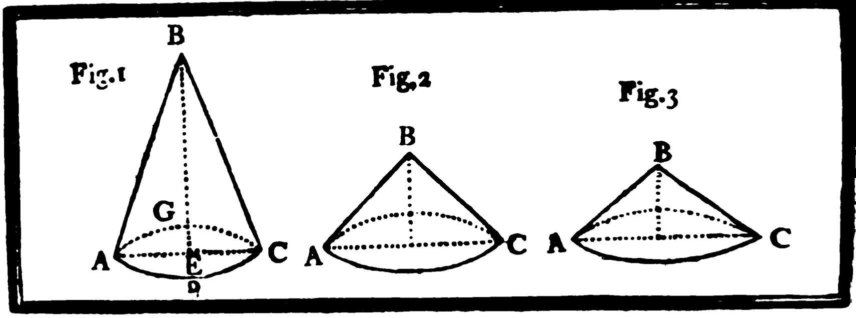
The Axis of a Sphere is the fixed diameter (AB) about which the semicircle revolves whilst it describes the superficies of the sphere.

XVI.

The Center of a Sphere is the same with that of the semicircle which described its superficies.

XVII.

The Diameter of a Sphere is any straight line which passes thro' the center, and is terminated both ways by the superficies of the sphere.



DEFINITIONS.

XVIII.

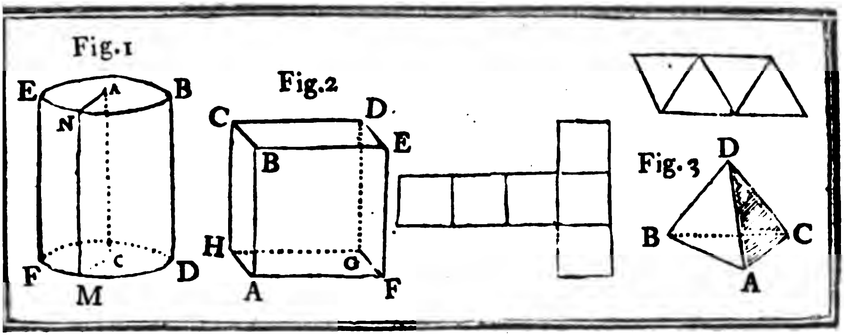
A *Cone* is a ſolid figure (A B C D) (*Fig.* 1, 2, & 3.) deſcribed by the revolution of a right angled triangle (A B E), about one of the ſides (B E) containing the right angle, which ſide remains fixed.
If the fixed ſide (B E) of the triangle (A B E) (*Fig.* 2.) be equal to the other ſide (A E) containing the right angle, the cone is called a right angled cone; if (B E) be leſs than (A E) (*Fig.* 3.) an obtuſe angled, and if (B E) be greater than (A E) (*Fig.* 1.) an acute angled cone.

XIX.

The Axis of a Cone is the fixed ſtraight line (B E) about which the triangle (A B E) revolved whilſt it deſcribed the ſuperficies of the cone.

XX.

The Baſe of a Cone is the circle (A G C D) (*Fig.* 1.) deſcribed by that ſide (B E) containing the right angle, which revolves.



DEFINITIONS.

XXI.

A *Cylinder* is a ſolid figure (E B D F) *(Fig. 1.)* deſcribed by the revolution of a right angled parallelogram (A N M C) about one of its ſides (A C) which remains fixed.

XXII.

The Axis of a Cylinder is the fixed ſtraight line (A C) about which the parallelogram revolved, whilſt it deſcribed the ſuperficies of the cylinder.

XXIII.

The Baſes of a Cylinder (viz. E N B, & F M D) are the circles deſcribed by he two oppoſite ſides (N A, M C) of the parallelogram, revolving about the oints A & C.

XXIV.

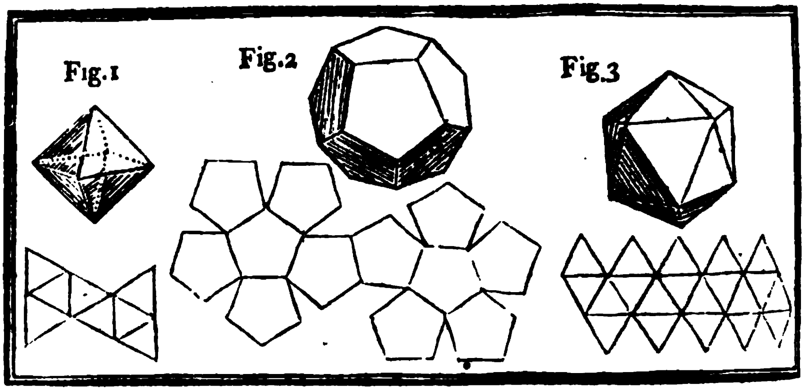
Similar Cones and Cylinders are thoſe which have their axes and the diameters of their Baſes proportionals.

XXV.

A Cube or Exahedron (Fig. 2.) is a ſolid figure contained by ſix equal ſquares.

XXVI.

A Tetrahedron is a pyramid (B D C A) *(Fig. 3.)* contained by four equal and equilateral triangles (viz. △ B D C, B A D, A D C & B A C).



DEFINITIONS.

XXVII.

AN *Octahedron (Fig.* 1.*)* is a solid figure contained by eight equal and equilateral triangles.

XXVIII.

*A Dodecahedron (Fig.*2.*)* is a solid figure contained by twelve equal pentagons which are equilateral and equiangular.

XXIX.

An Icosahedron (Fig. 3.*)* is a solid figure contained by twenty equal and equilateral triangles.

XXX.

A Parallelepiped is a solid figure contained by six quadrilateral figures whereof every opposite two are parallel.

XXXI.

A Solid is said to be inscribed in a Solid, when all the angles of the inscribed solid touch the angles, the sides, or the planes of the solid in which it is inscribed.

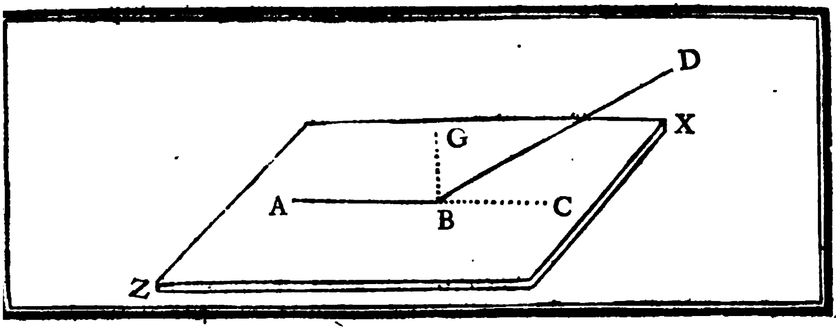
XXXII.

A Solid is said to be circumscribed about a Solid, when the angles, the sides, or the planes of the circumscribed solid touch all the angles of the inscribed solid.

EXPLICATION of the SIGNS.

∽ .... Similar.

▱ .... Parallelepiped.



PROPOSITION I. THEOREM I.

ONE part (A B) of a straight line cannot be in a plane (Z X); and nother part above it.

Hypothesis.
A B *is a part of a straight line situated in the plane* Z X.

Thesis.
Another part of this straight line (as B C*) will be in the same plane* Z X.

Demonstration.

If not
It will be above the plane as B D is.

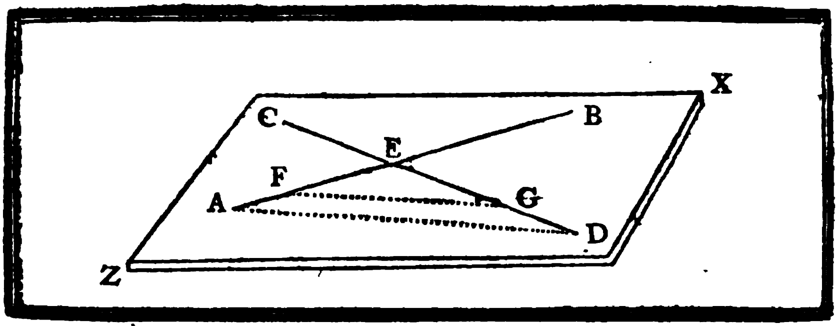
Preparation.

1. At the point B in A B erect in the plane Z X the ⊥ G B. } *P.*11. *B.*1.
2. At the point B in B G erect in the plane Z X the ⊥ B C. }

BECAUSE ∀ A B G is a ∟, likewise ∀ G B C, & they meet in the same point B.

1. The lines A B & B C are in the same straight line A C. *P.*14. *B.*1.
But the line B D is a part of the straight line above the plane *(Sup.)*.
2. Therefore the lines B D & B C have a common segment A B.
3. Consequently, ∀ D B G = ∀ G B A = G B C, *that is*, the part = to the whole. *Ax.*10.*B.*1.
4. Which is impossible. *Ax.* 8.*B.*1.
5. Therefore, B D cannot be a part of the straight line A B *(Arg.* 1*)*.
And as the same demonstration may be applied to any other part of B C.
6. It follows, that all the parts of a straight line are in the same plane.

Which was to be demonstrated.



PROPOSITION II. THEOREM II.

TWO straight lines which cut one another in (E), are in one plane (ZX) and three straight lines which constitute a triangle (E A D) are in the same plane (Z X).

Hypothesis.	Thesis.
I. A B & C D *cut one another in* E.	*I.* AB & CD, *are in the same plane.*
II. E A D *is a* △.	*II. The whole* △ E A D *is in the plane* Z X.

Demonstration.

If not,

The lines A B & C D are not in the same plane,

Likewise a part of the △ E A D, as A F G D.

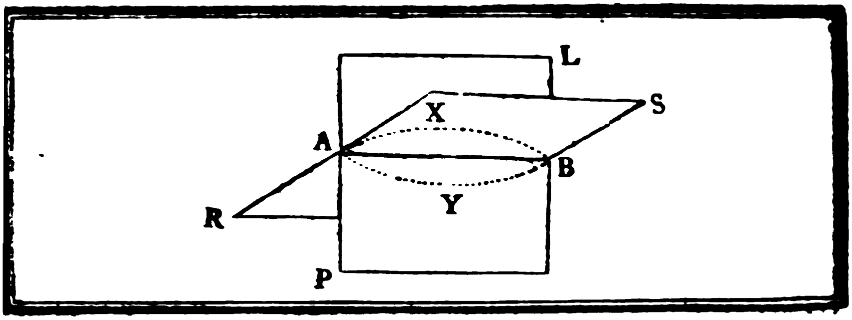
Preparation.

Draw G F.

BECAUSE the part A F G D of the △ E A D is not in one plane (Z X) with E F G (*Sup.*).

1. It follows, that the parts G D, C G of the line CD are in different planes, & the parts A F, F B of the straight line A B, are in different planes, as also A F G D & F E G.
2. Which is impossible. *P. 1. B. 11.*
3. Since then the parts of the two lines & of the △ can not be in different planes.
4. They must consequently be in the same plane.

Which was to be demonstrated, I. & II.



PROPOSITION III. *THEOREM III.*

IF two planes (R S & P L) cut one another, their common ſection is a ſtraight line (A B).

Hypotheſis.	Theſis.
R S & P L *are two planes which cut one another.*	*Their common ſection* A B, *is a ſtraight line.*

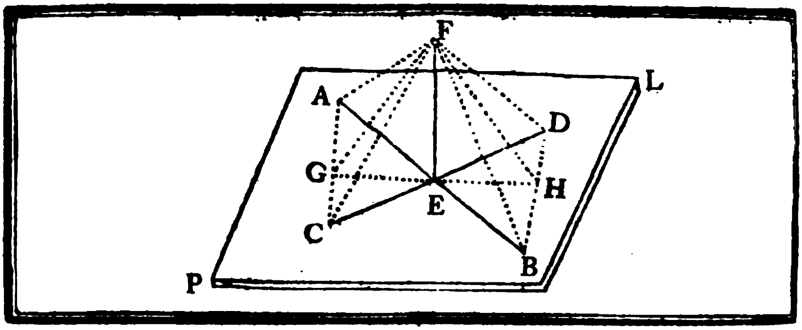
Demonstration.

If it be not,
The ſection will be two ſtraight lines.
As A X B for the plane R S ; & A Y B for the plane P L.

BECAUSE the ſtraight lines A X B & A Y B have the ſame extremities A & B.

1. Theſe two ſtraight lines A X B & A Y B include a ſpace A X B Y.
2. Which is impoſſible. *Ax.*12. *B.*1.
3. Conſequently, the ſection of the planes P L & R S can not be two ſtraight lines A X B & A Y B.
4. Therefore their common ſection, is a ſtraight line A B.

Which was to be demonſtrated.



PROPOSITION IV. *THEOREM IV.*

IF two ſtraight lines (A B & C D) interſect each other, and at the point (E) of their interſection a perpendicular (E F) be erected upon thoſe lines (A B & C D): it will be alſo perpendicular to the plane (P L) which paſſes through thoſe lines (A B & C D).

Hypotheſis.	Theſis.
I. A B *&* C D *are ſtraight lines ſituated in the plane* P L. *II. They interſect each other in* E. *III.* E F *is* ⊥ *to thoſe lines at the point* E.	E F *is* ⊥ *to the plane* P L.

Preparation.

1. Take E C at will, & make E B, E D & A E each equal to E C.
2. Join the points A & C, alſo B & D.
3. Thro' the point E in the ſame plane P L, draw the ſtraight line G H, terminated by the ſtraight lines A C & B D, at the points G & H.
4. Draw A F, G F, C F, D F, H F & B F.

DEMONSTRATION.

THE △ A E F, C E F, B E F, & D E F have the ſide E F common. The ſides A E, C E, B E, & D E eqnal *(Prep.* 1*)* & the adjacent ∀ A E F, C E F, B E F, & D E F equal *(Hyp.* 3*)*.

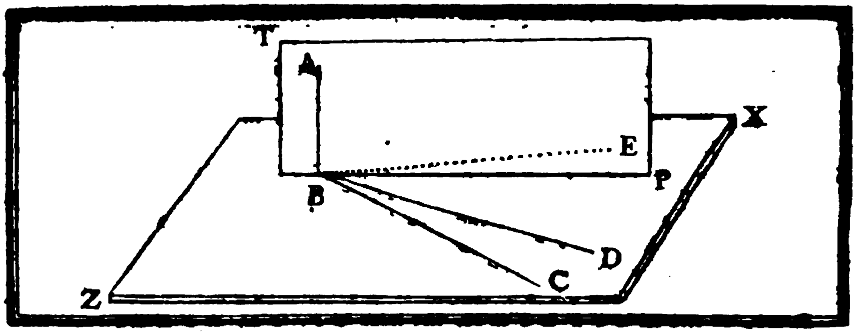
1\. Conſequently the baſes A F, C F, B F, & D F are equal. *P.* 4. *B.*1.
In the △ A E C & D E B, the ſides A E, C E, E D & E B are = *(Prep.* 1.*)* & the ∀ A E C & D E B alſo equal. *P.*15. *B.*1.

2\. Therefore, A C = B D.
3\. And ∀ E A C = ∀ E B D. } *P.* 4. *B.*1.

The △ G A E & E B H have ∀ A E G = ∀ H E B. *P.*15. *B.*1.
∀ E A G = ∀ E B H *(Arg.* 3.*)* & A E = E B *(Prep.* 1*)*.

4. Consequently, the sides G A & G E are = to the sides H B & E H. *P.26. B.1.*
In the △ A F C & F D B, the three sides A F, F C & A C of the first are = to the three sides F B, F D & D B of the second *(Arg. 1, & 2).*
5. Therefore, the three ∀ of the △ A F C are = to the three ∀ of the △ F D B each to each, *that is* ∀ F A G = ∀ F B H, &c. *P. 8. B.1.*
The △ G A F & H B F have the two sides A F & A G = to the two sides F B & B H *(Arg. 1. & 4).*
Moreover, ∀ F A G = ∀ F B H *(Arg. 5).*
6. Therefore, G F = F H. *P. 4. B.1.*
Infine, in the △ G F F & F E H, the sides G F, G E, & F E are = to the sides F H, E H, & E F *(Arg. 4. & 6).*
7. Consequently, the three ∀ of the △ G F E are = to the three ∀ of the △ F E H, each to each, *that is* ∀ F E G = ∀ F E H, &c. *P. 8. B.1.*
But those ∀ F E G & F E H are formed by the straight line E F falling upon G H (because G E & E H are in the same straight line) *(Prep. 3).*
8. Therefore, those ∀ F E G & F E H are ∟, & F E ⊥ upon G H. { *P.13. B.1.* *D.10. B.1.*
But H G is in the same plane, with the lines A B & C D *(Prep. 3).*
And E F is ⊥ upon those lines *(Hyp. 3).*
9. Consequently, E F is ⊥ upon the same plane P L. *D. 3. B.11.*

Which was to be demonstrated.



PROPOSITION V. *THEOREM V.*

IF three straight lines (B C, B D, & B E) meet all in one point (B), And a straight line (A B) is perpendicular to each of them in that point; these three straight lines (B C, B D, & B E) are in one and the same plane (Z X).

Hypothesis.	Thesis.
I. B C, B D, & B E *meet in* B.	B C, B D, & B E *are in the same plane* Z X.
II. A B *is* ⊥ *to those lines.*	

DEMONSTRATION.

If not,
One of those three as B E is in a different plane.

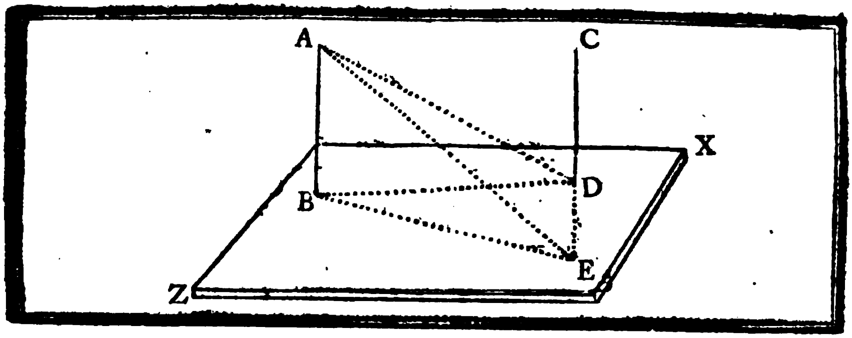
Preparation.

Let a plane T P pass thro' the ⊥ A B & the line B E.

BECAUSE TP & ZX are different planes which meet in B.

1. They will cut each other when produced, & their common section will be a straight line B P, common to the two planes. *P.* 3. *B.*11.
   But A B is ⊥ to B D & B C (*Hyp.* 11).
2. Consequently, A B will be also ⊥ to the plane Z X, in which those lines are. *P.* 4. *B.*11.
3. Therefore, A B is ⊥ to B P & ∀ A B P a ∟ (*Arg.* 1).
   But ∀ A B E is a ∟ (*Hyp.* 11).
   And B E is in the same plane with A B & B P (*Prep.* & *Arg.* 1).
4. Consequently, ∀ A B E = ∀ A B P, *that is,* the part = the whole.
5. Which is impossible. *Ax.*8. *B.* 1.
6. Therefore, B E is not in a different plane from that in which B D & B C are.
7. Consequently, those three lines are in the same plane Z X.

Which was to be demonstrated.



PROPOSITION VI. THEOREM VI.

IF two straight lines (A B & C D) be perpendicular to a plane (Z X), they shall be parallel to one another.

Hypothesis.	Thesis.
A B & C D are ⊥ to the plane Z X.	A B & C D are parallel.

Preparation.

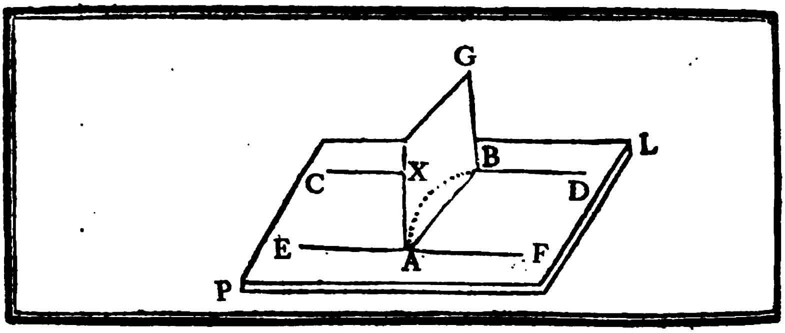
1. Join the points B & D in the plane Z X.
2. At the point D in B D in this same plane, erect the ⊥ D E. *P.* 11. *B.* 1.
3. Make D E = A B. *P.* 3. *B.* 1.
4. Draw A D, A E, & B E.

Demonstration.

BECAUSE in the △ A B D & B D E, the side D E is = A B (*Prep.* 3.), B D is common to the two △, & the ∀ A B D & B D E are ∟ (*Hyp. prep.* 2. & *D.* 3. 11.).

1. The side A D is = B E. *P.* 4. *B.* 1.
   In the △ A B E & A D E, the side A E is common, A B is = D E, & B E = A D (*Prep.* 3. & *Arg.* 1.)
2. Consequently, ∀ A B E is = ∀ A D E. *P.* 8. *B.* 1.
   But ∀ A B E is a ∟. *D.* 3. *B.* 11.
3. Therefore, ∀ A D E is also a ∟. *Ax.* 1. *B.* 1.
   But ∀ C D E is a ∟. *D.* 3. *B.* 11.
4. Consequently, D E is ⊥ to C D, D A & D B (*Hyp. prep.* 2. & *Arg.* 3).
5. Therefore, those lines C D, D A & D B are in the same plane, *that is* C D is in the plane which passes thro' D A & D B. *P.* 5. *B.* 11.
6. Likewise A B is also in the same plane which passes thro' D A & D B. *P.* 2. *B.* 11.
7. Therefore, A B & C D are in the same plane.
   But the interior ∀ A B D & B D C are ∟ (*Hyp.*)
8. Consequently, A B is parallel to C D. *P.* 28. *B.* 1.

Which was to be demonstrated.



PROPOSITION VII. *THEOREM VII.*

IF two points (A & B) in two parallels (D C & F E) be joined by a ſtraight line (A B); it will be in the ſame plane *(*P L*)* with the parallels.

Hypotheſis.	Theſis.
I. A *&* B *are two points taken at will in the parallels* E F *&* C D.	A B *is in the ſame plane* P L *with the plles.* C D *&* E F.
II. A B *is a ſtraight line which joins thoſe points.*	

DEMONSTRATION.

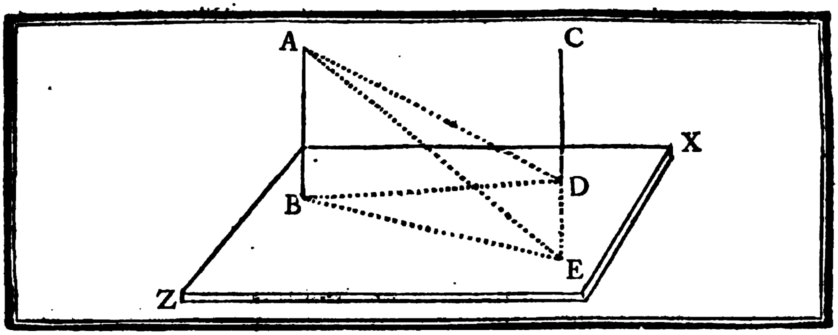
If not,

It will be in a different plane A G, as the line A X B is.

BECAUSE A X B is in the plane A G, different from the plane P L, & its extremities A & B are in the lines C D & E F, ſituated in the plane P L.

1. The line A X B will be common to the two planes, *that is*, A X B is the common ſection of the two planes A G & P L. *P.* 3. B. 11.
But A B is alſo a ſtraight line having the ſame extremities A & B *(Hyp.* 11*)*.
3. Which is impoſſible. *Ax.* 12. B. 1.
4. Wherefore, the ſtraight line (A B) which joins the points A & B, is not in a plane A G different from that in which the parallels C D & E F are.
5. Therefore, A B is in the ſame plane P L with the plles. C D & E F.

Which was to be demonſtrated.



PROPOSITION VIII. *THEOREM VIII.*

IF two ſtraight lines (A B & C D) be parallel, and one of them (as A B) is perpendicular to the plane (Z X); the other C D ſhall be perpendicular to the ſame plane.

Hypotheſis.	Theſis.
I. A B & C D *are plles.*	C D *is* ⊥ *to the plane* Z X.
II. A B *is* ⊥ *to the plane* Z X.	

Preparation.

1. Join the points B & D in the plane Z X. *Poſ.* 1. *B.* 1.
2. At the point D in B D, erect in the plane Z X the ⊥ D E. *P.* 12. *B.* 1.
3. Make D E = A B. *P.* 3. *B.* 4.
4. Draw A D, A E, & B E. *Poſ.* 1. *B.* 1.

DEMONSTRATION.

BECAUSE B D is in the plane X Z, & A B is ⊥ to this plane *(Hyp.* 11).

1. ∀ A B D is a ∟. *D.* 3. *B.* 4.
2. Conſequently, ∀ B D C is alſo a ∟. *P.* 29. *B.* 1.

But ∀ B D E is a ∟, D E is = A B *(Prep.* 2. & 3.) & B D being common to the two △ A B D & B D E.

3. The baſe A D is = to the baſe B E. *P.* 4. *B.* 1.

In the two △ A D E & A B E, A B is = D E *(Prep.* 3.) A D = B E *(Arg.* 3.) & A E common.

4. Conſequently, ∀ A B E = ∀ A D E. *P.* 8. *B.* 1.

But ∀ A B E is a ∟. *D.* 3. *B.* 1.

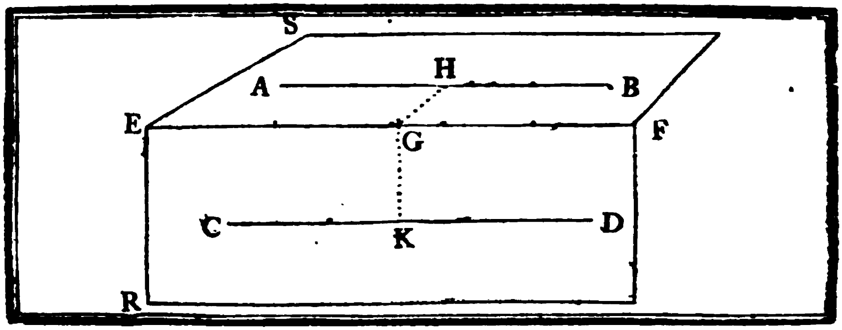
5. Therefore, ∀ A D E is alſo a ∟. *Ax.* 1. *B.* 1.
6. Conſequently, D E is ⊥ to B D & A D (*Prep.* 2. & *Arg.* 5).
7. Wherefore, D E is alſo ⊥ to the plane paſſing thro' thoſe lines B D & A D. *P.* 4. *B.* 1.

But A D joins two points A & D taken in A B & C D which are parallel (*Hyp.* 1).

8. Therefore C D is in the ſame plane with A B & A D. *P.* 7. *B.* 11.
9. Conſequently, D E is ⊥ to D C, or D C is ⊥ to D E, *D.* 3. *B.* 11.

Since then C D is ⊥ to D B & E D *(Arg.* 2. & 9).

10. C D will be alſo ⊥ to the plane paſſing thro' thoſe lines (that is) to the plane Z X. *P.* 4. *B.* 11.



PROPOSITION. IX. *THEOREM IX.*

THE lines (A B & CD) which are each of them parallel to the same straight line (E F) though situated in different planes (S F & R F) are parallel to one another.

Hypothesis.	Thesis.
I. A B *is in the plane* S F, *&* C D *in the plane* R F.	A B *is plle. to* C D.
II. A B *&* C D *are each plle. to* E F.	

Preparation.

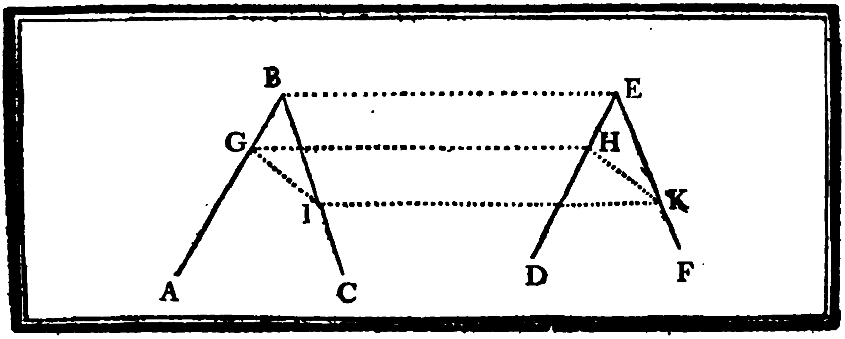
1. From the point H of the line A B in the plane F S let fall a ⊥ H G upon E F.
2. From the point G in the plane R F let fall the ⊥ G K upon C D. *P.* 11. *B.* 1.

Demonstration.

BECAUSE E G or E F is ⊥ to G H & G K *(Prep.* 1. *&* 2*)*.

1. E G will be ⊥ to the plane which passes thro' those lines. *P.* 4. *B.*11
   But A B is plle. to E F *(Hyp.* 2*)*.
2. Therefore, A B is ⊥ to the plane which passes thro' those lines H G & G K. *P.* 8. *B.*11.
3. In like manner, C D is also ⊥ to this same plane.
   Therefore, the lines A B & C D being ⊥ to the same plane *(Arg.* 2. *&* 3*)*.
4. They are plle. to one another. *P.* 6. *B.*11.

Which was to be demonstrated.



PROPOSITION X. THEOREM X.

If two straight lines (A B & B C) which meet one another (in B) be parallel to two others (D E & E F) which meet one another in (E); and are not in the same plane with the first two; the first two and the other two shall contain equal angles (A B C & D E F).

Hypothesis.	Thesis.
A B & C D *meet one another in* B, *in a different plane from that in which* D E & E F *are, which also meet one another in* E.	∀ A B C *is* = ∀ D E F.

Preparation.

1. Cut off at will from the straight lines A B & B C the parts B G & B I. *P.* 3. *B.* 1.
2. Make H E = B G, & E K = B I.
3. Join the points B E, G H, G I, H K & I K. *Pos.*1. *B.* 1.

Demonstration.

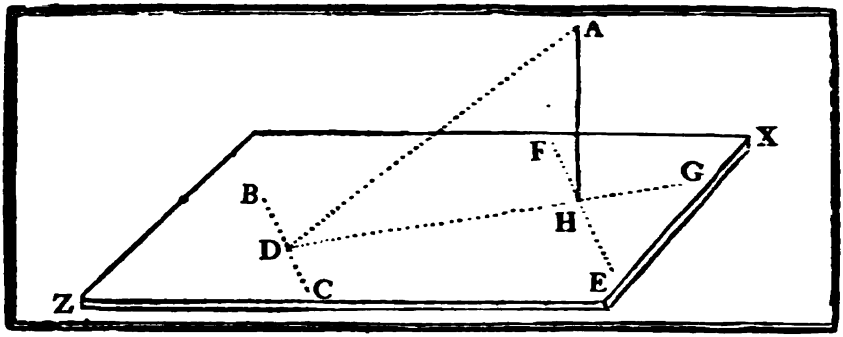
The line B G being = & plle. to H E *(Prep.* 2. *& Hyp).*

1. G H will be = & plle to B E. *P.*33. *B.* 1.
2. In like manner, I K is = & plle. to B E. { *P.* 9. *B.*11.
3. Consequently, G H is = & plle. to I K . . . . { *Ax.*1. *B.* 1.
4. Therefore, G I is = & plle. to K H. *P.*33. *B.* 1.

And because in the △ G B I & H E K the three sides B G, B I, & G I of the first, are = to the three sides H E, E K, & H K of the last, each to each, *(Prep.* 2. *& Arg.* 4*).*

5. ∀ G B I or A B C is = to ∀ H E K or D E F. *P.* 8. *B.* 1.

Which was to be demonstrated.



PROPOSITION XI. *PROBLEM I.*

TO draw a ſtraight line (A H) perpendicular to a plane (Z X) from a given point (A) above it.

Given.	Sought.
I. The plane Z X.	*The ſtraight line* A H *let fall from*
II. A point A *above it.*	*the point* A, ⊥ *to the plane* Z X.

Reſolution.

1. In the plane Z X draw at will the ſtraight line B C:
2. From the point A let fall upon B C the ⊥ A D. *P.*12. *B.* 1.
3. At the point D in the plane Z X erect upon B C the ⊥ D G. *P.*11. *B.* 1.
4. From the point A let fall upon D G the ⊥ A H. *P.*12. *B.* 1.

Preparation.

Thro' the point H draw the ſtraight line F E plle. to B C. *P.*31. *B.* 1.

DEMONSTRATION.

BECAUSE the ſtraight line B C is ⊥ to D A & D G *(Ref.*2.*&*3*)*.

1. It will be ⊥ to the plane which paſſes thro' thoſe lines. *P.* 4. *B.*11.

But F E is plle. to B C *(Prep)*.

2. Therefore, F E is alſo ⊥ to this ſame plane which paſſes thro' D G & D A. *P* 8. *B.*11.

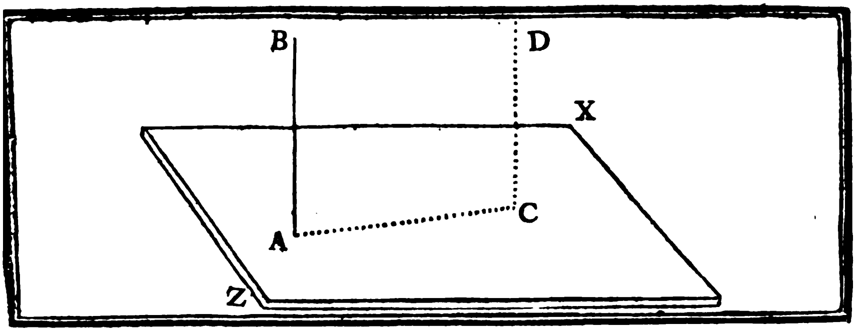
But A H is in the ſame plane with D A & D G *(P.* 2. *B.* 11*)* & meets F E in H *(Ref.* 4. *& Prep)*.

3. Therefore, ∀ F H A is a ∟. *D.* 3. *B.*11.

And becauſe ∀ A H D is a ∟ *(Ref.* 4*)*.

4. A H is ⊥ to the two lines F E & D G ſituated in the plane Z X which interſect each other in H.
5. Therefore, A H is ⊥ to the plane Z X. *P.* 4. *B.*11.

Which was to be done.



PROPOSITION XII. *PROBLEM II.*

FROM a given point (A) in a plane (X Z) to erect a perpendicular (B A).

Given.	Sought.
A point A *in the plane* X Z.	*A straight line* B A *drawn from the point* A ⊥ *to the plane* X Z.

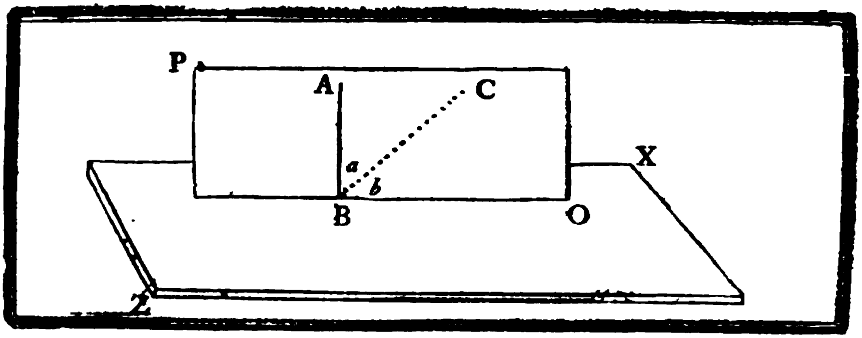
Resolution.

1. Take at will a point D above the plane X Z.
2. From this point D; let fall upon this plane the ⊥ D C. *P.*11. *B.*11.
3. Join the points A & C. *Pos.*1. *B.* 1.
4. From the point A draw A B plle. to D C. *P.*31. *B.* 1.

Demonstration.

BECAUSE the line A B is plle. to D C *(Ref. 4)*.
And that D C is ⊥ to the plane X Z *(Ref. 2)*.
1. A B will be also ⊥ to the same plane X Z. *P.* 8. *B.*11.

Which was to be done.



PROPOSITION XIII. *THEOREM XI.*

FROM the same point (B) in a given plane (Z X) there cannot be drawn on the same side of it more than one perpendicular (A B).

Hypothesis.	Thesis.
A B *is* $\perp$ *at the point* B, *to the plane* X Z.	*It is impossible to draw from the point* B *another* $\perp$ *to the plane* X Z *on the same side that* A B *is.*

Demonstration.

If not,
There may be drawn from the point B another $\perp$.

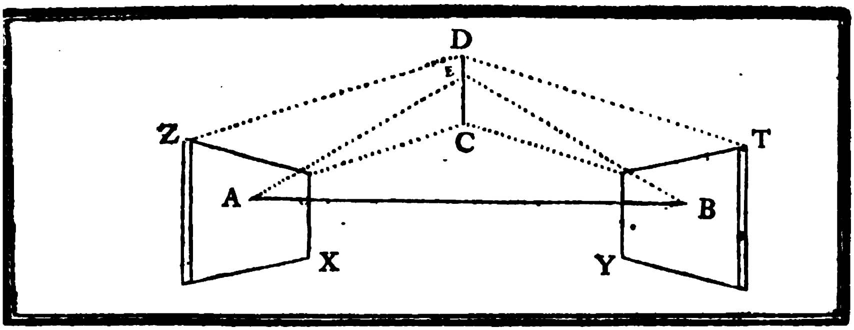
Preparation.

From the point B erect a $\perp$ B C different from A B.

BECAUSE the lines A B & B C meet at the point B.

1. They are in the same plane P O. — *P.* 2. *B.* 11.
   But they are each $\perp$ to the plane X Z *(Sup).*
2. Consequently, the $\angle$ *a* + *b*, & *b* are each $\llcorner$. — *D.* 10. *B.* 1.
3. Therefore, $\angle a + b = \angle b$, *that is,* the whole = to the part.
4. Which is impossible. — *Ax.* 8. *B.* 1.
   But A B is $\perp$ to the plane X Z *(Hyp).*
5. Therefore, B C is not $\perp$ to X Z.
6. Consequently, it is impossible to draw from a point B any other line on the same side as A B, that will be $\perp$ to the plane X Z.

Which was to be demonstrated.



PROPOSITION XIV. THEOREM XII.

PLANES (Z X & T Y) to which the same straight line (A B) is perpendicular; are parallel to one another.

Hypothesis.	Thesis.
A B *is* ⊥ *to the planes* X Z *&* T Y.	*The plane* X Z *is plle. to the plane* T Y.

DEMONSTRATION.

If not,
The planes X Z & T Y produced will meet one another. *D.* 8. *B.*11.

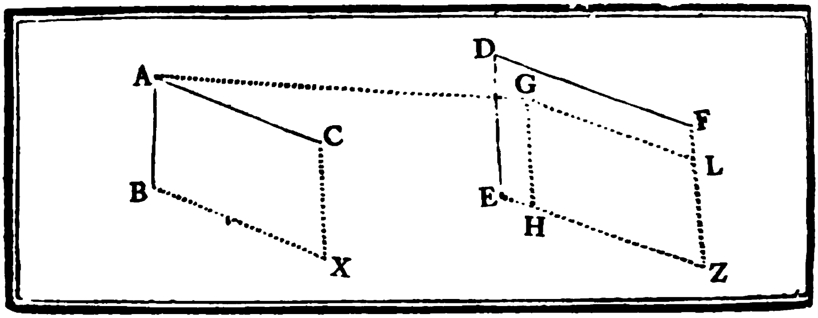
Preparation.

1. Produce the planes X Z & T Y until they meet in D C.
2. Take a point E in the section D C.
3. Draw E A & E B.

BECAUSE A B is ⊥ to the plane T Y *(Hyp.)* & E B is in this plane *(Prep.* 3*)*. *D.* 3. *B.*11.

1. ∀ A B E is a ∟.
2. Likewise ∀ B A E is a ∟.
3. Consequently, the △ B A E has two ∟.
4. Which impossible. *P.*17. *B.* 1.
5. From whence it follows that the lines A E & E B do not meet one another, no more than the planes T Y & X Z. *P.* 1. *B.*11.
6. Therefore, those planes are plle. *D.* 8. *B.*11.

Which was to be demonstrated.



PROPOSITION XV. *THEOREM XIII.*

IF two straight lines (A B & A C) situated in the same plane (A X), and meeting one another (in A), be parallel, to two straight lines (D E & D F) meeting one another, and situated in another plane (D Z); *those planes* (A X & D Z) will be parallel.

Hypothesis.
A B & A C *situated in the plane* A X & *meeting each other in* A, *are plle. to* D E & E F *meeting each other in* D, & *situated in the plane* D Z.

Thesis.
The plane A X *in which are the lines* A B & A C *is plle. to the plane* D Z *in which are the lines* D E & D F.

Preparation.

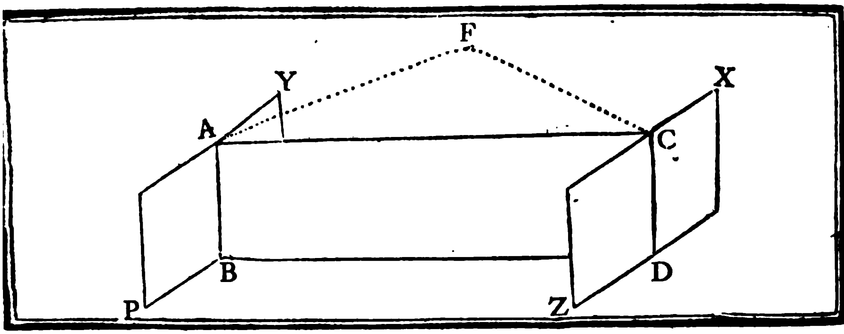
1. From the point A let fall upon the plane D Z the ⊥ A G. *P.* 11. *B.* 11.
2. Draw G H plle. to D E, & G L plle. to D F. *P.* 31. *B.* 1.

Demonstration.

BECAUSE the lines G H & G L are plle. to D E & D F *(Prep. 2)*.

1. They will be also plle. to A B & A C. *P.* 9. *B.* 11.
   And G L being plle. to A C.
2. The ∀ C A G + A G L are = 2 ∟. *P.* 29. *B.* 1.
   But ∀ A G L is a ∟ *(Prep. 1)*.
3. Consequently, ∀ C A G is also a ∟.
4. It may be demonstrated after the same manner that ∀ B A G is a ∟.
5. Therefore, G A is ⊥ to the plane A X. *P.* 4. *B.* 11.
   But G A is also ⊥ to the plane D Z *(Prep. 1)*.
6. Wherefore, the plane A X is plle. to the plane D Z. *P.* 14. *B.* 11

Which was to be demonstrated.



PROPOSITION XVI. THEOREM XIV.

IF two parallel planes (Z X & Y P) be cut by another plane (A B D C), the common sections with it (C D & A B) are parallels.

Hypothesis.	Thesis.
I. The planes Z X *&* P Y *are plle.*	*The common sections* C D *&* A B *are plle.*
II. They are cut by the plane A B C D.	

Demonstration.

If not,
The lines A B & C D being produced will meet somewhere.

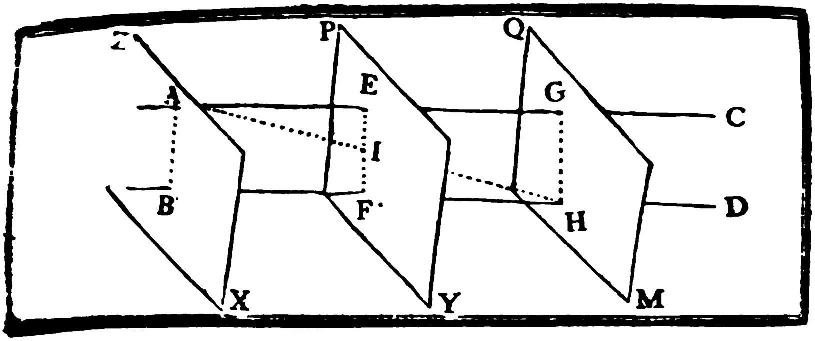
Preparation.

Produce them until they meet in F. *Pos. 2. B. 1.*

BECAUSE the straight lines B A F & D C F meet in F.

1. The planes P Y & Z X in which those lines are, will also meet one another: (B A F being entirely in the plane P Y, & D C F entirely in the plane Z X). *P. 1. B. 11.*
2. Which is impossible *(Hyp. 1)*.
3. Wherefore, A B & C D do not meet one another.
4. Therefore, A B & C D are plle. *D. 35. B. 1.*

Which was to be demonstrated.



PROPOSITION XVII. *THEOREM XV.*

If two straight lines (AC & BD) be cut by parallel planes (XZ, PY & QM): they shall be cut in the same ratio, *(that is,* A E : E F = B F : F H &c).

Hypothesis.	Thesis.
AC & B D are two straight lines.	AE : EG = BF : FH.
cut by the plle. planes X Z, P Y *&* QM.	

Preparation

1. Join the points A & B, alſo G & H.
2. Draw A H which will paſs thro' the plane P Y in the point I. } *Pof.* 1. *B.* 1.
3. Draw E I & I F.

Demonstration.

BECAUSE the plle. planes Z X & P Y are cut by the plane of the △ A B H,

B A is plle. to I F. *P.* 16. *B.* 11.

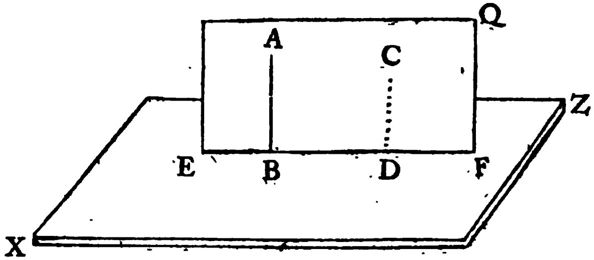
Likewiſe, E I is plle. to G H.

Conſequently, AI : IH = B F : F H. } *P.* 2. *B.* 6.

And, AI : IH = AE : EG.

Therefore, AE : EG = B F : F H. *P.* 11. *B.* 5.

Which was to be demonſtrated.



PROPOSITION XVIII. *THEOREM XVI.*

IF a straight line (A B) is perpendicular to a plane (Z X): every plane as Q E) which passes thro' this line (A B) shall be perpendicular to this plane (Z X).

Hypothesis.	Thesis.
A B *is* ⊥ *to the plane* Z X.	*Every plane (as* Q E) *which passes thro' the* ⊥ A B *is* ⊥ *to the plane* Z X.

Preparation.

1. Let a plane Q E pass thro' A B, which will cut the plane Z X in E F. *P.* 3. *B.* 1.
2. Take in this straight line E F, a point D at will.
3. From this point D, draw in the plane Q E, the line D C plle. to A B. *P.*31. *B.* 1.

DEMONSTRATION.

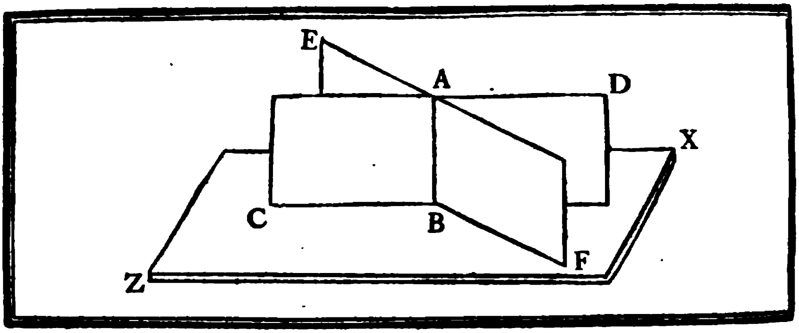
BECAUSE the straight line A B is ⊥ to the plane Z X, & D C is plle. to A B *(Hyp. & Prep. 3).*

1. The line D C is ⊥ to the plane Z X. *P.* 8. *B.*11.
2. Consequently, C D is also ⊥ to the common section E F. *D.* 3. *B.*11.
3. Therefore, the plane E Q in which the lines A B & C D are, is ⊥ to the plane Z X. *D.* 4. *B.*11.

And as the same demonstration may be applied to any other plane which passes thro' the ⊥ A B, we may conclude,

4. That every plane which passes thro' this line is ⊥ to the plane Z X.

Which was to be demonstrated.



PROPOSITION XIX. *THEOREM XVII.*

IF two planes (C D & E F) cutting one another be each of them perpendicular to a third plane (Z X); their common section (A B) shall be perpendicular to the same plane (Z X).

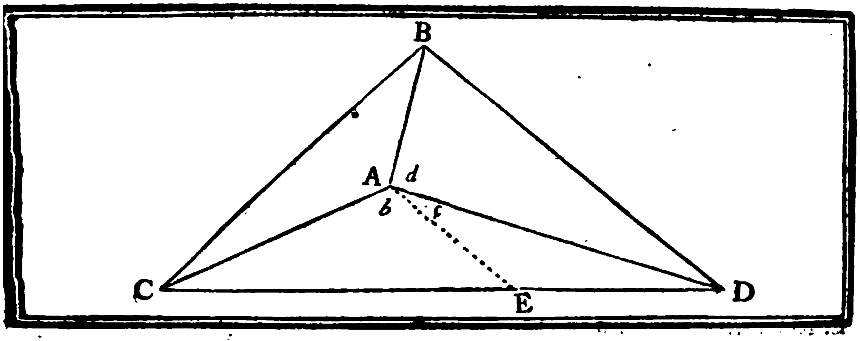
Hypothesis.	Thesis.
I. The planes C D & E F *are* ⊥ *to the plane* Z X.	*The common section* A B *is* ⊥ *to the plane* Z X.
II. They cut one another in A B.	

DEMONSTRATION.

BECAUSE CB, the common section of the plane C D with the plane X Z is also in the plane X Z. *P. 3. B. 11.*

1. There may be erected at the point B in C B a ⊥ *(P. 11. B. 11.)* which will be in the plane C D *(Hyp. 1.)* *P. 18. B. 11.*
And because the line F B the common section of the planes F E & X Z is also in the plane X Z. *P. 3. B. 11.*
2. There may be erected at the same point B & at the same side with the foregoing another ⊥ which will fall in the plane F E. *P. 18. B. 11.*
But from the point B only one ⊥ can be raised. *P. 13. B. 11.*
3. Consequently, those ⊥ must coincide, *that is,* those two lines must form but one which is common to the two planes.
But those planes have only the line A B in common *(Hyp. 2.)*
4. Therefore A B is ⊥ to the plane X Z.

Which was to be demonstrated.



PROPOSITION XX. THEOREM XVIII.

IF three plane angles (C A B, B A D & D A C) form a solid angle A: any two of those angles (as B A D & C A B) are greater than the third (C A D).

Hypothesis.	Thesis.
The three plane ∀ C A B, *d* & *c* + *b* *form a solid* ∀ A.	∀ C A B + *d* > ∀ *b* + *c*.

Demonstration.

CASE I.

When the three angles C A B, *d*, & *c* + *b* are equal.

BECAUSE the ∀ C A B, *d* & *c* + *b* are equal.

1. It follows that ∀ C A B + *d* will be > ∀ *c* + *b*. *Ax.* 4. *B.* 1.

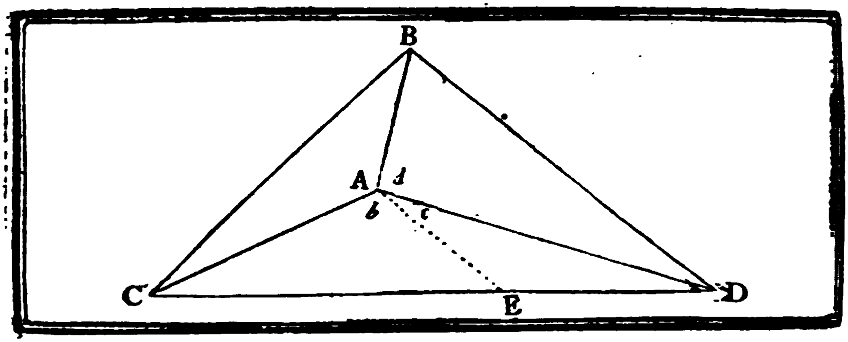
CASE II.

When of the three angles C A B, *d* & *c* + *b* two as C A B & *d* are equal, & the third *c* + *b* is less than either of them.

BECAUSE ∀ C A B is > ∀ *c* + *b*. *Ax.* 4. *B.* 1.

1. ∀ C A B + ∀ *d* will be much > ∀ *c* + *b*.

Which was to be demonstrated.



CASE III.

When the three angles are unequal, & $b + c$ is > C A B or d.

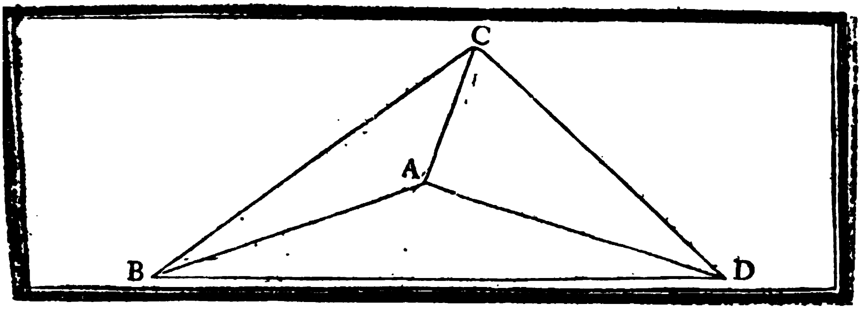
Preparation.

1. At the point A in A C make ∀ b = ∀ C A B in the plane C A D. *P. 23. B. 1.
2. Make A E = A B. P. 3. B. 1.
3. From the point C draw thro' E the straight line C E D.
4. From the points C & D draw C B & B D. } Pos. 1. B. 1.

THE △ B C A & C A E have the sides A B & A E equal (*Prep.* 2). The side C A common & ∀ b = ∀ C A B (*Prep.* 1).

1. Consequently, the side B C is = to the side C E. P. 4. B. 1.
But in the △ C B D the sides C B + B D are > C D. P. 20. B. 1.
Therefore, if from C B + B D be taken away the part C B, & from C D a part = to C E.
2. The remainder B D will be > E D. Ax. 5. B. 1.
In the △ B A D & E A D, the sides A B & A E are = (*Prep.* 2). & A D common.
But the base B D is > the base E D (*Arg.* 2).
3. Therefore, ∀ d is > ∀ c. P. 25. B. 1.
If therefore, ∀ C A B be added one side, & its equal ∀ b on the other.
4. ∀ C A B + d will be > ∀ b + c or C A D. Ax. 4. B. 1.

Which was to be demonstrated.



PROPOSITION XXI. *THEOREM XIX.*

ALL the plane angles (B A C, C A D & D A B) which form a solid angle (A); are less than four right angles.

Hypothesis.	Thesis.
The ∀ B A C, C A D & D A B *form a solid* ∀ A.	*The plane* ∀ B A C + C A D + D A B *are* < 4 ∟.

Preparation.

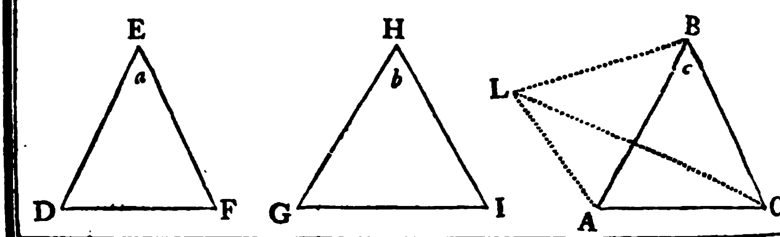
1. In the sides B A, A C, & A D take the three points B, C, D.
2. Draw B C, B D & C D. *Pos.* 1. *B.* 1.
3. Let a plane B C D pass thro' those lines, which will form with the planes B A C, C A D & B A D, three solid ∀; viz. the solid ∀ B, formed by the plane ∀ C B A, A B D & C B D; the solid ∀ C, formed by the plane ∀ B C A, A C D & B C D, & infine, the solid ∀ D, formed by the plane ∀ C D A, A D B & B D C. *D.* 11. *B.* 11.

DEMONSTRATION.

BECAUSE the solid ∀ D, is formed by the plane ∀ C D A, A D B & B D C.

1. The ∀ C D A + A D B are > ∀ B D C.
2. Likewise, ∀ A B D + A B C are > ∀ D B C.
3. And ∀ A C B + A C D are > ∀ B C D. } *P.* 20. *B.* 11.
4. Hence, the six plane ∀ C D A + A D B + A B D + A B C + A C B + A C D are > the three plane ∀ B D C + D B C + B C D.
But those three plane ∀ B D C + D B C + B C D are = 2 ∟. *P.* 32. *B.* 1.
5. Therefore, the six plane ∀ C D A + A D B + A B D + &c. are > 2 ∟. (*Arg.* 4.)
But the nine ∀ of the △ B C A, C A D & D A B viz. the six already mentioned (*Arg.* 5.) & the three remaining ∀ B A C, C A D & D A B are together = to 6 ∟. *P.* 32. *B.* 1.
If therefore the six ∀ C D A + A D B + A B D + A B C + A C B + A C D which are together > 2 ∟ be taken away.
6. The remaining plane ∀ B A C + C A D + D A B will be < 4 ∟.
But those plane ∀ B A C, C A D & D A B form a solid ∀ A.
7. Consequently, the plane ∀ which form a solid ∀ A are < 4 ∟.

Which was to be demonstrated.



PROPOSITION XXII. THEOREM XX.

IF every two of three plane angles be greater than the third, and if the straight lines which contain them be all equal; a triangle may be made of the straight lines (D F, G I & A C) which subtend those angles.

Hypothesis.	Thesis.
I. *Any two of the three given* ∀ *a, b, c, are* $>$ *the third, as* $b + a > c$, *or* $a + c > b$, *or* $b + c > a$. II. *The sides* H G, H I, D E, E F, A B & B C *which contain those* ∀, *are equal.*	*A* △ *may be made of the straight lines* G I, D F & A C, *which subtend those* ∀.

DEMONSTRATION.

The three given ∀ *a, b, & c* are either equal, or unequal.

CASE I If the ∀ *a, b, & c* be equal.

BECAUSE the sides which contain the ∀, are equal *(Hyp. 2.)*

1. The △ D E F, G H I & A B C are equal. *P. 4. B. 1.*
2. Therefore D F = G I = A C.
3. Consequently, D F + A C > G I. *Ax. 4. B. 1.*
4. Wherefore a △ may be made of those straight lines D F, A C & G I. *P. 22. B. 1.*

CASE. II. If the given ∀ *a, b, & c* be unequal

Preparation.

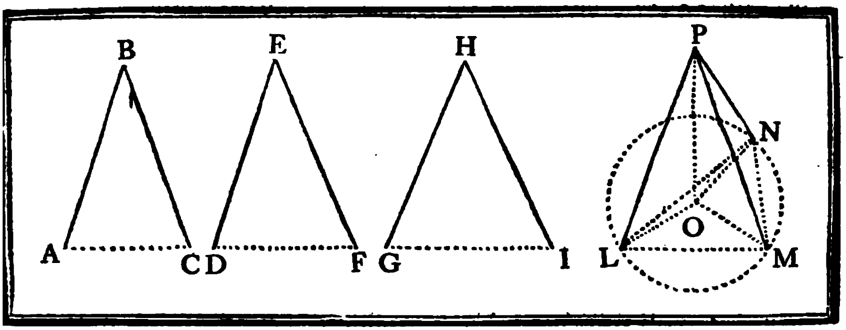
1. At the vertex of one of the ∀ as B, make ∀ A B L = ∀ *a*. *P. 23. B. 1.*
2. Make B L = D E. *P. 3. B. 1.*
3. Draw L C & L A. *Pos. 1. B. 1.*

DEMONSTRATION.

BECAUSE the two ∀ *a* + *c* are > ∀ *b* *(Hyp. 1.)* & L B = H G = B C = H I *(Prep. 2. & Hyp. 2.)*

1. The base L C will be > G I. *P. 24. B. 1.*
   But L C < L A + A C. *P. 20. B. 1.*
2. Much more then G I is < L A + A C.
   But L A = D F *(Prep. 1. & P. 4. B. 1).*
3. Therefore G I is < D F + A C. *Ax. 1. B. 1.*
4. Consequently, a △ may be made of the straight lines D F, A C & G I.

Which was to be demonstrated.



PROPOSITION XXIII. *PROBLEM III.*

TO make a folid angle (P), which fhall be contained by three given plane angles (A B C, D E F & G H I), any two of them being greater than the third, and all three together (∀ A B C + ∀ D E F + ∀ G H I) lefs than four right angles.

Given.

I. Three ∀ A B C, D E F & G H I, *any two of which are greater than the third, as* ∀ B + E > ∀ H, ∀ B + H > ∀ E, & ∀ E + H > ∀ B.

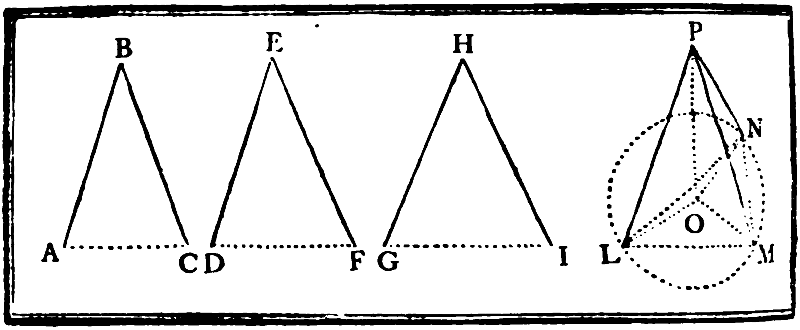
II. ∀ B + E + H < 4 ∟.

Sought.

A folid ∀ P, *contained by the three plane* ∀ B, E & H.

Refolution.

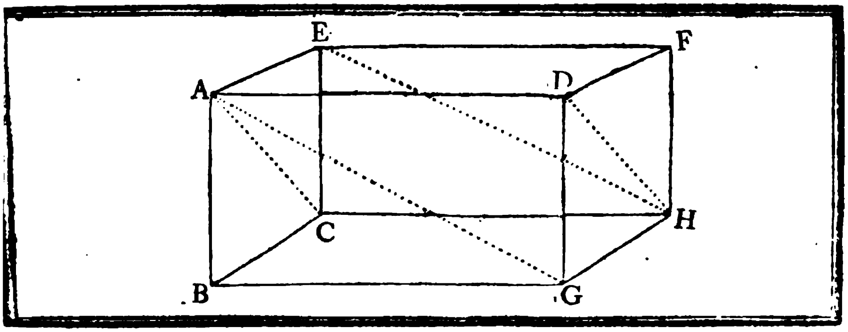
1. Take A B at will, & make the fides B C, D E, E F, G H & H I equal to one another & to A B. *P.* 3. *B.* 1.
2. Draw the bafes A C, D F, & G I. *Pof.* 1. *B.* 1.
3. With thofe three bafes A C, D F & G I make a △ L M N fo that N M be = G I, N L = A C, & L M = D F. { *P.* 27. *B.* 1. *P.* 22. *B.* 11.
4. Infcribe the △ L M N in a ⊙ L M N. *P.* 5. *B.* 4.
5. From the center O, to the ∀ L, M & N, draw the ftraight lines L O, O N & O M.
6. At the point O, erect the ⊥ O P to the plane of the ⊙ L M N. *P.* 12. *B.* 11.
7. Cut O P fo that the □ of L O + the □ of P O be = to the □ of A B.
8. Draw the ftraight lines L P, P N & P M.



Demonstration.

BECAUSE P O is ⊥ to the plane of the ⊙ L M N *(Ref. 6.)*
1. The △ P O L will be right angled in O *(Ref. 5. & 8.)*
2. Consequently, the □ of P O + the □ of O L is = to the □ of L P. *P. 47. B. 1.*
But the □ of P O + the □ of O L = □ A B, *(Ref. 7.)* *Ax 1.*
3. Therefore the □ of A B is = to the □ of L P, & A B = L P. *P. 40. B 1.*
4. Likewise P N & P M are each = to A B. *Cor. 3.*
But N M is = to G I, N L = A C, & L M = D F, *(Ref. 3).*
5. Consequently, △ N M P is = to the △ G H I, △ N P L = △ A B C, △ L P M = △ D E F, ∀ N P M = ∀ H, ∀ L P N = ∀ B, & ∀ L P M = ∀ E. *P. 8. B. 1.*
But those three ∀ N P M, L P N & L P M form a solid ∀ P.
6. Therefore a solid ∀ P has been made, contained by the three given plane ∀ B, E & H.

Which was to be done.



PROPOSITION XXIV. *THEOREM XXI.*

IN every parallelepiped (A H); the opposite planes (B D & C F; B E & F G; A F & B H) are similar & equal parallelograms.

Hypothesis.	Thesis.
In the given ⌸ B F, *the plane* B D *is opposite to* C F, B E *to* F G & A F *to* B H.	*The opposite planes* B D, C F; B E & F G; A F & B H *are* = & ∾ *pgrs. each to each.*

Preparation.

Draw the opposite diagonals E H & A G, also A C & D H.

DEMONSTRATION.

BECAUSE the plle. planes B D & C F are cut by the plane A B C E.

1. The line B A is plle. to E C. *P.*16. *B.*11.
2. Likewise C H is plle. to G B.

And the same plle. planes B D & C F being also cut by the plane D G H F.

3. The line D G will be plle. to F H.
4. Likewise A E is plle. to B C & D F plle. to G H.

And because those plle. planes (*Arg.* 1. 2. & 4.) are the opposite sides of the quadrilateral figures A E C B & D F H G.

5. Those quadrilateral figures A E C B & D F H G, are pgrs. *D.*35. *B.* 1.
6. Likewise the other opposite planes B D & C F; A F & B H are pgrs.

And since A B & B G are plle. to E C & C H, each to each (*Arg.* 1. & 2).

7. ∀ A B G is = to ∀ E C H. *P.*10. *B.*11.

But A B is = to E C & B G = C H. *P.*34. *B.* 1.

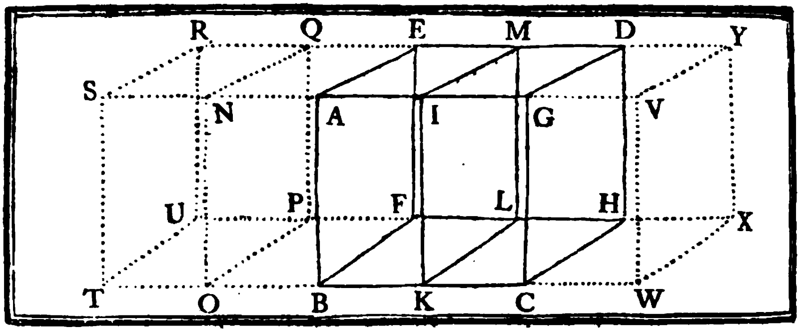
8. Therefore the △ A B G is = & ∾ to the △ E C H. { *P.* 4. *B.* 1. *P.* 4. *B.* 6. }

But the pgr. B D is double of the △ A B G. }
And the pgr. C F is double of the △ E C H. } (*P.*41. *B.*1.)

But those pgrs. have each an ∀ common with the equiangular △.

9. Consequently, the pgrs. B D & C F are = & ∾. *D.* 1. *B.* 6.
10. It may be demonstrated after the same manner that the pgr. B D is = & ∾ to the pgr. C F, & pgr. A F is = & ∾ to the pgr. B H.
11. Therefore the opposite planes of a ⌸ are = & ∾ pgrs.

Which was to be demonstrated.



PROPOSITION XXV. *THEOREM XXII.*

IF a parallelepiped (B E D C) be cut by a plane (K I M L) parallel to the opposite planes (A E F B & C G D H); it divides the whole into two parallelepipeds (*viz.* the ⏢ B E M K & K M D C), which shall be to one another as their bases (B F L K & K L H C).

Hypothesis.
The ⏢ B E D C *is divided into two* ⏢ B M & M C, *by a plane* K M, *plle. to the opposite planes* B E & C D.

Thesis.
The ⏢ B M : ⏢ M C = *base* B L : *base* L C.

Preparation.

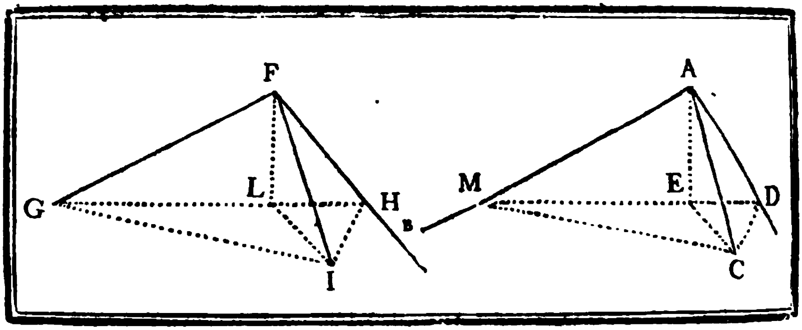
1. Produce B C both ways, as also F H. *Pos. 2. B. 1.*
2. In B C produced take any number of lines = to B K & C K : as B O & T O each = to B K & C W = K C. *P. 3. B. 1.*
3. Thro' those points T, O & W, draw the straight lines T U, O P & W X plle. to B F or C H, until they meet the other plle. produced in the points U, P & X. *P. 31. B. 1.*
4. Thro' the lines T U, O P & W X let the planes T R, O Q & W Y pass, plle. to the planes B E & C D, which will meet the plane A E D G in S R, N Q & V Y.

DEMONSTRATION.

BECAUSE the lines B O & T O, are each = to B K & C W = K C (*Prep.* 2.) & the lines O P, T U & W X plle. to B F or C H, meet F H produced, in the points, P, U & X (*Prep.* 3).

1. The pgrs. T P & B P are = to the pgr. B L; & pgr. C X = pgr. K H. *P.*35. *B.* 1.
The planes A R or A Q & T F or O F being plle; & the plane N P plle. to the plane A F; moreover the lines S A & R E being plle. to the lines B T or F U.
2. The solid O Q E B will be a ▭ = & ∽ to the ▭ B E M K. *D.*10. *B.*11.
3. It may be demonstrated after the same manner that the solid T R Q O is = & ∽ to ▭ B E M K; also the solid C D Y W is = & ∽ to ▭ K M D C.
But there are as many equal ▭ O Q E B, &c. as there are equal pgrs. O F, T P, &c. & those ▭ together compose the ▭ T E: moreover there are as many equal pgrs. O F, &c. as there has been taken straight lines, each = to B K, which together are = to T B.
4. Consequently, the ▭ T E is the same multiple of the ▭ B E M K that the parts (T O, O B) of the line T B taken together, are multiples of the line B K.
5. Likewise the ▭ C D Y W is the same multiple of the ▭ K M D C that the line W C is of the line K C.
6. Therefore according as the ▭ T R E B is >, = or < the ▭ B E M K, the line T B will be >, = or < the line B K.
And according as the ▭ C D Y W is >, = or < ▭ K M D C, the line C W will be >, = or < the line K C.
7. Consequently, the ▭ B E M K : ▭ K M D C = B K : K C. *D.* 5. *B.* 5.
But B K : K C = base B L : base K H. *P.* 1. *B.* 6.
8. Therefore ▭ B E M K : ▭ K M D C = base B L : base K H. *P.*11. *B.* 5.

Which was to be demonstrated.



PROPOSITION XXVI. *PROBLEM IV.*

AT a given point (A) in a given ſtraight line (A B), to make a ſolid angle equal to a given ſolid angle (F).

Given.	Sought.
I. A point A *in a ſtraight line* A B.	*At the point* A, *a ſolid angle* = *to the ſolid angle* F.
II. A ſolid angle F.	

Reſolution.

1. From any point I in one of the ſections about the ſolid ∀ F, let fall a ⊥ I L upon the oppoſite plane G F H. *P.* 11. *B.* 11.
2. Draw L F, L G, L H, H I & G I in the planes which form the ſolid ∀. *Poſ.* 1. *B.* 1.
3. In the given ſtraight line A B, take A M = F G. *P.* 3. *B.* 1.
4. At the point A, make a plane ∀ M A D = the plane ∀ G F H. *P.* 23. *B.* 1
5. Cut off A D = F H. *P.* 3. *B.* 1.
6. In the ſame plane M A D, make a plane ∀ M A E = to the plane ∀ G F L. *P.* 23. *B.* 1.
7. Cut off A E = F L. *P.* 3. *B.* 1.
8. At the point E, in the plane M A D erect the ⊥ E C. *P.* 12. *B.* 11.
9. Make E C = L I, *P.* 3. *B.* 1.
10. Draw A C. *Poſ.* 1. *B.* 1.

Preparation.

Draw M E, E D, C D & C M in the planes, M A D, C A D & M A C.

Demonstration.

BECAUSE in the △ G F H & M A D, the ſides F G & F H are = to the ſides A M & A D, each to each, (*Ref.* 3. & 5.) & ∠ G F H is = to ∠ M A D, (*Ref.* 4).

1. G H will be = to M D. } *P.* 4. *B.* 1.
2. Likewiſe in the △ G F L & A M E, G L is = to M E. } *P.* 4. *B.* 1.

Therefore if G L be taken from G H & M E from M D.

3. L H will be = to E D. *Ax.* 3. *B.* 1.

And ſince in the △ L H I & E D C, E D is = to L H, L I = E C & the ∠ D E C & H L I, are ∟, (*Arg.* 3. *Ref.* 9. & *D.* 3. *B.* 11).

4. I H will be = to C D. *P.* 4. *B.* 1.

Likewiſe in the △ F L I & A E C, L I is = to E C, & L F = A E, beſides ∠ F L I & ∠ A E C, are ∟, (*Ref.* 7. 9. & *D.* 3. *B.* 11).

5. Therefore F I = A C. *P.* 4. *B.* 1.
6. It may be demonſtrated after the ſame manner that G I is = M C.

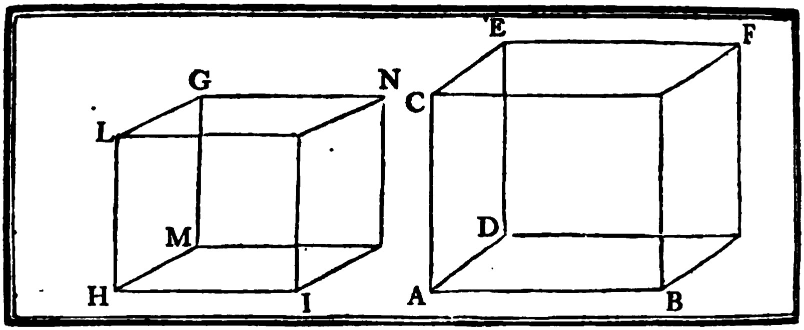
Since then the three ſides H I, F I & F H of the △ I F H are = to the three ſides D C, A C & A D, of the △ C A D (*Arg.* 4. & 5).

7. ∠ I F H will be = to ∠ C A D. *P.* 8. *B.* 1.
8. Likewiſe △ G F I is = to the △ M A C & ∠ G F I = ∠ M A C.

Therefore the plane ∠ G F H being = to the plane ∠ M A D, (*Ref.* 4.)
The plane ∠ I F H = to the plane ∠ C A D (*Arg.* 7).
And the plane ∠ G F I = to the plane ∠ M A C, (*Arg.* 8).
Beſides the plane ∠ G F H, I F H & G F I, form a ſolid ∠ F.
And the plane ∠ M A D, C A D & M A C, ſimilarly ſituated as theſe already mentioned, form the ſolid ∠ A.

9. It follows that the ſolid ∠ A is = to the ſolid ∠ F. *D.* 9. *B.* 11.

Which was to be done.



PROPOSITION XXVII. *PROBLEM V.*

TO describe from a given ſtraight line (A B), a parallelepiped ſimilar, & ſimilarly ſituated to one given (H N).

Given.	Sought.
I. A ſtraight line A B.	*From* A B *to deſcribe a* ▭ A F, ∽
II. The ▭ H N.	*&* *ſimilarly ſituated to a* ▭ H N.

Reſolution.

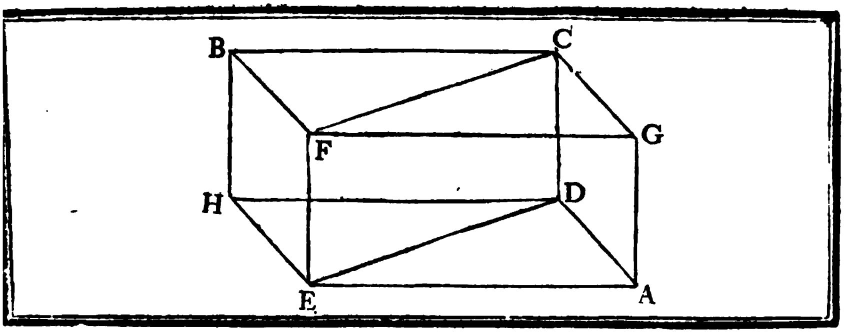
1. At the point A in the line A B make a ſolid ∀ C A D B, = to the ſolid ∀ H, or L H M I. *P.26. B.11.*
2. Cut A C ſo that H I : H L = A B : A C. } *P.12. B.6.*
3. Alſo A D ſo that H L : H M = A C : A D. }
4. Complete the pgrs. A E, B D & B C. *P.31. B.1.*
5. Complete the ▭ A F.

DEMONSTRATION.

THE three pgrs. A E, B D & B C being ∽ & ſimilarly ſituated with the three pgrs. H G, M I & L I of the ▭ H N, each to each *(Reſ. 1. 2. 3. & 4. & D. 1. B. 6).*
As alſo their oppoſite ones. *P.24. B.11.*

1. Conſequently, the ſix planes or pgrs. which form the ▭ A F, are ∽, & ſimilarly ſituated to the ſix planes or pgrs. which form the given ▭ H N.
2. Therefore the ▭ A F deſcribed from A B, is ſimilar & ſimilarly ſituated to the given ▭ H N. *D. 9. B.11.*

Which was to be done.



PROPOSITION XXVIII. *THEOREM XXIII.*

IF a parallelepiped (A B) be cut by a plane (F C D E) passing thro' the diagonals (F C & E D) of the opposite planes (B G & A H): it shall be cut into two equal parts.

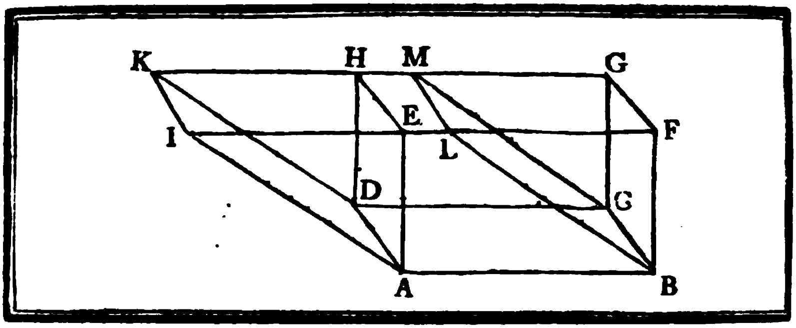
Hypothesis.
The ▭ A B *is cut by a plane* F D *passing thro' the diagonals* F C & E D *of the opposite planes* B G & A H.

Thesis.
The plane F D *cuts the* ▭ A B *into two equal parts.*

DEMONSTRATION.

BECAUSE the plane F A is a pgr. *P.24. B.11.*
1. The sides E F & G A are = & plle. *P.33. B.11.*
2. Likewise C D & G A are = & plle. *P. 9. B.11.*
3. Consequently, E F is = & plle. to C D. *Ax.1. B. 1.*
4. Therefore E D is = & plle. to F C. *P.33. B. 1.*
5. From whence it follows that F C D E is a pgr. *D.35. B. 1.*
But the pgr. B C G F is = & plle. to the pgr. H D A E. *P.24. B.11.*
6. Consequently, the △ B C F & F G C are = & ∽ to the △ H D E & E D A. *P.34. B. 1.* *P. 4. B. 1.*
Moreover, the pgrs. F E A G & G A D C, are = & ∽ to the pgrs. B H D C & B H E F, each to each. *P.24. B.11.*
7. Therefore all the planes which form the prism B F D are = & ∽ to all the planes which form the prism D F G.
8. Therefore the prism B F D or B H E D C F is = & ∽ to the prism D F G or D E F C G A. *D.10. B.11.*
9. Consequently, the plane F C D E, cuts the ▭ A B into two equal parts.

Which was to be demonstrated.



PROPOSITION XXIX. THEOREM XXIV.

PARALLELEPIPEDS (HB & KB) upon the same base (BD), and of the same altitude (A E), the insisting straight lines of which (A E, A I; B F, B L; D H, D K; C G, C M) are terminated in the same straight line (I F, G K) in the plane opposite the base, are equal to one another.

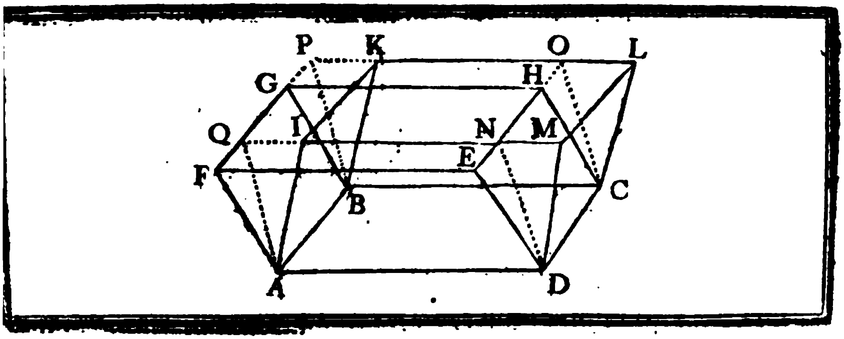
Hypothesis.	Thesis.
I. The ▭ K B & H B *have the same base* B D.	▭ H B *is* = ▭ K B.
II. They have the same altitude A E.	
III. The insisting lines A E, A I, *&c. of which, are terminated in the lines* I F, G K.	

Demonstration.

BECAUSE the pgrs. K C or K M C D, & H C or H G C D, have the same base D C, & their opposite sides K D, M C, & D H, C G, are terminated in K G which is plle. to D C *(Hyp. 3)*.

1. The pgr. K C is = to the pgr. H C. *P.35. B. 1.*
Therefore if from those equal pgrs. be taken away the common trapezium H M C D.
2. The remainders, *viz*. the △ K H D & M G C will be equal. *Ax.3. B. 1.*
3. Likewise △ I E A is = to the △ L F B.
4. The pgr. K E or K H E I, is also = to the pgr. M F or M G F L.
Because they are each = to the pgr. D C B A, less the pgr. H M L E, *(D. 30. & P. 24. B. 11)*.
But the plane G B or C F is = to the plane H A or D E, & the plane M B or L C is = to the plane K A or I D. *P.24. B.11.*
5. Consequently, the prism H A K D is = to the prism G B M C. *D.10. B.11.*
Therefore if to those equal prisms the part H M C B L E A D be added.
6. The prism H A K D + part H M C B L E A D is = prism G B M C + part H M C B L E A D.
But prism H A K D + part H M C B L E A D = ▭ K B. } *Ax.1. B. 1.*
And prism G B M C + part H M C B L E A D = ▭ H B. }
7. Therefore the ▭ K B is = ▭ H B. *Ax.1. B. 1.*

Which was to be demonstrated.



PROPOSITION XXX. THEOREM XXV.

PARALLELEPIPEDS (F G H E D C B A & I M L K B C A) upon the same base (A B C D) and of the same altitude, the insisting straight lines of which (A F, A I; D E, D M; B G, B K; C H, C L), are not terminated in the same straight lines in the plane opposite the base, are equal to one another.

Hypothesis.	Thesis.
I. The ⏢ H A & L A *are upon the same base* A C.	⏢ F H C *is* = ⏢ I L C A.
II. They have the same altitude.	
III. The insisting straight lines A F, A I, *&c. are not terminated in the same straight lines.*	

Preparation.

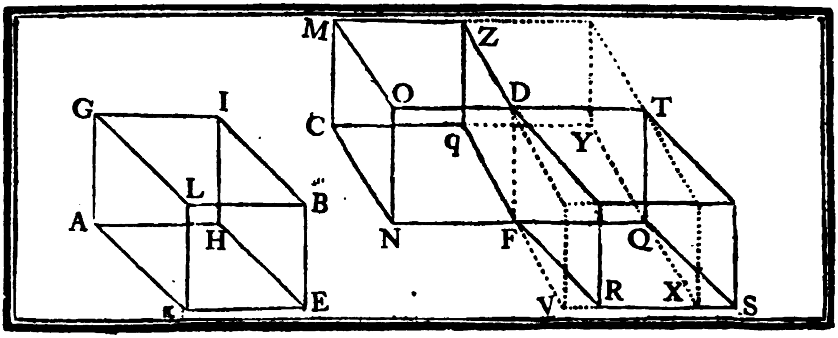
1. Produce L K & F G until they meet in P.
2. Produce I M until it meets F G in Q. } *Pos.* 2. *B.* 1.
3. And E H to O.
4. Draw Q A, P B, O C & N D. *Pos.* 1. *B.* 1.

Demonstration.

BECAUSE the ⏢ F H C A & Q O C A have the same base A B C D, & their insisting straight lines A F, A Q; D E, D N; B G, B P; & C H, C O are terminated in the lines F P & E O.

1. The ⏢ F H C A is = to the ⏢ Q O C A. *P.* 29. *B.* 11.
2. Likewise the ⏢ Q O C A is = to ⏢ I L C A.
3. Therefore the ⏢ F H C A is = to the ⏢ I L C A. *Ax.* 1. *B.* 1.

Which was to be demonstrated.



PROPOSITION XXXI. THEOREM XXVI.

PARALLELEPIPEDS (K I & N Z) which are upon equal bases (K H & N q), and of the same altitude, are equal to one another.

Hypothesis.	Thesis.
I. The □ K I & N Z, *have their bases* K H & N q *equal.* *II. They have the same altitude.*	*The* □ K I *is = to the* □ N Z.

Demonstration.

CASE I.

If the insisting lines A G, &c. of the □ K I; & the insisting lines C M, &c. of the □ N Z, are ⊥ to their bases; or if the inclinations of the insisting straight lines A G & M C are the same.

Preparation.

1. PRODUCE NF, & make F Q = A H. *Pos.* 1. *B.* 1. *P.* 3. *B.* 1.
2. At the point F in F Q, make the plane ∠ Q F R = plane ∠HAK. *P.* 23. *B.* 1.
3. Make F R = A K.
4. Complete the pgr. F Q S R. *P.* 31. *B.* 1.
5. Complete likewise with the lines F Q & F D; F R & F D, the pgrs Q T D F & D F R. *P.* 31. *B.* 1.
6. Complete the □ D S.
7. Produce the straight lines F q & R S until they meet in V. *Pos.* 2. *B.* 1.
8. Thro' the point Q, draw X Q Y, plle. to V q. *P.* 31. *B.* 1.
9. Produce C q, until it meets X Y, in the point Y.
10. Complete the □ Z Q & V D T X.

BECAUSE the lines F Q & F R are = to AH & AK. *(Prep.* 1. & 3*)*.
And the ∠ Q F R is = to the ∠ H A K *(Prep.* 2*)*. *P.* 36. *B.* 1.

1. The pgr. F S is = & ∼ to the pgr. K H *D.* 1. *B.* 6.
2. It may be demonstrated after the same manner that the pgrs. F T & D R are = & ∼ to the pgrs. A I, & A L.

Therefore, since the three pgrs. F S, F T, & D R, of the ▱ D S are = & ∽ to the three pgrs. A E, A I, & A L, of the ▱ K I, (*Arg*. 1. & 2).

And the remaining pgrs. of the ▱ D S, likewise those of the ▱ K I are = & ∽ to those already mentioned; each to each, *P*.24. *B*.11.

3. The ▱ D S, will be = & ∽ to the ▱ K I. *D*.10. *B*.11.

The ▱ D X & D S, have the same base D Q, & their insisting lines F V &. F R, &c. are in the same plle. directions V S, &c.

4. Consequently, ▱ D S is = to the ▱ D X. *P*.29. *B*.11.

But the ▱ D S is = to the ▱ K I (*Arg*. 3).

5. Therefore the ▱ D X is also = to the ▱ K I. *Ax*.1. *B*. 1.

The ▱ M Q is cut by the plane F Z, plle. to the plane M N.

6. Consequently, the base N q : base q Q = ▱ M F : ▱ Z Q. *P*.25. *B*.11.

The ▱ Z X is cut by the plane D Q, plle. to the plane Z Y.

7. Consequently, the base F X : base q Q = ▱ D X : ▱ Z Q. *P*.25. *B*.11.

But the pgr. F X is = to the pgr. F S. *P*.35. *B*. 1.

And the pgr. F S is = to the pgr. H K. (*Arg*. 1).

8. Consequently, the pgr. F X is = to the pgr. H K. *Ax*.1. *B*. 1.

But the base H K is = to the base q N (*Hyp*. 1).

9. Hence the base q N = to the base F X.

But the base q N : base q Q = ▱ M F : ▱ Z Q (*Arg*. 6).

And the base q Q : base F X = ▱ Z Q : ▱ D X. (*Conv*.*Arg*.7).

10. Hence the base q N : base F X = ▱ M F : ▱ D X. *P*.22. *B*. 5.

But the base q N is = to the base F X (*Arg*. 9).

11. Consequently, the ▱ M F is = to the ▱ D X. *P*.14. *B*. 5.

But the ▱ D X & K I are equal (*Arg*. 5).

12. Therefore, the ▱ M F is = to the ▱ K I. *Ax*.1, *B*. 1.

Which was to be demonstrated.

CASE II.

If the angles of inclination of the insisting straight lines, A G &c. of ▱ K I are not equal to the angles of inclination of the insisting straight lines C M, &c. of the ▱ M F.

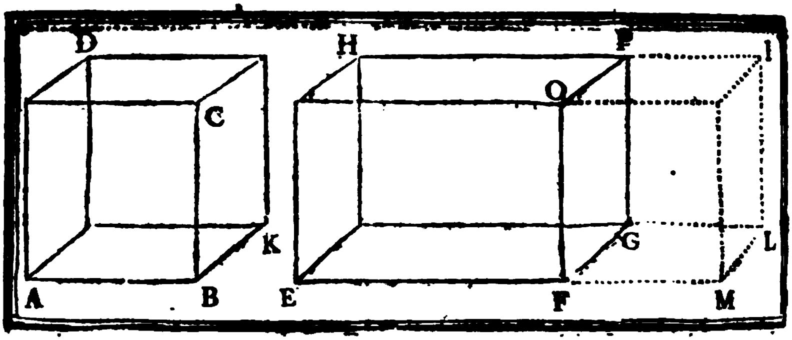
UPON the base K I, make a ▱ having its insisting straight lines, either ⊥ : or equally inclined with the insisting straight lines of the ▱ M F, & in the same direction as those of ▱ K I.

And consequently, which will be equal to it (*P*. 30. *B*. 11).

The remainder of the construction, & of the demonstration, are the same as in the foregoing case.

COROLLARY.

E*QUAL parallelepipeds which have the same altitude, have equal bases*



PROPOSITION XXXII. *THEOREM XXVII.*

PARALLELEPIPEDS (BD & EP) which have equal altitudes (B C & F O), are to one another as their bases (A K & E G).

Hypothesis.
The altitudes B C & F O, *of the* ▱ B D & E P, *are equal.*

Thesis.
▱ B D : ▱ E P = *base* A K : *base* E G.

Preparation.

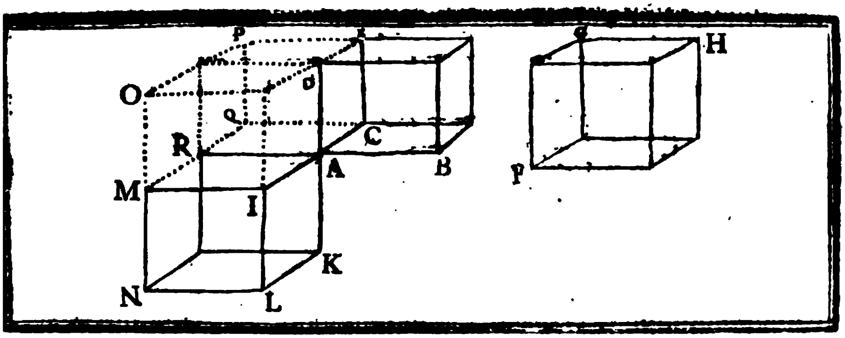
1. Produce E F to M. *Pof.* 2. *B.* 1.
2. Upon F G with F M, make the pgr. F L = pgr. K A, which will be in the same direction with the pgr. E G.
   So that the pgrs. E G & F L together, form the pgr. EL. *P.* 44. *B.* 1.
3. Complete the ▱ F I.

DEMONSTRATION.

BECAUSE the base F L of the ▱ F I, is = to the base A K of the ▱ B D *(Prep. 2)*.

1. The ▱ F I is = to the ▱ B D. *P.* 31. *B.* 11.
2. Consequently, ▱ F I : ▱ E P = ▱ B D : ▱ E P. *P.* 7. *B.* 5.
   But, ▱ F I : ▱ E P = base F L : base E G. *P.* 25. *B.* 11.
   And the base F L is = to the base A K *(Prep. 2)*.
3. Therefore, ▱ B D : ▱ E P = base A K : base E G. *P.* 11. & 7. *B.* 5.

Which was to be demonstrated.



PROPOSITION XXXIII. *THEOREM XXVIII.*

SIMILAR parallelepipeds (E B & F H) are to one another in the triplicate ratio of their homologous sides (A B & G H).

Hypothesis.

The ▱ E B *&* F H *are* ∽, *& the sides* A B *&* G H *are homologous.*

Thesis.

The ▱ E B *is to the* ▱ F H *in the triplicate ratio of* A B *to* G H, *or as* $AB^3 : GH^3$.*

Preparation.

1. Produce A B & make A R = G H. { *Pos.* 2. *B.* 1. / *P.* 3. *B.* 1.
2. From A R describe the ▱ R L = & ∽ to the ▱ F H, so that the lines A C & A I; D A & A K be in the same straight line. *P.* 27. *B.* 11.
3. Complete the ▱ A O, so as to form with ▱ R L the ▱ O K.
4. Complete likewise the ▱ A P, so as to form with ▱ O A, the ▱ O C, & with the ▱ E B the ▱ P B.

Demonstration.

BECAUSE the ▱ E B & R L, are ∽ *(Prep. 2).*

1. The pgr. A M is ∽ to the pgr. C B. *D.* 9. *B.* 11.
2. Consequently, A B : A C = A R : A I. *D.* 1. *B.* 6.
3. And alternando A B : A R = A C : A I. *P.* 16. *B.* 5.
4. Likewise A B : A D = A R : A K. *D.* 1. *B.* 6.
5. And alternando A B : A R = A D : A K. *P.* 16. *B.* 5.

And since A R is = to G H *(Prep. 1).*

6. The three ratios A B : A R, A C : A I, & A D : A K, are equal to one another & equal to the ratio of A B to G H.

But the ▱ P B is cut by the plane A E *(Prep. 4).*

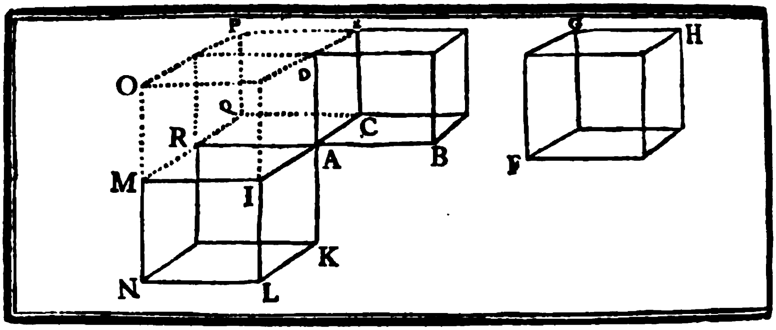
7. Consequently, the base C B : base Q A = ▱ B E : ▱ A P. *P.* 25. *B.* 11.

And the base C B : base Q A = A B : A R. *P.* 1. *B.* 6.

8. Therefore A B : A R = ▱ B E : ▱ A P. *P.* 11. *B.* 5.

Which was to be demonstrated.

* *See Cor. 2. of this proposition.*



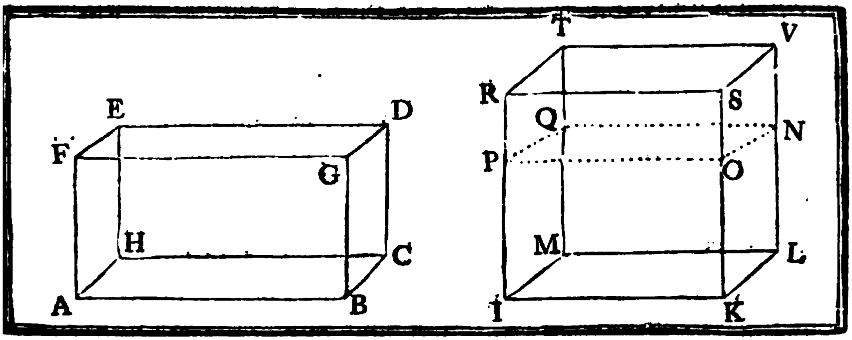
The ▱ O C is cut by the plane R D (*Prep.* 4).
9. Consequently, the base R C : base A M = ▱ A P : ▱ O A. P.25. B.11.
And, the base R C : base A M = A C : A I. P. 1. B. 6.
10. Therefore, A C : A I = ▱ A P : ▱ O A. P.11. B. 5.
Infine, the ▱ O K being cut by the plane A M (*Prep.* 4).
11. It may be demonstrated after the same manner.
That A D : A K = ▱ A O : ▱ A N.
But the three ratios A B : A R, A C : A I, & A D : A K are = to the ratio A B : G H (*Arg.* 6).
12. Consequently, the four ▱ B E, A P, A O, & A N form a series of magnitudes in the same ratio (A B : G H). P.11. B. 5.
13. Therefore, they are proportionals. D. 6. B. 5.
14. Consequently, the ▱ B E is to the ▱ A N in the triplicate ratio of A B to G H. D.11. B. 5.
But the ▱ B E is to the ▱ F H in the triplicate ratio of A B to G H, (or as AB^3 to GH^3). *

COROLLARY I.

FROM this it is manifest, that if four straight lines be continual proportionals, as the first is to the fourth, so is the parallelepiped described from the first to the similar & similarly described parallelepiped from the second; because the first straight line has to the fourth, the triplicate ratio of that which it has to the second.

* COROLLARY II.

ALL cubes being similar parallelepipeds (D. IX & XXX. B. 11), *similar parallelepipeds* (A B & F H) *are to one another as the cubes of their homologous sides* (A B & G H) *(expressed thus* $AB^3 : GH^3$*); because they are in the triplicate ratio of those same sides.*



PROPOSITION XXXIV. THEOREM XXIX.

THE bases, (pgrs. A C & I L) and altitudes (G B & I R) of equal parallelepipeds, (A D & I V) are reciprocally proportional; and if the bases, (pgrs. A C & I L) and altitudes (G B & I R) be reciprocally proportional, the parallelepipeds are equal.

Hypothesis.	Thesis.
▭ A D *is* = *to* ▭ I V.	*Base* A C : *base* I L = *alt.* I R : *alt.* G B.

I. Demonstration.

The given parallelepipeds may be either.

CASE 1. Of the same altitude } and equally inclined on their bases.
CASE 2. Of different altitudes } and equally inclined on their bases.
CASE 3. Having different inclinations : as if one was ⊥ to the base, and the other oblique.

CASE I.

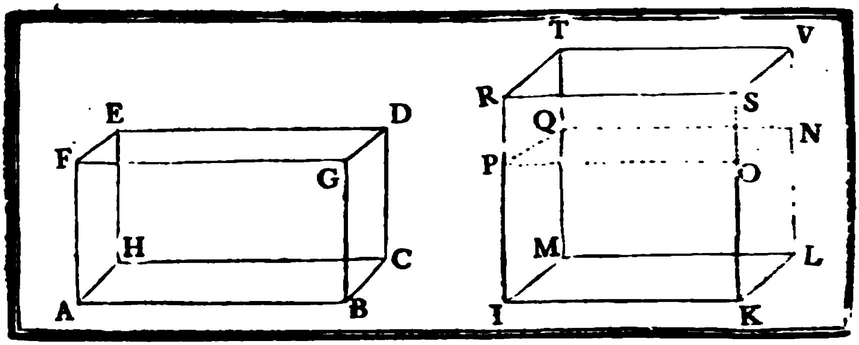
When the ▭ have the same altitude, *that is*, I R = G B.

BECAUSE the given ▭ are equal, & have the same altitude.

1. Their bases are equal *(Cor. of P. 31. B. 11).*
2. Therefore, the base A C : base I L = altitude I R : altitude G B. *D. 6. B. 5.*

CASE II.

When I R is > G B.



I. Preparation.

1. From the alt. R I, cut off the part P I = to the alt. B G.
2. Thro' the point P, pass the plane P O N Q, plle. to the base I L.

BECAUSE the parallelepipeds A D & I N have the same altitude *(I. Prep. 1)*.

1. The ▱ A D : ▱ I N = base A C : base I L. *P.32. B.11.*
   But the ▱ A D is = to the ▱ I V *(Hyp)*.
2. Therefore, ▱ A D : ▱ I N = ▱ I V : ▱ I N *P. 7. B. 5.*
3. Consequently, ▱ I V : ▱ I N = base A C : base I L. *P.11. B. 5.*
   The ▱ I V is cut by the plane P O N Q *(I. Prep. 2)*.
4. Therefore, ▱ P V : ▱ I N = base P S : base K P. *P.25. B.11.*
5. Therefore, componendo ▱ I V : ▱ I N = base K R : base K P. *P.18. B. 5.*
   But the base K R : base K P = R I : P I. *P. 1. B 6.*
6. Wherefore, ▱ I V : ▱ I N = R I : P I. *P.11. B. 5.*
   But, ▱ I V : ▱ I N = base A C : base I L *(Arg. 3)*.
   And P I = G B *(I. Prep. 1)*.
7. Consequently, base A C : base I L = I R : B G. *P.11. B. 5*

CASE III.

When the ▱ I V has a different inclination from the ▱ A D.

II. Preparation.

Describe a ▱ of the same altitude with the ▱ I V, having the same inclination as the ▱ A D.

BECAUSE the described ▱, has the same base & the same altitude with the ▱ A D *(II. Prep)*.

1. This ▱ will be = to the given ▱ I V. *P.31. B.11.*
   But this described ▱ is in the reciprocal ratio of its base, & of its altitude with the ▱ A D *(Case II.)*.
2. Therefore, the ▱ I V will be also in reciprocal ratio with the ▱ A D. *P. 7. B. 5.*

Which was to be demonstrated.

Hypothesis.	Thesis.
Base I L : *base* A C = *alt.* G B : *alt.* I R.	▱ A D *is* = ▱ I V.

II. Demonstration.

The preparation is the same as for the foregoing case.

BECAUSE the ▱ I N & AD have the same altitude *(I. Prep.* 1).

1. The ▱ I N : ▱ A D = base I L : base A C. *P.*32. *B.*11.
   But the base I L : base A C = alt. G B : alt. I R. *(Hyp).*
2. Therefore ▱ I N : ▱ A D = alt. G B : alt. I R. *P.*11. *B.* 5.
   And as P I is = B G. *(I. Prep.* 1).
3. The ▱ I N : ▱ A D = alt. P I : alt. I R. *P.* 7. *B.* 5.
   But P I : I R = pgr. P K : pgr. K R. *P.* 1. *B.* 6.
   And pgr. K P : pgr. K R = ▱ I N : ▱ I V. *P.*32. *B.*11.
4. Therefore the ▱ I N : ▱ A D = ▱ I N : ▱ I V. *P.*11. *B.* 5.
   But the ▱ I N is the first & third terms of the proportion.
5. Consequently, the ▱ A D is = to the ▱ I V. *P.*14. *B.* 5.

Which was to be demonstrated.

The demonstrations of the first and third cases in this hypothesis, are the same, for which reason we have omitted them.

REMARK I.

WHAT has been demonstrated in the propositions 25, 29, 30, 31, 33 *&* 34. *concerning parallelepipeds, is also true with respect to triangular prisms; because such a prism is the half of its parallelepiped;* (P. 28. B. 11.) *from whence we may conclude.*

I. *If a triangular prism be cut by a plane plle. to the opposite planes; the two prisms resulting from thence, will be to one another as the parts of the pgr., base of the whole prism.*

II. *Triangular prisms which have the same, or equal bases, & have equal altitudes, are equal.*

III. *Triangular prisms which have the same altitude, are to one another as their bases.*

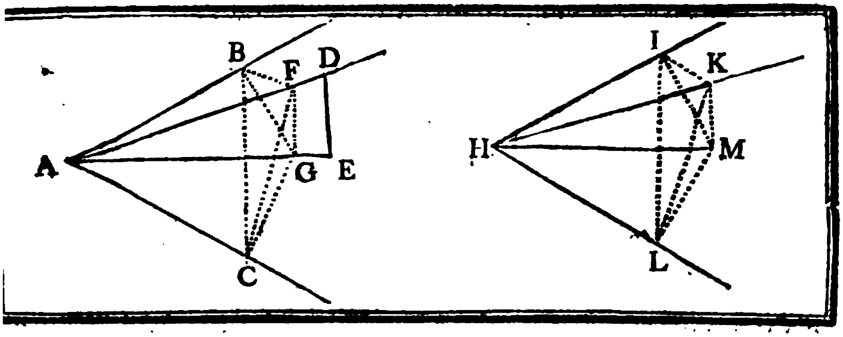
IV. *Similar triangular prisms, are to one another in the triplicate ratio, of their homologous sides.*

V. *Equal triangular prisms, have their bases and altitudes reciprocally proportional, & triangular prisms whose bases and altitudes, are reciprocally proportional, are equal.*

REMARK II.

WITH the same properties prisms are endued, whose opposite planes are polygons. Since it has been demonstrated, (*P.* 20. *B.* 6.) that those opposite & similar polygons may be divided into the same number of similar triangles; therefore if thro' the homologous diagonals which form those triangles, planes, be passed: those planes will divide the polygon prisms, into as many triangular prisms as there are triangles in their opposite & plle, planes.

But what has been observed in the foregoing remark, is applicable to those triangular prisms. Consequently, we may conclude (*P.* 12. *B.* 5.) that *polygon prisms are endued with the same properties.*



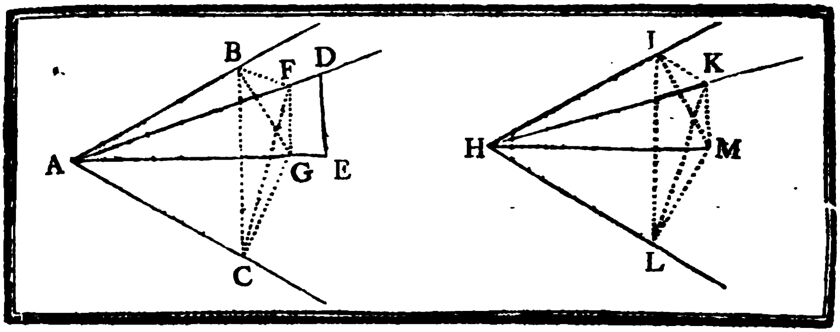
PROPOSITION XXXV. THEOREM XXX.

IF, from the vertices (A & H) of two equal plane angles (B A C & I H L), here be drawn two straight lines (A D & H K) above the planes in which he angles are, and containing equal angles (∀ B A D = ∀ I H K & ∀ D A C = ∀ K H L), with the respective sides of those angles, (viz. A D with A B & A C; H K with I H & H L), and from any two points (D & K) n those lines, (A D & H K), above the planes, there be let fall the perpendiculars (D E & K M), on the planes of the first named angles (B A C & I H L), and from the points (E & M), in which the perpendiculars meet hose planes, the straight lines (A E & H M), be drawn to the vertices (A & H), of the angles first named: those straight lines (A E & H M), shall contain equal angles (D A E & K H M), with the straight lines (A D & H K) which are above the planes of the angles.

Hypothesis.	Thesis.
I. *Above the planes of the equal* ∀ B A C *&* I H L, *& from their vertices* A *&* H, *there has been drawn* A D *&* H K, *containing* ∀BAD *&* DAC=∀IHK *&* KHL, *each to each.*	∀ D A E = ∀ K H M.
II. *From the two points* D *&* K, *in* AD *&* HM, *there has been let fall the* ⊥ D E *&* K M, *on the planes* B A C *&* I H L.	
III. *From the points* E *&* M, *where the* ⊥ *meet those planes, there has been drawn* A E *&* M H, *to the vertices* A *&* H.	

Preparation.

1. Make A F = H K. — *P.* 3. *B.* 1.
2. Draw FG, plle. to D E, until it meets the plane BAC in G. — *P.*31. *B.* 1.
3. From the point G, in the plane B A C, draw C G, ⊥ to A C; & G B, ⊥ to A B. — *P.*12. *B.* 1.
4. From the point K in the plane I H L, draw I M, ⊥ to H I; & M L, ⊥ to H L. — *P.*12. *B.* 1.
5. Draw B F, B C & F C; I K, I L & L K. — *Pof.*1. *B.* 1.



DEMONSTRATION.

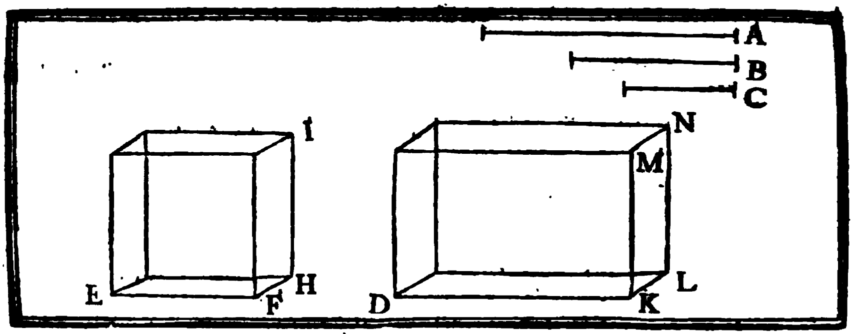
BECAUSE FG is plle. to DE which is ⊥ to the plane BAC. *(Hyp.* 1 1.)
1. The line GF is ⊥ to the same plane BAC. *P.* 8. *B.*11.
And the ∠ FGB, FGA & FGC are ∟. *D.* 3. *B.*11.
2. Consequently, the □ of AF is = to □ of FG + □ of GA. *P.*47. *B.* 1.
But the □ of AG is = to □ of AB + □ of BG. *(Prep.*3*).* & *P.*47. *B.* 1.
3. Therefore, the □ of AF is = to □ FG + □ AB + □ BG. *Ax.*1. *B.* 1.
But the □ GB + □ FG are = to the □ BF *(Prep.*3*).* *P.*47. *B.* 1.
4. Consequently, the □ AF is also = to the □ BF + □ AB.
5. Therefore, ∠ ABF, is a ∟. *P.*48. *B.* 1.
6. It may be demonstrated after the same manner that ∠ FCA, is a ∟.
7. That also the ∠ KIH & KLH, are ∟.
In the △ FCA & KLH; the line HK is = to AF *(Prep.* 1.) the ∠ ACF & KLH, are ∟ *(Arg.* 6. & 7.), & the ∠ FAC = ∠ KHL, *(Hyp.* 1). *P.*26. *B.* 1.
8. Therefore the sides AC & CF are = to the sides HL & LK, each to each.
9. Likewise AB is = to HI & BF = IK.
10. Consequently, in the △ BAC & IHL; the bases BC & IL are equal and the ∠ ACB & ABC = to the ∠ HLI & HIL, each to each. *P.* 4. *B.* 1.
Therefore if those equal ∠, be taken from the four ∟ ACG, ABG, HLM & HIM.
11. The remaining ∠ will be equal, *viz.* ∠ BCG = ∠ ILM & ∠ CBG = ∠ LIM. *Ax.*5. *B.* 1.
Since then the △ GBC & IML have their bases BC & IL equal (*Arg.* 10).
And the ∠ at those bases are equal, each to each, (*Arg.* 11).
12. The sides BG & CG will be = to the sides IM & ML. *P.*26. *B.* 1.
In the △ BAG & HIM, AB is = to HI (*Arg.* 9.) BG = IM, *(Arg.* 12.) & the ∠ ABG & HIM are ∟. *(Prep.* 3 & 4).

3. Consequently, A G = H M. *P.* 4. *B.* 1.
But the □ of A F (= □ A G + □ G F) (*Arg.* 2.) is = to the □ of H K (= □ H M + □ K M) (*Hyp.* 1. & *P.* 47. *B.* 1.) because A F is = H K. (*Prep.* 1).
If therefore from the □ A F be taken the □ G A, & from the □ H K, the □ H M = □ G A, (*Arg.* 13. & *P.* 46 *B.* 1. *Cor* 3).
14. The remainder, *viz.* the □ of G F will be = to the □ of K M. *Ax.* 3. *B.* 1.
15. Consequently, G F = K M (*Cor.* 3. *of P.* 46. *B.* 1).
Infine, because in the two △ A G F & H K M, the sides A F, A G & F G are = to the sides H K, H M & K M, each to each, (*Prep.* 1. & *Arg.* 13. & 15).
16. The ∀ F A G or D A E is = to the ∀ K H M. *P.* 8. *B.* 1.

Which was to be demonstrated.

C O R O L L A R Y.

If from the vertices A & H *of two equal plane angles* B A C & I H L, *there be elevated two equal straight lines* A F & H K; *containing with the respective sides, the* ∀ B A F & F A C *equal to the* ∀ I H K & K H L; *each to each, & there be let fall from those points* F & K *(of those elevated straight lines) the perpendiculars* F G & K M *on the planes* B A C & I H L: *those* ⊥ F G & K M *will be equal.* (*Arg.* 15).



PROPOSITION XXXVI. *PROBLEM XXXI.*

IF three ſtraight lines (A, B, C) be proportionals, the parallelepiped (D N), deſcribed from theſe three lines as its ſides, is equal to the equiangular parallelepiped (E I), deſcribed from the mean proportional (B).

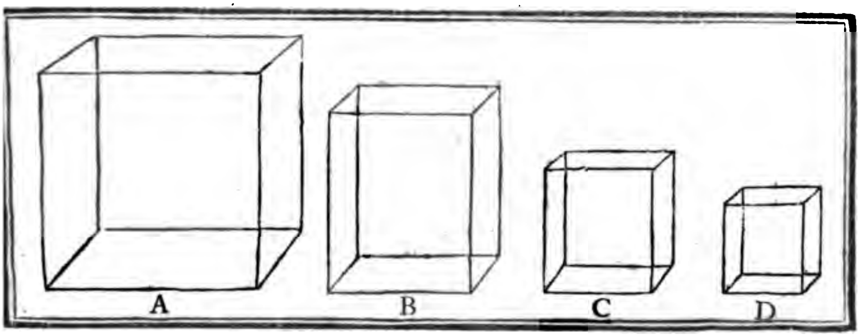
Hypotheſis.	Theſis.
I. The ſtraight lines A, B, *&* C *are proportionals*, that is, A : B = B : C.	*The* ▱ E I *is = to the* ▱ DN.
II. The ▱ DN, *is deſcribed from thoſe three lines*, that is, D K = A, M K = B, *&* K L = C.	
III. The equiangular ▱ E I, *is deſcribed from the mean proportional* B, that is, EF = FG = FH = B.	

Demonstration.

BECAUSE DK : EF = EF or FH : KL *(Hyp. 2)*.
And the plane ∀ EFH is = to the plane ∀ DKL *(Hyp. 3)*.

1. The pgr. D L, baſe of ▱ DN is = to the pgr. EH, baſe of ▱ EI *P. 14. B. 6.*
Moreover, the plane ∀ G F E & G F H contained by the elevated line F G, & the ſides E F & F H, being = to the plane ∀ M K D, & M K L, contained by the elevated line K M, & D K, & K L, each to each, *(Hyp. 3.)*, & F G = K M, *(Hyp. 2. & 3)*.
2. The ⊥ let fall from the point G, on the baſe E H, will be = to the ⊥ let fall from the point M on the baſe D L. *(Cor. of P. 35. B. 11)*.
3. Conſequently, ▱ E I has the ſame altitude with the ▱ D N. *D. 4. B. 6.*
But the baſe E H of ▱ E I is = to the baſe D L of ▱ D N, *(Arg. 1)*.
4. Therefore, ▱ E I is = to the ▱ D N. *P. 31. B. 11.*

Which was to be demonſtrated.



PROPOSITION XXXVII. *THEOREM XXXII.*

IF four ſtraight lines (A, B, C, & D) be proportionals, *(that is, if,* A : B = C : D) : the ſimilar and ſimilarly deſcribed parallelepipeds, from the two firſt (A & B), will be proportional to the ſimilar and ſimilarly deſcribed parallelepipeds, from the two laſt (C & D); and if the two ſimilar and ſimilarly deſcribed parallelepipeds, from the two lines (A & B); be proportional to the two other ſimilar and ſimilarly deſcribed parallelepipeds, from the two other ſtraight lines (C & D); the homologous ſides of the firſt (A & B), will be proportional to the homologous ſides (C & D) of the laſt.

Hypotheſis.

I. A : B = C : D.
II. From A & B *there has been deſcribed* ∽ ▱.
III. Alſo from C & D.

Theſis.

▱A : ▱B = ▱C : ▱D.

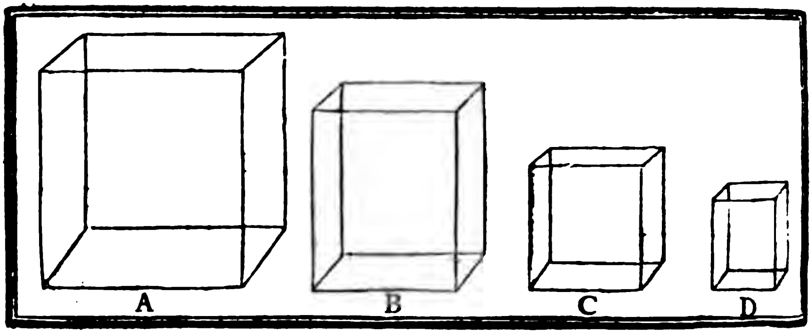
DEMONSTRATION.

BECAUSE the ▱A is ∽ to the ▱B *(Hyp. 2).*

1. The ▱A : ▱B = A^3 : B^3. { *P.*33. *B.*11.
2. Likewiſe, the ▱C : ▱D . C^3 : D^3. { *Cor.*

But the ratio of A to B being = to the ratio of C to D *(Hyp. 1).*

3. It follows, that three times the ratio of A to B is = to three times the ratio of C to D, *that is,* A^3 : B^3 = C^3 : D^3. *Ax.*6. *B.* 1.
4. Conſequently, the ▱A : ▱B = ▱C : ▱D. *P.*11. *B.* 5.



Hypothesis.

I. The ▱ A *is* ∾ *to the* ▱ B.
II. Also the ▱ C *is* ∾ *to the* ▱ D.
III. The ▱ A : ▱ B = ▱ C : ▱ D.

Thesis.

A : B = C : D

II. Demonstration.

BECAUSE the ▱ A is ∾ to the ▱ B *(Hyp. 1).*

1. The ▱ A : ▱ B = $A^3 : B^3$. *P. 33. B. 11.*

Likewise the ▱ C is ∾ to the ▱ D *(Hyp. 2).*

2. The ▱ C : ▱ D = $C^3 : D^3$. *P. 33. B. 11.*

But the ▱ A : ▱ B = ▱ C : ▱ D *(Hyp. 3).*

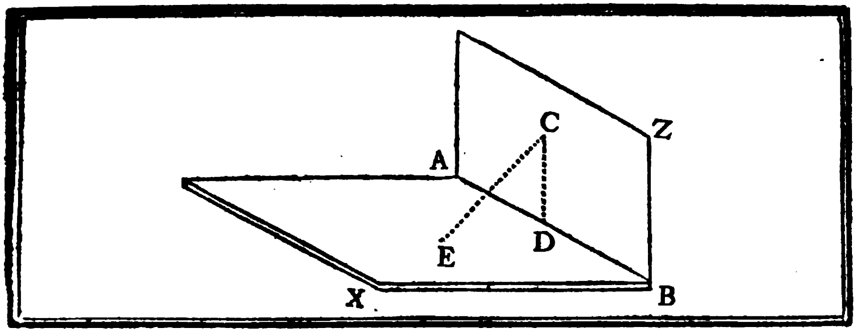
3. Therefore, $A^3 : B^3 = C^3 : D^3$. *P. 11. B. 5.*
4. Consequently, A : B = C : D. *Ax. 7. B. 1.*

Which was to be demonstrated.

REMARK.

I. *BECAUSE the triangular prism is the half of its parallelepiped (P. 28. B. 11.), it follows (Ax. 7. B. 1), that the same truth is applicable to similar triangular prisms.*

II. It may be also applied to similar polygon prisms; because they may be divided by planes into triangular prisms. (Remark 2. of P. 34. B. 11).



PROPOSITION XXXVIII. *THEOREM XXXIII.*

IF two planes (A Z & A X) be perpendicular to one another; and a straight line (C D) be drawn from the point (C) in one of the planes (A Z) perpendicular to the other (A X): this straight line shall fall on the common section (A B) of the planes.

Hypothesis.	Thesis.
The plane A Z *is* ⊥ *to the plane* A X.	*The line* C D *drawn from the point* C, *situated in the plane* AZ, ⊥ *to the plane* AX, *falls on the common section* A B.

Demonstration.

If not,
There may be drawn a ⊥ as C E, which will not fall on the common section A B.

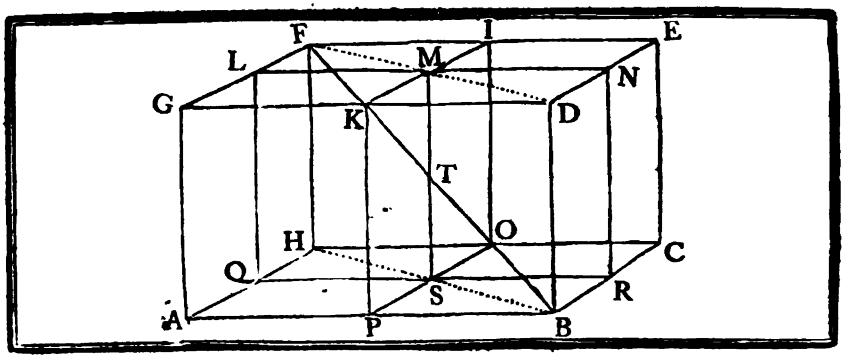
Preparation.

From the point C, let fall on A B, in the plane A Z, a ⊥ C D. *P.* 12. *B.* 1.

BECAUSE C D is ⊥ to the common section A B *(Prep)*.

1. C D will be ⊥ to the plane A X. *D.* 4. *B.* 11.
   But E C is ⊥ to the same plane. *(Sup.)*.
2. Therefore, from the same point C, there has been drawn to the plane A X, two ⊥ E C & C D.
3. Which is impossible. *P.* 13. *B.* 11.
4. Consequently, the ⊥ C D let fall from the point C, of the plane A Z, to the plane A X (which is perpendicular to it) passes thro' their common section A B.

Which was to be demonstrated.



PROPOSITION XXXIX. *THEOREM XXXIV.*

IN a parallelepiped (A E) if the ſides (G D, A B; G F, A H; F E, H C; E D, & B C) of the oppoſite planes, (F A & E B; F C & G B) be divided each into two equal parts, the common ſection (M S) of the planes (I P & L R), paſſing thro' the points of ſection (K, P, O, I & L, Q, R, N) and the diameter (F B) of the parallelepiped (A E) cut each other into two equal parts in the point (T).

Hypotheſis.

I. *In the* ▭ AE, *having for diam* FB; *the ſides* DG, AB, *&c. are biſected in the points* K, P, *&c.*

II. *The planes* KO *&* LR, *have been paſſed thro' the points,* K, P, O, I, *&* L, Q, R, N.

Theſis.

The common ſection MS *of thoſe planes, & the diam.* FB, *cut each other into two equal parts in the point* T.

Preparation.

Draw SB, SH, FM, & MD. *Poſ.*1. *B.* 1.

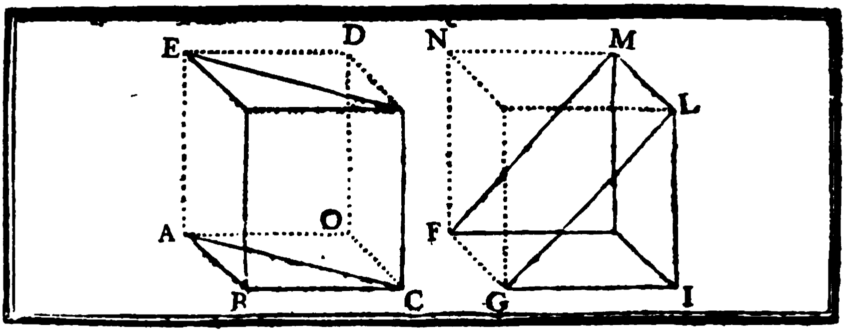
DEMONSTRATION.

THE ſides HQ & SQ being = to the ſides BR & SR (*Hyp.*1). *P.*34. *B.* 1.
And the ∀ HQS = ∀ SRB. *P.*29. *B.* 1.

1. The baſe HS of the △ HSQ will be = to the baſe SB of the △ BSR, & ∀ HSQ = ∀ RSB. *P.* 4. *B.* 1.
   But the ∀ RSH & HSQ together, are = 2 ∟. *P.*13. *B.* 1.
2. Conſequently, ∀ RSH + ∀ RSB = 2 ∟. *Ax.*1. *B.* 1.
3. Wherefore, ∀ HSB is a ſtraight line. *P.*14. *B.* 1.
4. It may be demonſtrated after the ſame manner, that FD is a ſtraight line.
   Moreover, BD being = & plle. to AG & AG = & plle. to FH. *P.*34. *B.* 1.
5. The line BD will be = & plle. to FH. { *P.* 9. *B.*11. *Ax.*1. *B.* 1.

6. And, consequently, F D is = & plle. to H B. *P.*33. *B.* 1.
7. From whence it follows, that F B & M S are in the same plane F D B H. *P.* 7. *B.*11.
But in the △ F M T, & T S B, the sides F M & S B are equal, (because the △ F M T is = & ∾ to the △ H S O, H S = S B), *(Arg.* 1*).* Moreover, ∠ S T B = ∠ F T M, & ∠ F M T = ∠ T S B. { *P.*15. *B.* 1. *P.*29. *B.* 1.
8. Therefore, M T = T S, & F T = T B *(P.* 26. *B.*1.*) that is*, the common section M S of the planes K O & L R, & the diameter F B of the parallelepiped, cut each other into two equal parts, in the point T.

Which was to be demonstrated.



PROPOSITION XL. *THEOREM XXXV.*

IF two triangular prisms (F L & E C) have the same altitude (L I & A E), and the base of one (as C L) is a parallelogram (F I), and the base of the other (E C) a triangle (A B C): if the parallelogram be double of the triangle, the first prism (L F) will be equal to the second (E C).

Hypothesis.	Thesis.
I. In the prisms F L *&* E C; *the alt.* L I *is* = *to the alt.* A E. *II. The base of the prism* L F *is a pgr.* F I, *& the base of the prism* E C *a* △ A B C. *III. The pgr.* F I *is double of the* △ A B C.	*The prism* F L *is* = *to the prism* E C.

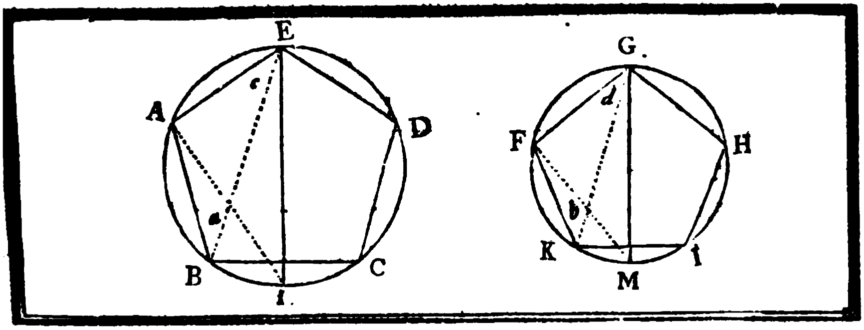
Preparation.

Complete the ▭ N I & B D.

Demonstration.

BECAUSE the pgr. F I, base of the prism F L, is double of the △ A B C, base of the prism E C (*Hyp.* 2. *&* 3).
And the pgr. B O is also double of the △ A B C. — *P.*41. *B.* 1.
1. The pgr. F I is = to the pgr. B O.
Moreover, the altitude L I being = to the altitude A F (*Hyp.* 1),
2. The ▭ B D is = to the ▭ N I. — *P.*31. *B.*11.
The given prism L F is the half of the ▭ N D. }
And the prism E C is the half of the ▭ B D. } — *P.*28. *B.*11.
3. Consequently, the prism F L is = to the prism E C. — *Ax.*7. *B.* 1.

Which was to be demonstrated.



PROPOSITION I. *THEOREM I.*

SIMILAR polygons (ABCDE & FGHIK), inscribed in circles are to one another as the squares of their diameters (EL & GM).

Hypothesis.	Thesis.
I. The polygons ABCDE *&* FGHIK *are* ∽.	*Polyg.* : ACE : *polyg.* FIH = *the* □ *of the diam.* EL : □ *of the diam.* GM,
II. They are inscribed in circles.	*or as diam.* EL^2 : *diam.* GM^2.

Preparation.

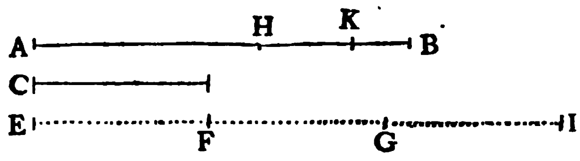
1. In the ⊙ ACD, draw AL, & BE, also diam. EL.
2. In the ⊙ FMH, draw the homologous lines FM & GK; also the diameter GM. } *Pos.* 1. *B.* 1.

DEMONSTRATION.

BECAUSE the polygons ABCDE & GFKIH are ∽ (*Hyp.* 1).
And the ∀ A or EAB is = to ∀ GFK, & AE : AB = FG : FK (*D.* 1. *B.* 6).

1. The △ ABE is equiangular with the △ FGK. *P.* 6. *B.* 6.
2. Wherefore, △ ABE is ∽ to △ GFK, & ∀ *a* = ∀ *b*, also ∀ *c* = ∀ *d*.
   But ∀ ELA is = ∀ EBA, or *a*, & ∀ GMF = ∀ GKF or *b*. *P.* 21. *B.* 6.
3. Consequently, ∀ ELA is = to ∀ GMF. *Ax.* 1. *B.* 1.
4. Likewise, ∀ EAL = ∀ GFM. *P.* 31. *B.* 3.
   And, because, in the two △ ALE & GFM, the two ∀ ELA & EAL of the first are = to the two ∀ GMF & GFM of the second (*Arg.* 3. *&* 4).
5. The third ∀ AEL of the △ EAL will be = to the third ∀ FGM of the △ FMG. *P.* 32. *B.* 1.
6. Therefore, EL : AE = GM : GF. *P.* 4. *B.* 6.
7. And alternando EL : GM = AE : GF. *P.* 16. *B.* 5.
   But AE & GF are homologous sides of the polygons ABD & FHK. Besides, EL & GM are the diameters of the ⊙ in which those polygons are inscribed.
8. Wherefore, polyg. ABCDE : polyg. FKIHG = EL^2 : GM^2. *P.* 22. *B.* 1.

Which was to be demonstrated.



LEMMA.

IF from the greater (A B), of two unequal magnitudes (A B & C), there be taken more than its half (viz. A H), and from the remainder (H B) more than its half (viz. H K), and so on: there shall at length remain a magnitude (K B), less than the least (C), of the proposed magnitudes.

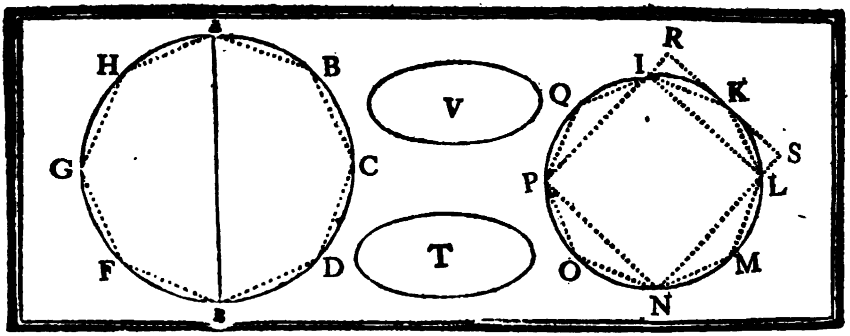
Preparation.

1. Take a multiple E I of the least C, which may surpass A B, & be > 2 C. *Pos. 1. B. 5.*
2. From A B, take a part H A > the half of A B. *Pos. 2. B. 5.*
3. From the remainder H B, take H K > the half of H B.
4. Continue to take more than the half from those successive remainders, until the number of times, be equal to the number of times, that C is contained in its multiple E I. *Pos. 2. B. 5.*

DEMONSTRATION.

BECAUSE the magnitude E I is a multiple greater than twice the least magnitude C *(Prep. 1)*.
If there be taken from it a magnitude G I = C.
1. The remainder E G will be > the half of E I.
But E I is > A B *(Prep. 1)*.
2. Consequently, the half of E I is > the half of A B. *P. 19. B. 5.*
3. Therefore, G E will be much > the half of A B.
But H B is < the half of A B *(Prep. 2)*.
4. Much more then G E is > H B.
5. Therefore, E F, the half of E G, is > the half of H B.
And K B is < the half of H B *(Prep. 3)*.
6. Consequently, E F is > K B.
And as the same reasoning may be continued until a part (E F) of the multiple of the magnitude C be attained, which will be equal to C *(Prep. 4)*.
7. It follows, that the magnitude C will be > the remaining part (K B) of the greater A B.

Which was to be demonstrated.



PROPOSITION II. *THEOREM II.*

CIRCLES (A F D & I L P), are to one another as the squares of their diameters (A E & I N).

Hypothesis.	Thesis.
In the circles A F D & I L P *there has been drawn the diameters* A E & I N.	⊙ A F D : ⊙ I L P = $AE^2 : IN^2$.

Demonstration.

If not,

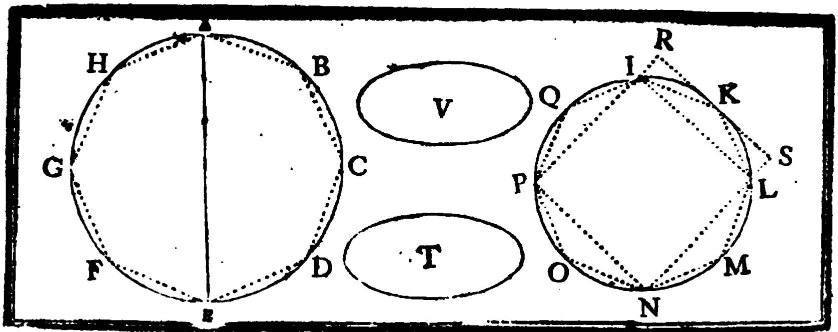
AE^2 is to IN^2 as the ⊙ A F D is to a space T (which is < or > the ⊙ I L P).

I. Supposition.

Let T be < ⊙ I L P by the space V. *that is,* T + V = ⊙ I L P.

I. Preparation.

1. In the ⊙ L I P describe the □ I L N P. *P.* 6. *B.* 4.
2. Divide the arches I L, L N, N P, & P I into two equal parts in the points K, M, O, & Q. *P.*30. *B.* 3.
3. Draw the lines I K, K L, L M, M N, N O, O P, P Q & Q I. *Pos.*1. *B.* 1.
4. Thro' the point K, draw S R plle. to L I. *P.*31. *B.* 1.
5. Produce N L & P I to R & S; which will form the rgle. S R I L.
6. Inscribe in the ⊙ A D F a polygon ∾ to the polygon of the ⊙ I L P.



BECAUSE the □ defcribed about the ⊙ I L P is > the ⊙ itfelf. *Ax.* 8. *B.* 1.

1. The half of this □ will be > the half of the ⊙ I L P. *P.* 19. *B.* 5.
   But the infcribed □ I L N P is = to half of the circumfcribed □ (the fide of the circumfcribed □ being = to the diameter, & the □ of the diameter = □ L I + □ L N = 2 □ L I). *P.* 47. *B.* 1.
2. Therefore, the □ L I P N is > the half of the ⊙ I L P. *Ax.* 1. *B.* 1.
   The rgle. S I is > the fegment L K I (*Prep.* 5. & *Ax.* 8. *B.* 1).
3. Confequently, the half of the rgle. S I is > the half of the fegment L K I. *P.* 19. *B.* 5.
   The △ L K I is = to half of the rgle. S I. *P.* 41. *B.* 1.
4. Therefore, the △ L K I is > the half of the fegment L K I. *P.* 19. *B.* 5.
5. It may be proved after the fame manner, that all the △ L M N, N O P, &c. are each > the half of the fegment in which it is placed.
6. Wherefore, the fum of all thofe triangles will be > the fum of the half of all thofe fegments.
   Continuing to divide the fegments K I, I L, &c. as alfo the fegments arrifing from thofe divifions.
   It will be proved after the fame manner.
7. That the triangles formed by the ftraight lines drawn in thofe fegments, are together > the half of the fegments in which thofe triangles are placed.
   Therefore, if from the ⊙ I L P be taken more than its half, viz. the □ I L N P, & from the remaining fegments (L K I, I Q P, &c.) be taken more than the half, & fo on.
8. There will at length remain fegments which together, will be < V. *Lem.* *B.* 12.
   But the ⊙ I L P is = T + V (1. *Sup.*).
   Therefore, taking thofe fegments L K I, &c. from the ⊙ I L P.
   And the fpace V, from T + V (which is > thofe fegments).
9. The remainder, viz. the polygon I K L M N O P Q will be > T. *Ax.* 5. *B.* 1.
   But the polyg. A D F K : polyg. I L O Q = □ of A E : □ of I N. *P.* 1. *B.* 12.

And the □ of A E : □ of I N = ⊙ A C E G : T. *(Sup.)*.
10. Therefore, the polyg. A D F H : polyg. I L O Q = ⊙ A C E G : T. *P.* 11. *B.* 5.
But the polygon A D F H is < ⊙ A C E G. *Ax.* 8. *B.* 1.
11. Consequently, the polygon I L O Q is < T. *P.* 14. *B.* 5.
But the polygon I L O Q is > T. *(Arg. 9)*.
12. Therefore, T will be > & < the polyg. I L O Q *(Arg. 9. & 11)*.
13. Which is impossible.
14. Therefore, T is not < ⊙ I L P.
15. From whence it follows, that the □ of the diameter (A E) of a ⊙ (A C E G), is not to the □ of the diameter (I N) of another ⊙ (I L P), as the first ⊙ (A C E G) to a space < the second ⊙ (I L P).

II. Supposition.

Let the space T be > the circle I L P.

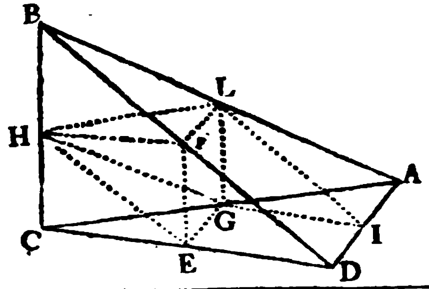
II. Preparation.

Take a space V, such that
T : ⊙ A C E G = ⊙ I L P : V.

BECAUSE the □ of A E : □ of I N = ⊙ A C E G : T.
16. Invertendo T : ⊙ A C E G = □ of I N : □ of A E. { *P.* 4. *B.* 5. *Cor.*
But T : ⊙ A C E G = ⊙ I L P : V. *(II. Prep.)*.
Moreover, T is > ⊙ I L P. *(II. Sup.)*.
17. Consequently, the ⊙ A C E G is also > V. *P.* 14. *B.* 5.
Besides T : ⊙ A C E G = □ of I N : □ of A E *(Arg. 16)*.
And T : ⊙ A C E G = ⊙ I L P : V. *(II. Prep.)*.
18. Therefore, the □ of I N : □ of A E = ⊙ I L P : V. *P.* 11. *B.* 5.
But V < ⊙ A C E G. *(Arg. 17)*.
And it has been demonstrated *(Arg. 15)*, that the □ of the diameter (I N) of a ⊙ (I L P), is not to the □ of the diameter of another ⊙ (A C E G); as the first ⊙ (I L P) to a space < the second ⊙ (A C E G).
19. Consequently, V is not < the ⊙ I L P.
20. Therefore, T is not > the ⊙ I L P.
Therefore, the space T being neither < nor > the ⊙ I L P, *(Arg. 14. & 19)*.
21. T will be = to this ⊙ I L P.
22. Consequently, the ⊙ A C E G : ⊙ I L P = □ of A E : □ of I N. *P.* 7. *B.* 1.
Which was to be demonstrated.

COROLLARY.

CIRCLES *are to one another as the polygons inscribed in them* (P. 1. B. 12. & P. 11. B. 5).



PROPOSITION III. THEOREM III.

EVERY pyramid (A B C D) having a triangular base (A C D), may be divided into two equal and similar prisms, (I D E F L G & G L F H C E), and into two equal and similar pyramids, (L G I A & L F H B), which are similar to the whole pyramid; and the two prisms together are greater than half of the whole pyramid (A B C D).

Hypothesis.	Thesis.
A B C D *is a pyramid whose base* A D C *is a* △.	I. *The part* I D E F L G *is a prism* = & sim. *the part* G L F E C H.
	II. *The part* A L G I *is a pyramid* = & sim. *the part* B L F H.
	III. *Those pyramids* A L G I & B L F H *are sim. to the pyramid* A B C D.
	IV. *The prisms* I D E F L G & G L F C H *are together* > *than the half of the pyr.* A B C D.

I. Preparation.

1. Cut all the sides of the pyramid A B C D into two equal parts, in the points L, F, H, E, G, & I. *P.10. B. 1.*
2. Draw the lines L F, F H, F E, G E, G I & I L, also L G, & L H. *Pof.1. B. 1.*

Demonstration.

BECAUSE in the △ B C D the sides B D & B C are divided into two equal parts in the points F & H (*Prep.* 1).

1. B H : H C = B F : D F. *P.19. B. 5.*
3. Consequently, F H is plle. to D C. } *P. 2. B. 6.*
3. Likewise, F E is plle. to B C. }
4. Therefore, F E C H is a pgr. *D.35. B. 1.*
5. It may be proved after the same manner, that L F E G & L G C H are pgrs.

And since F H & H L are plle. to E C & G C. (*Arg.* 2. & 5).

6. The planes passing thro' L F H & E C G will be plle. *P.15. B.11.*
7. Therefore, L G E C H F will be a prism. } *D.13. B.11.*
8. Likewise, L F E D I G will be also a prism. }

But thoſe two priſms have the ſame altitude LG, & the pgr. GIDE which is the baſe of the priſm L D is double of the △ C E G, baſe of the priſm L C. *P.41. B. 1.*

9. Therefore, the priſm L D is = to the priſm L C. *P.40. B.11.*

Which was to be demonſtrated. I.

BECAUSE the ſide B D is cut into two equal parts in F, that F E & DE are plle. to B C & F H, each to each. *(Prep. 1. & Arg. 2. & 3).* *P.26. B. 1.*

10. The △ F D E is = & ∾ to △ B F H. *P. 7. B. 6.*

11. The △ F E D & I L G are alſo equal. *D.13. B.11.*

12. Therefore, △ B F H = △ L I G. *Ax.1. B. 1.*

And ſince the other ſides of the pyramid A B C D are divided into two equal parts.

It may be eaſily proved that,

13. △ B L F is = to the △ L A I, △ B L H = △ A G L, & △ L F H = △ A G I.

14. From whence it follows, that thoſe parts B L H F & A L G I are equal & ∾ pyramids. *D.10. B.11.*

Which was to be demonſtrated. II.

THE line F H, is plle. to D C. *(Arg. 2).*

15. Therefore, △ B F H is ∾ △ B D C. *P. 2. B. 6.*

Likewiſe, all the triangles which form the pyramids BLHF & ALGI are ∾ to all the triangles of the whole pyramid A B C D.

16. Therefore, the pyramids B L H F & A L G I, are ∾ to the pyramid A B C D.

Which was to be demonſtrated. III.

II. Preparation.

Draw G H & E H.

THE line B H being = to H C *(I. Prep. 1.)* F H = E C *(Arg. 4.)* & ∀ E C H = ∀ F H B *(P. 29. B. 1).*

17. Conſequently, the △ E C H is = to the △ B F H. *P. 4. B. 1.*

18. Alſo the △ H G C & G E C are = & ∾ to the △ B L H & L H F. *P. 4. B. 1.* *D.13. B.11.*

19. Therefore, the pyramid L F H B is = to the pyramid H G E C. *D.10. B.11.*

But the pyramid E C H G is only a part of the priſm E C H F L G.

20. Therefore, the priſm E C H F L G is > the pyramid E C H G. *Ax.8. B. 1.*

21. Conſequently, this priſm E C H F L G is alſo > the pyramid LFHB. *P. 7. B. 5.*

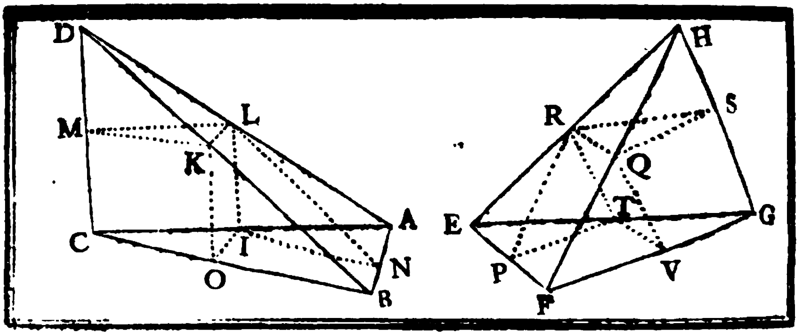
The priſm L G E C H F is = to the priſm E F L G I D, & the pyramid L F H B = to the pyramid A I G L *(Arg. 9. & 14).*

22. Therefore, the priſm E F L G I D is alſo > the pyramid A I G L.

23. Therefore, the two priſms E C H F L G & E F L G I D together, will be > the two pyramids B L F H & L A I G together. *Ax.4. B. 1.*

24. From whence it follows, that the two priſms E C H F L G & E F L G I D together, are > the half of the given pyr. A B C D.

Which was to be demonſtrated. IV.



PROPOSITION IV. *THEOREM IV.*

IF there be two pyramids (A BCD & EFGH) of the fame altitude, upon triangular bafes (A B C & E F G), and each of them be divided into two equal pyramids fimilar to the whole pyramid, (viz. the pyramid ABCD into the pyramids DLKM & ANIL, and the pyramid EFGH into the pyramids HRQS & REPT); and alfo into two equal prifms, (viz. the pyramid ABCD into the prifms LB & LC, and the pyramid EFGH into the prifms RF & RG); and if each of thefe pyramids (DLKM, ANIL, HRQS, & REPT) be divided in the fame manner as the firft two, and fo on. The bafe (A B C), of one of the firft two pyramids (A B C D), is to the bafe (E F G) of the orher pyramid (E F G H), as all the prifms contained in the firft pyramid (A B C D), is to all the prifms contained in the fecond (E F G H), that are produced by the fame number of divifions.

Hypothefis.	Thefis.
I. *The triangular pyramids* ABCD & EFGH, *have the fame altitude.* II. *Each of them are cut into two equal prifms* LB & LC; *alfo* RF & RG, & *into two equal pyramids fimilar to the whole pyramid.* III. *Each of thofe pyramids* LDMK, LNIA, RTPE & RQSH, *are fuppofed to be divided in the fame manner as the firft two,* & *fo on.*	*The fum of all the prifms contained in the pyramid* ABCD *is to the fum of thofe contained in the pyramid* EFGH, *being equal in number; as the bafe* ABC, *of the pyramid* ABCD *is to the bafe* EFG, *of the pyramid* EFGH.

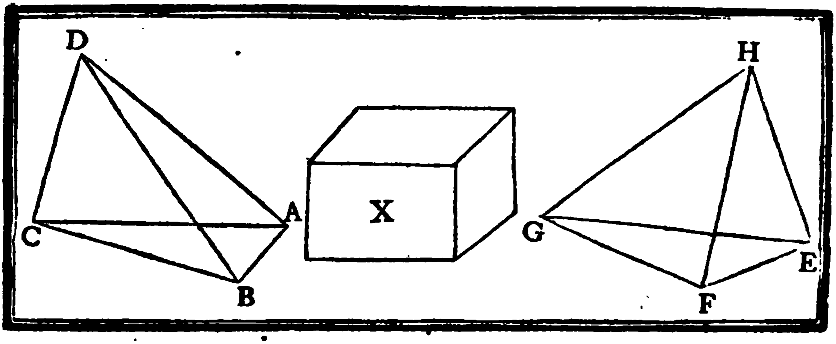
DEMONSTRATION.

BECAUSE the pyramids ABCD & EFGH have equal altitudes, & the prifms LB, LC, RF & RG have each the half of this altitude, *(Hyp. 1. & P. 3. B. 12).*

1. Thofe prifms LB, LC, RF & RG have the fame altitude. *Ax. 7. B. 1.*
The lines BC & FG are cut into two equal parts in the points O & V. *P. 3. B. 12.*

2. Therefore, C B : C O = G F : G V. { P.19. B. 5. P.16. B. 5.
3. Consequently, △ A B C : △ I O C = △ E F G : △ T V G. P.22. B. 6.
4. And alternando △ A B C : △ E F G = △ I O C : △ T V G. P.16. B. 5.
5. Moreover, base I O C : base T V G = prism L K M C O I : prism R Q S G V T. { Cor. 3. Rem. of P. 35. B. 11.
6. And prism L K O B N I : prism L K M C O I = prism R Q V F P T : prism R Q S G V T (having the same altitude *(Arg. 1.)* & being equal taken two by two *(Hyp. 11)*. P. 7. B. 5.
7. Consequently, prism L B + prism L C : prism L C = prism R F + prism R G : prism R G. P.18. B. 5.
8. And alternando, prism L B + prism L C : prism R F + prism R G = prism L C : prism R G. P.16. B. 5.
But prism L C : prism R G = base I O C : base T V G *(Arg. 5)*.
And base I O C : base T V G = base A B C : base E F G *(Arg. 4)*.
9. Therefore, the prism L B + pr. L C : pr. R F + pr. R G = base A B C : base E F G. P.11. B. 5.
If the remaining pyramids L K M D & L I N A ; also R Q S H & E P T R, be divided after the same manner as the pyramids A B C D & E F G H : it may be proved after the same manner.
10. That the four pyramids resulting from the first pyramids L K M D & A N I L, will have the same ratio to the four prisms resulting from the last R Q S H & E P T R, that the bases L K M & A N I have to the bases R Q S & E P T *(Hyp. 111. & Arg. 9)*.
And it has been demonstrated, that the bases L K M & A N I, are each = I O C; also R Q S & E P T, each = T V G.
Moreover, △ A B C : △ E F G = △ I O C : △ T V G *(Arg. 4)*.
11. Wherefore, the sum of all the prisms contained in the pyramid A B C is to the sum of all the prisms contained in the pyramid E F G H, as the base A B C is to the base E F G. P.12. B. 5.

Which was to be demonstrated.



PROPOSITION V. *THEOREM V.*

PYRAMIDS (A B C D & E F G H) of the same altitude, which have triangular bases (A B C & E F G): are to one another as their bases, (A B C & E F G).

Hypothesis.	Thesis.
I. *The pyramids* A B C D & E F G H *have for bases the* △ A B C & E F G.	*Pyram.* A B C D : *pyram.* E F G H = *base* A B C : *base* E F G.
II. *They have the same altitude.*	

Demonstration.

If not,

Pyramid A B C D : pyramid E F G H > base A B C : base E F G.

Preparation.

1. Take a solid X which may be > the pyramid A B C D, so that X : pyram. E F G H = base A B C : base E F G.
2. Divide the pyramids A B C D & E F G H as directed in *P.* 3. *B.* 12.

BECAUSE the two prisms resulting from the first division, are > the half of the pyramid A B C D; & the four following, resulting from the second division, are > than the halves of the pyramids resulting from the first division, & so on. *P.* 3. *B.* 12.

1. It is evident, that the sum of all the prisms contained in the pyramid A B C D, will be > the solid X, which was supposed to be < the pyramid A B C D. *Lem. B.* 12.

But all the prisms contained in the pyramid A B C D, are to all the prisms contained in the pyramid E F G H, as the base A B C is to the base E F G. *P. 4. B.12.*

And the solid X : pyramid E F G H = base A B C : base E F G *(Prep. 1).*

2. Consequently, all the prisms contained in the pyramid A B C D are to all the prisms contained in the pyramid E F G H, as the solid X is to the pyramid E F G H. *P.11. B. 5.*

But all the prisms contained in the pyramid A B C D, are > the solid X. *(Arg. 1).*

3. Therefore, all the prisms contained in the pyramid E F G H, are > the pyramid E F G H itself. *P.14. B. 5.*

4. Which is impossible. *Ax.8. B. 1.*

5. Consequently, a solid (as X) which is < the pyramid A B C D, cannot have the same ratio to the pyramid E F G H, which the base A B C, has to the base E F G.

And as the same demonstration holds for any other solid greater than the pyramid A B C D.

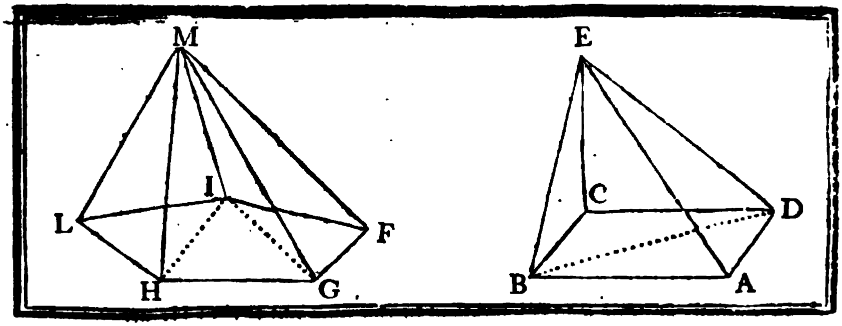
6. It follows, that the pyramid A B C D : pyramid E F G H = base A B C : base E F G.

COROLLARY I.

P*YRAMIDS of the same altitude, & which have equal triangles for their bases : are equal.* (P. 14. & 16. B. 5.).

COROLLARY II.

E*QUAL pyramids which have equal triangles for their bases : have the same altitude.*



PROPOSITION VI. *THEOREM VI.*

PYRAMIDS (FGLIM & ABCDE) of the same altitude, which have polygons (FGHLI, & ABCD) for their bases: are to one another as their bases.

Hypothesis.	Thesis.
I. The pyramids FGHLI *&* ABCD, *have polygons for their bases.*	*Pyram.* MFGHLI : *pyram.* ABCDE = *base* FILHG : *base* ABCD.
II. They have the same altitude.	

Preparation.

1. Divide the bases FILHG & ABCD into triangles, by drawing the lines GI, FH; & DB.
2. Let planes be passed thro' those lines & the vertices of the pyramids, which will divide each of those pyramids into as many pyramids as each base contains triangles.

DEMONSTRATION.

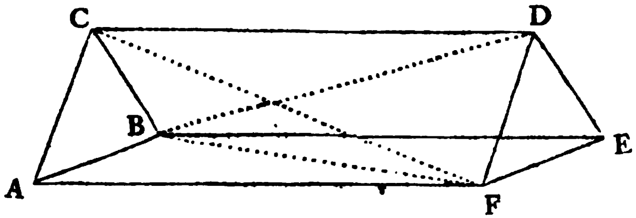
BECAUSE the triangular pyramids ILHM & ABDE have the same altitude. *(Hyp. II. & Prep. 2).*

1. The pyramid IHLM : pyr. ABDE = base HIL : base ABD. } P. 5. B.12.
2. Likewise, pyr. GIHM : pyr. ABDE = base HIG : base ABD. }
3. Consequently, pyr. IHLM + pyr. GIHM : pyr. ABDE = base HIL + base HIG : base ABD. *P.24. B. 5.*
4. Moreover, pyr. FIGM : pyr. ABDE = base FIG : base ABD. *P. 5. B.12.*
5. Therefore, pyr. IHLM + pyr. GIHM + pyr. FIGM : pyr. ABDE = base HIL + base HIG + base FIG : base ABD. *P.24. B. 5.*

   But pyr. IHLM + pyr. GIHM + pyr. FIGM are = to the pyr. MFGHLI, & the base HIL + base HIG + base FIG = base FILHG. *Ax.1. B. 2.*
6. Consequently, pyr. MFGHIL : pyr. ABDE = base FILHG : base ABD. *P. 7. B. 5.*

   It may be proved after the same manner, that
7. Pyr. MFGHLI : pyr. BDCB = base FILHG : base BDC.
8. Therefore, pyr. MFGHLI : pyr. ABCDE = base FILHG : base ADCB. *P.25. B. 5.*

Which was to be demonstrated.



PROPOSITION VII. *THEOREM VII.*

EVERY triangular prifm (A D E) : may be divided (by planes paffing hrough the △ B C F & B D F) into three pyramids (A C B F, B D E F & D C B F) that have triangular bafes, and are equal to one another.

Hypothefis.	Thefis.
The given prifm A D E *has a triangular bafe.*	*The prifm* A D E *may be divided into three equal triangular pyramids,* A C B F, B D E F, D C B F.

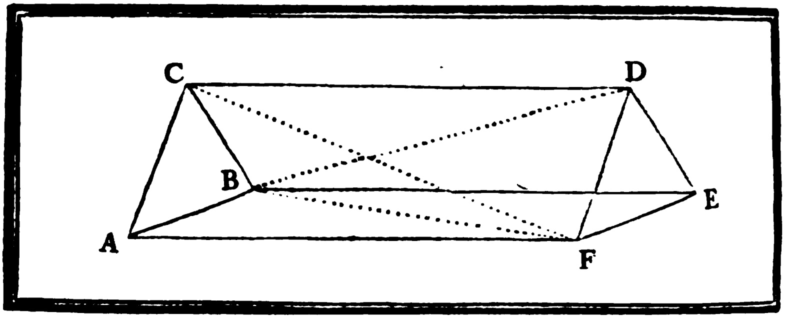
Preparation.

1. In the pgr. D A draw any diagonal C F.
2. From the point F in the pgr. A E, draw the diag. B F. } *Pof.* 1. *B.* 1.
3. From the point B in the pgr. C E, draw the diag. B D.
4. Let a plane be paffed thro' C F & B F, alfo thro' B F & B D.

DEMONSTRATION.

BECAUSE A D is a pgr. cut by the diagonal C F. *(Prep. 1).*

1. The △ A C F bafe of the pyramid A B C F is = to the △ C F D, bafe of the pyramid B C F D. *P.* 34. *B.* 1.
   But thofe pyramids A B C F & B C F D, have their vertices at the point B.
2. Therefore, the pyramid A B C F is = to the pyramid B C F D. { *P.* 5. *B.* 12. *Cor.* 1.
   Likewife, the pgr. E C is cut by its diagonal B D. *(Prep. 3).*
3. Therefore, the △ C B D, bafe of the pyramid B C F D is = to the △ B D E, bafe of the pyramid D E F B. *P.* 34. *B.* 1.
   And thofe pyramids B C F D, &c. have their vertices at the point F.
4. Confequently, the pyramid B C D F is = to the pyramid B D E F. { *P.* 5. *B.* 12. *Cor.* 1.
   But the pyramid A B C F is alfo = to the pyramid B C D F. *(Arg. 2).*
5. Therefore, the pyramids A B C F, B C D F, & B D E F are equal. *Ax.* 1. *B.* 1.



6. Consequently, the triangular prism (ADE) may be divided into three triangular pyramids.

Which was to be demonstrated.

COROLLARY I.

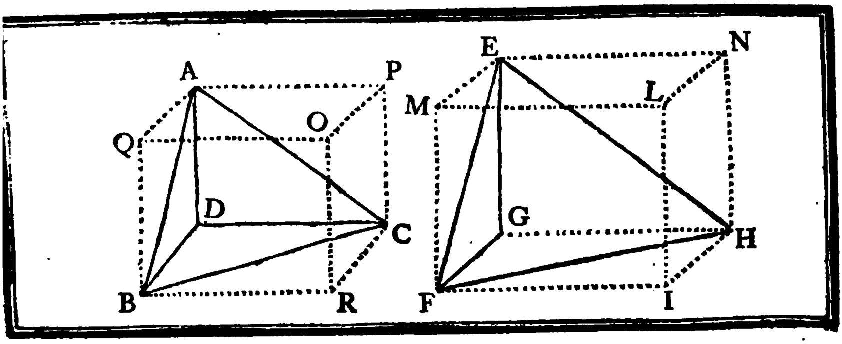
FROM this it is manifest, that every pyramid which has a triangular base, is the third part of a prism which has the same base, & is of an equal altitude with it.

COROLLARY II.

EVERY pyramid which has a polygon for base, is the third part of a prism which has the same base, & is of an equal altitude with it; since it may be divided into prisms having triangular bases.

COROLLARY III.

PRISMS of equal altitudes are to one another as their bases, because pyramids upon the same bases, & of the same altitude, are to one another as their bases. (P. 6. B. 12).



PROPOSITION VIII. *THEOREM VIII.*

SIMILAR pyramids (A B C D & E F G H) having triangular bases (B D C & F G H) : are to one another in the triplicate ratio of that of their homologous sides.

Hypothesis.	Thesis.
The ∾ pyramids A B C D & E F G H *have triangular bases* D B C & G F H, *whose homologous sides are* B D & F G, &c.	*The pyramid* ABCD *is to the pyramid* EFGH, *in the triplicate ratio of* BD *to* F G, that is, *as* D B³ : F G³.

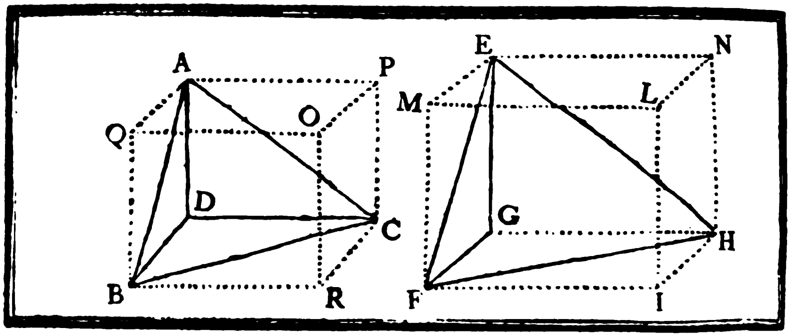
Preparation.

1. Produce the planes of the △ B D C, A B D & A D C; complete the pgrs. D R, D Q & D P. *P.*31. *B.* 1.
2. Draw P O & O Q plle. to A Q & A P, & produce them to O. *P.*31. *B.* 1.
3. Join the points O & R; & Q C will be a ▱ which will have the same altitude with the pyramid A B C D.
4. After the same manner describe the ▱ M H.
5. Infine, Join the points Q & P, also M & N, homologous to the points B & C; also F & H.

Demonstration.

BECAUSE the pyramids A B C D & E F G H are ∾ *(Hyp.)*.

1. All the triangular planes which form the pyramid A B C D are ∾ to all the triangular planes which form the pyramid E F G H, each to each. *D.* 9. *B.*11.
2. Consequently, A D : B D = E G : G F, &c. *D.* 1. *B.* 6.
3. And the plane ∀ A D B is = to the plane ∀ E G F. *P.* 5. *B.* 6.
4. Therefore the pgr. D Q is ∾ to the pgr. M G. *D.* 1. *B.* 6.
5. Likewise, the pgr. D R & G I; D P, & G N are ∾; as also their opposite ones A O, E L; Q R, M I. *P.*24. *B.*11.

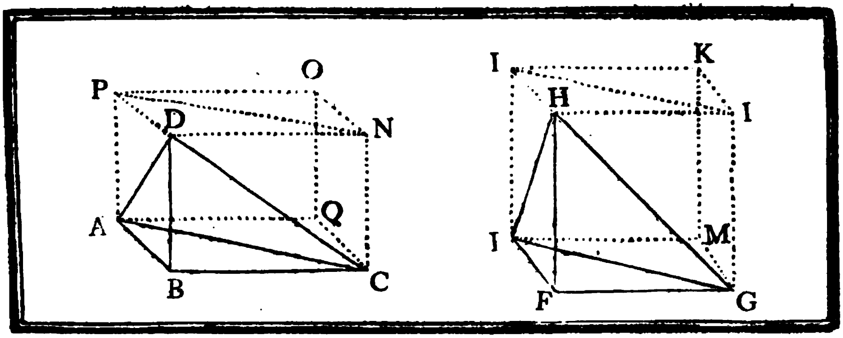


6. Consequently, A R & E I are ∽ ▱. *D. 9. B. 11.*
7. Therefore, ▱ A R : ▱ E I = D B³ : F G³. *P. 33. B. 11.*
And since the lines Q P & B C ; M N & F H, are diagonals similarly drawn in the equal & plle. pgrs. O A & R D ; E L & I G. *(Prep. 5).*
8. The parts B Q A P C D & F M E N H G will be ∽ prisms : & each equal to the half of its ▱. *D. 9. B. 11. P. 28. B. 11.*
9. Consequently, the prism B P Q C : prism FNMH = BD³ : FG³. *P. 15. B. 5. P. 34. B. 11. Rem. 1.*
But the pyramid A B D C is the third part of the prism B Q P C, & the pyramid E F G H is the third part of the prism F M N H. *P. 7. B. 12. Cor. 1.*
10. Therefore, the pyramid ABCD : pyramid EFGH = BD³ : FG³. *P. 15. B. 5.*

Which was to be demonstrated.

COROLLARY.

FROM this it is evident, that similar pyramids which have polygons for their bases, are to one another in the triplicate ratio of their homologous sides, (because they may be divided into triangular pyramids ; which are similar, taken two by two.



PROPOSITION IX. *THEOREM IX.*

THE bases (A B C & E F G), and altitudes (B D & F H), of equal pyramids, (A B C D & E F G H), having triangular bases, are reciprocally proportional, (*that is*, the base A B C : base E F G = altitude F H : altitude B D), and triangular pyramids (A B C D & E F G H), of which the bases (A B C & E F G), and altitudes (B D & F H), are reciprocally proportional: are equal to one another.

Hypothesis.	Thesis.
I. *The pyram.* ABCD & EFGH *are triangular.*	*Base* ABC : *base* EFG = *altitude*
II. *The pyram.* ABCD *is = to the pyram.* EFGH.	F H : *altitude* B D.

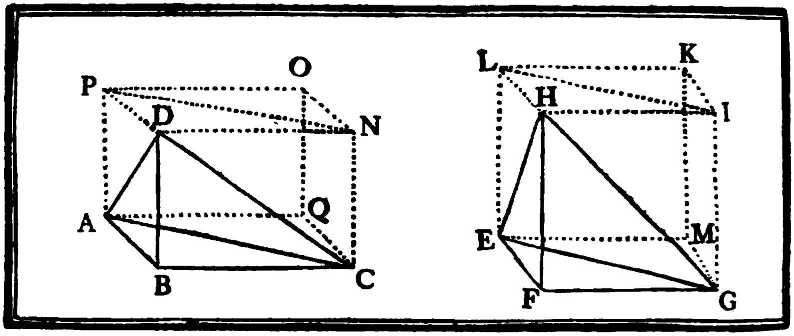
Preparation.

Complete the ▱ B O & F K having the same altitude with the pyramids A B C D & E F G H; as also the prisms B A P N C & F E L I G.

I. Demonstration.

BECAUSE the prisms P N B & L I F, have the same base & altitude with the given pyramids A B C D & E F G H. *(Prep).*

1. Each prism will be triple of its pyramid, (*that is*, the prism P N B triple of the pyramid A B C D, & the prism L I F triple of the pyramid E F G H). { *P.* 7. *B.*12. *Cor.* 1.
2. Consequently, the prism P N B is = to the prism L I F. *Ax.*6. *B.* 1.
   But the ▱ B O is double of the prism P N B, & the ▱ F K double of the prism L I F. *P.*28. *B.*11.
3. Therefore, the ▱ B O is = to the ▱ F K. *Ax.*6. *B.* 1.
   But the equal ▱ (B O & F K) have their bases and altitudes reciprocally proportional (*that is*, base B Q : base F M = altitude F H : altitude B D).
   And those ▱ are each sextuple of their pyramids, (*that is*, the ▱ B O is = six pyramids A B C D, & the ▱ K F = six pyramids E F G H. *Arg.* 1. & 3.)



Moreover, the base of the pyramid A B C D is the half of the base of the ▱ B O. And the base of the pyramid E F G H is the half of the base of the ▱ F K. P.41. B. 1.

4. Consequently, base A B C : base E F G = alt. F H : alt. B D. P.15. B. 5. P.11. B. 5.

Which was to be demonstrated.

Hypothesis.	Thesis.
I. The pyramids ABCD *&* EFGH *are triangular.*	*The triangular pyramid* ABCD *is* = *to the triangular pyramid* EFGH.
II. Base ABC : *base* EFG = *alt.* FH : *alt.* BD.	

II. Demonstration.

BECAUSE the △ A B C : △ E F G = F H : B D. *(Hyp. 2).*
And the pgr. B Q is double of the △ A B C, the pgr. F M double of the △ E F G. P.41. B. 1.

1. It follows, that the pgr. B Q : pgr. F M = F H : B D. P.15. B. 5.
But ▱ B O has for base the pgr. B Q, & for alt. B D. And ▱ F K has for base the pgr. F M, & for alt. F H. *(Prep.).*

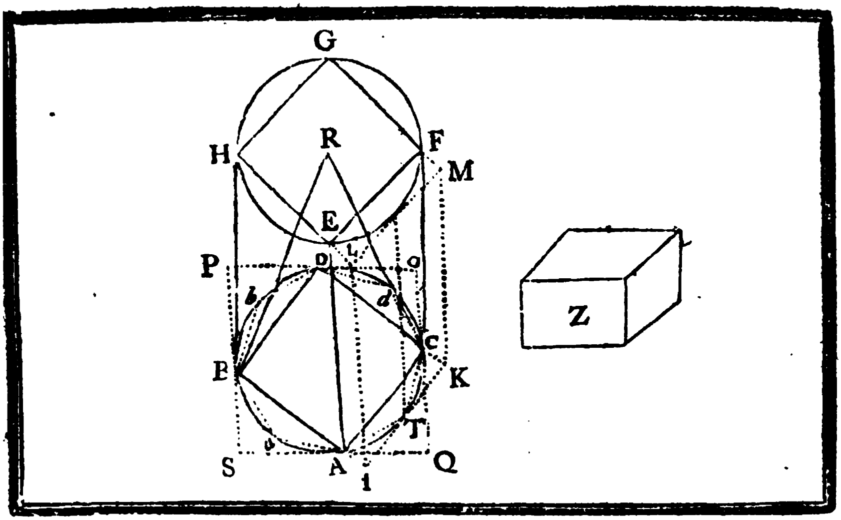
2. Consequently, the ▱ B O is = to the ▱ F K. P.34. B.11.
But the ▱ B O & F K are each double of the prisms P N B & L I F. P.18. B.11.
And those prisms P N B & L I F are each triple of their pyramids A B C D & E F G H. P. 7. B.12. Cor. 1.

3. Therefore, the triangular pyramid A B C D is = to the triangular pyramid E F G H. Ax.7. B. 1.

Which was to be demonstrated.

COROLLARY.

EQUAL polygon pyramids have their bases and altitudes reciprocally proportional; & polygon pyramids whose bases & altitudes are reciprocally proportional: are equal.



PROPOSITION X. *THEOREM X.*

EVERY cone (B R C) is the third part of the cylinder (H G F E A B D C) which has the same base, (B D C A) and the same altitude (B H) with it.

Hypothesis.
The cone B R C, *& the cylinder* H F A D C, *have the same base* B D C A, *& the same altitude* B H.

Thesis.
The cone B R C *is equal to the third part of the cylinder* H F C A B D.

Demonstration.

If not,

The cone will be $<$ or $>$ the third part of the cylinder, by a part $=$ Z.

I. Supposition.

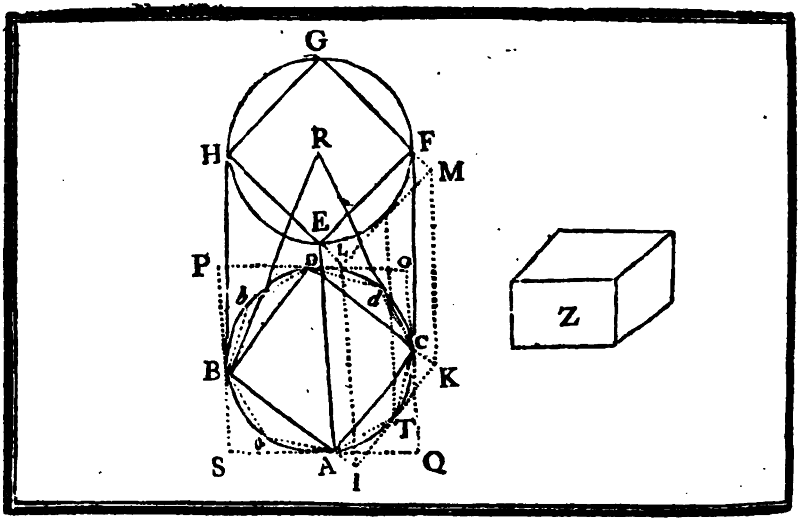
Let the third part of the cylinder H C be $=$ cone B R C $+$ Z.

I. Preparation.

1. IN the base A B D C of the cone & cylinder, describe the □ A B D C. *P. 6. B. 4.*
2. About the same base describe the □ P O Q S. *P. 7. B. 4.*
3. Upon those squares erect two ▭, the first ▭ F H B C, upon the inscribed □, & the second, on the circumscribed □, which will touch the superior base with its plle. planes, in the points H, G, F, & E, * having the same altitude with the cylinder, & the cone.

S-f

* *We have omitted a part of the preparation in the figure to avoid confusion.*



4. Bisect the arches A T C, C *d* D, D *b* B, & B *a* A, in T, *d*, *b*, & *a*. *P.*30. *B.* 3.
5. Draw A T, & T C, &c. *Pos.*1. *B.* 1.
6. Thro' the point T, draw the tangent ITK, which will cut BA & D C produced, in the points I & K & complete the pgr. A K. *P.*17. *B.* 3.
7. Upon the pgr. AK, erect the ▭ ALFK, & upon the △ AIT, T A C, & T C K the prisms E T I, E T F, & T F K, having all the same altitude with the cylinder & cone.
8. Do the same with respect to the other segments A *a* B, B *b* B, &c.

BECAUSE the □ P O Q S is described about, & the □ B D C A described in the ⊙. *(Prep.* 1. *&* 2*).*
1. The □ P O Q S is double of the □ B D C A. *P.*47. *B.* 1.
And the ▭ described upon those squares having the same altitude, *(Prep.* 3*).*
2. Therefore, the ▭ upon POQS is double of the ▭ upon BDCA. *P.*32. *B.*11.
But the ▭ upon POQS is > the given cylinder. *Ax.*8. *B.* 1.
3. Therefore, the ▭ upon BDCA is > the half of the same cylinder. *P.*19. *B.* 5.
And since the △ T A C is the half of the pgr. A K. *P.*41. *B.* 1.
4. The prism E T F, described upon this △ T A C, will be the half of the ▭ upon the pgr. A K. *P.*28. *B.*11. *P.*34. *B.*11.
The ▭ described upon the pgr. A K is > the element of the cylinder, which has for base the segment A T C. *Rem.*1. *Cor.*3. *Ax.*8. *B.* 1.
5. Consequently, prism E T F described upon △ T A C is > half of the element of the cylinder which has for base segment A T C. *P.*19. *B.* 5.
6. Likewise, all the other prisms described after the same manner, will be > the half of the corresponding parts or elements of the cylinder. Therefore, there may be taken from the whole cylinder more than the half, (viz. the ▭ upon the □ BDCA), & from those remaining elements (viz. C F E A T, &c.) more than the half; (viz. the prisms E T F, &c.), & so on.

7. Until there remains several elements of the cylinder which together will be < Z.. *Lem. B.12.*
But the cylinder is = to three times the cone B R C + Z. *(Sup.)*.
Therefore, if from the whole cylinder be taken those elements *(Arg. 7.)*.
And from three times the cone B R C + Z, the magnitude Z.
8. The remaining prism (viz. that which has for base the polygon A *a* B *b* D *d* C T) will be > the triple of the cone. *Ax.4. B. 1.*
But this prism is the triple of the pyramid of the same base & altitude (viz. of the pyramid T A *a* B *b* D *d* C T R). *P. 7. B.12. Cor. 2.*
9. Consequently, the pyramid A B D C R is > the given cone. *Ax.7. B. 1.*
But the base of the cone is the ⊙ in which this polygon A B D C is inscribed, (& which is consequently > this polygon), & this cone has the same altitude with the pyramid.
10. Therefore, the part is > the whole.
11. Which is impossible. *Ax.8. B. 1.*
12. Consequently, the cone is not < the third part of the cylinder.

II. Supposition.

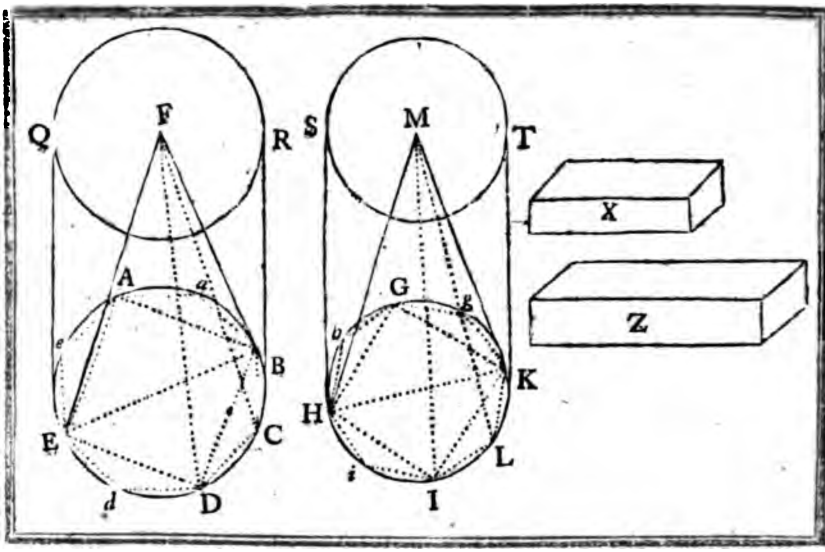
Let the cone be > the third part of the cylinder by the mgn. Z, *that is*, the cone = the third part of the cylinder + Z.

II. Preparation.

Divide the given cone into pyramids, *in the same manner that the cylinder was divided in the first supposition.*

IF from the given cone be taken the pyramid which has for base the □ A B D C, (which is greater than the half of the whole base of the given cone, being the half of the circumscribed □, *Arg.* 1. & this □ being > the base of the cone, *Ax.* 8. *B.* 1.), & from the remaining segments, the pyramids corresponding to those segments, *(as has been done in the cylinder Arg. 7.)*.
13. There will remain several elements of the cone which together will be < Z. *Lem. B.12.*
Therefore, if from the cone those elements be taken which are < Z, & from the cylinder + Z, the magnitude Z.
14. The remainder, viz. the pyramid A *a* B *b* D *d* C T R is = to the third part of the cylinder. *Ax.5. B. 1.*
But the pyr. A *a* B *b* D *d* C T R is = to the third part of the prism, which has for base the same polyg. A *a* B *b* D *d* C T, & the same alt. *P. 7. B.12. Cor. 2.*
15. Therefore, the given cylinder, is = to this prism. *Ax.6. B. 1.*
But the base of the given cylinder is > the base of the prism since this second is inscribed in the first. *(I. Prep. 4. & 5.)*.
16. Therefore, the part is = to the whole.
17. Which is impossible. *Ax.8. B. 1.*
18. Therefore, the third part of the cylinder is not < the cone.
And it has been demonstrated *(Arg. 12.)*, that the third part of the cylinder is not > the cone.
19. Therefore, the cone is the third part of the cylinder of the same base & altitude.

Which was to be demonstrated.



PROPOSITION XI. *THEOREM XI.*

CONES (E A B D F & H G K I M), and cylinders (Q R B E & S T K H) of the same altitude, are to one another as their bases.

Hypothesis,	Thesis.
The cones EABDF *&* HGKIM, *as likewise the cylinders* QRBE *&* STKH *have the same altitude.*	*I. Cone* EFB : *cone* HMK = *base* EABD : *base* HGKI. *II. Cylinder* QRBE : *cylinder* STKH = *base* EABD : *base* HGKI.

Demonstration.

If not, The cone E F B : Z (which is < or > the cone H M K) = base E A B D : base H G K I.

I. Supposition.

Let Z be < the cone H M K by a magnitude X, *that is*, let the cone H M K = Z + X.

I. Preparation.

1. In ⊙ G H I K base of cone H M K; describe □ G H I K. *P. 6. B. 4.*
2. Divide the cone into pyramids *(as in II. Sup. of P. 10.)*.
3. In the bases of the cones EFB & HMK, draw diam. EB & HK.
4. In the ⊙ E A B D base of the cone E F B, describe a polyg. as to the polyg. H *b* G *g* K L I *i* H, & divide it as the cone H M K.

BECAUSE the cone H M K has been divided into pyramids. *(Prep. 2.)*.
If those pyramids be taken from the cone (as was done in the foregoing proposition. *Arg.* 13.).

1. The sum of the remaining elements will be < X. *Lem. B.* 12.
Therefore, if those elements be taken from the cone H M K, & the magnitude X from Z + X.

2. The remaining pyramid H *b* G *g* K L I *i* M will be > Z.
But those polygons inscribed in the ⊙ E A B D & H G K I are ∾. *(Prep. 4.)*.
3. Therefore, ⊙ A E D B : ⊙ G H I K = polyg. C *d e a* : polyg. *i b g* L. { *P. 2. B.12. Cor.*
But, ⊙ A E D B : ⊙ G H I K = cone E F B : Z. *(Sup.)*.
And the pyramid D *d* E *e* A *a* B C F : pyramid H *b* G *g* K L I *i* M = polygon C *d e a* : polygon *i b g* L. *P. 6. B.11.*
4. Consequently, pyram. D *d* E *e* A *a* B C F : pyram. H *b* G *g* K L I *i* M = cone E F B : Z. *P.11. B. 5.*
But the pyramid D *d* E *e* A *a* B C F is < cone E F B. *Ax.8. B. 1.*
5. Therefore, the pyramid H *b* G *g* K L I *i* M is < Z. *P.14. B. 5.*
6. But this pyramid is > Z. *(Arg. 2.)*
7. Therefore, it will be > & < Z. *(Arg. 2. & 6)*.
8. Which is impossible.
9. Therefore, the supposition of Z < the cone H M K is false.
10. Wherefore, the base of the cone E F B is not to the base of the cone H M K (the cones having the same altitude) as the cone E F B to a magnitude Z < the cone H M K.

II. Supposition.

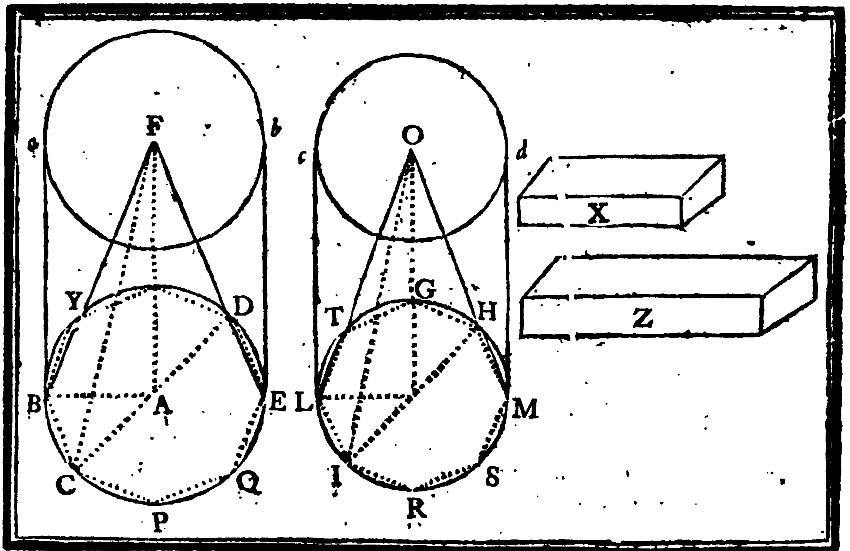
Let Z be > the cone H M K.

II. Preparation.

Take a magnitude X such that Z : cone E F B = cone H M K : X.

BECAUSE Z is > the cone H M K. *(I. Sup.)*.
11. The cone E F B is > X. *P.14. B. 5.*
But the cone E F B : Z = base E A B D : base H G K I. *(Sup.)*. { *P. 4. B. 5. Cor.*
12. Therefore, base H G K I : base E A B D = Z : cone E F B.
13. Consequently, base G H I K : base A E B D = cone H M K : X. *P.11. B. 5.*
But it has been demonstrated *(Arg. 10.)*, that the base of a cone is not to the base of another cone, having the same altitude, as the first cone is to a magnitude < the second.
14. Therefore, X is not < the cone E F B.
But X is < the cone E F B. *(Arg. 10.)*.
15. Consequently, X will be < & not < this cone EFB. *(Arg. 11. & 14.)*.
16. Which is impossible.
17. From whence it follows, that the supposition of Z > the cone H M K is false.
Therefore, the magnitude Z being neither < nor > the cone H M K. *(Arg. 9. & 17.)*.
18. It will be = to the cone H M K.
19. Hence cone E F B : cone H M K = base E A B D : base HGKI. *P. 7. B. 5.*
Which was to be demonstrated. 1.

BECAUSE the cone EFB is the third part of the cylin. QRBE
And the cone H M K is the third part of the cylin. H S T K. } *P.10. B.12.*
20. The cylin. Q R B E : cyl. H S T K = base E A B D : base HGKI. *P.15. B. 5.*
Which was to be demonstrated. 11.



PROPOSITION XII. *THEOREM XII.*

SIMILAR cones (B F E & L O M), and cylinders (B *a* *b* E & L *c* *d* M) have to one another the triplicate ratio of that which the diameters (C D & I H) of their bases (B Y D E P & L T H M R), have.

Hypothesis.

The cones BFE *&* LOM, *likewise the cylinders* B *a b* E *&* L *c d* M, *are* ∽.

Thesis.

I. *The cone* BFE *is to the cone* LOM *in the triplicate ratio of* C D *to* I H; *or as* $CD^3 : IH^3$.

II. *The cyl.* B *a b* E *is to the cyl.* L *c d* M, *in the triplicate ratio of* C D *to* I H; *or as* $CD^3 : IH^3$.

Demonstration.

If not,

The cone B F E is to a magnitude Z (which is < or > the cone L O M) as $CD^3 : IH^3$.

I. Supposition.

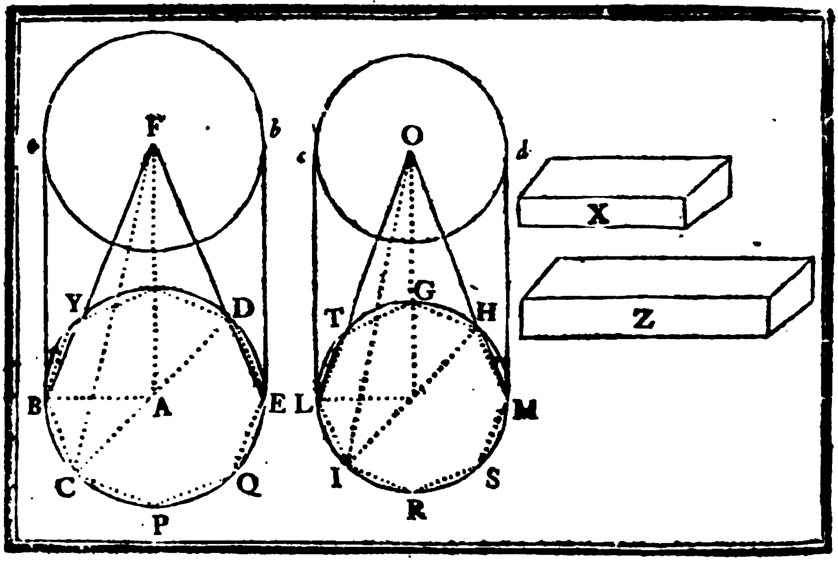
Let Z be < the cone L O M by the magnitude X, *that is*, the cone L O M = Z + X.

I. Preparation.

1. Divide the L O M into pyramids, as in the foregoing proposition.
2. In the base of the cone B F E describe a polygon ∽ to the polygon of the base of the cone L O M.
3. In the two cones draw the homologous diameters I H & C D; also the rays L N & B A.

BECAUSE the cone L O M has been divided into pyramids. If those pyramids be taken from this cone (in the same manner as in the foregoing proposition. *Arg.* 1.).

1. The sum of the remaining elements will be < X. *Lem. B.*12.
Therefore, if those elements be taken from the cone L O M, & the part X from the magnitude Z + X.
2. The remainder, viz. the pyramid L T G H M S R I O will be > Z. *Ax.*4. *B.* 1.
But the ∽ cones have their axes & the diameters of their bases proportional. *D.*24. *B.*11.
And the cones B F E & L O M are ∽. (*Hyp.*).
3. Consequently, C D : H I = F A : O N.
But, C D : H I = C A : I N. *P.*15. *B.* 5.
4. Therefore, C A : I N = F A : O N. *P.*11. *B.* 5.
5. And alternando C A : F A = I N : O N. *P.*16. *B.* 5.
The △ F A C & I O N have the ∀ C A F = to ∀ I N O. (*Prep.*3).
And the sides C A, A F; I N, O N about those equal angles proportional. (*Arg.* 5.).
6. Wherefore, the △ F A C is ∽ to the △ I O N. *D.* 1. *B.* 6.
7. Consequently, C F : C A = I O : I N. *P.* 4. *B.* 6.
8. Likewise, the △ B C A is ∽ to the △ L I N. (∀ B A C being = ∀ L N I). (*Prep.* 3.).
9. Therefore, C A : B C = I N : I L. *P.* 4. *B.* 6.
But, C F : C A = I O : I N. (*Arg.* 7.).
10. Consequently, C F : B C = I O : I L. *P.*22. *B.* 5.
In the △ C A F & B A F, the side C A is = to B A (*D.* 15. *B.* 1.) A F is common, & ∀ C A F = ∀ B A F. (*Prep.* 3.).
11. Therefore, the base B F is = to the base C F. *P.* 4. *B.* 1.
12. In like manner, L O is = to O I.
But, C F : B C = O I : I L. (*Arg.* 10.).
13. Therefore, B F : B C = L O : I L. *P.* 7. *B.* 5.
14. And invertendo, B C : B F = I L : O L. *P.* 4. *B.* 5.
15. Consequently, the three sides of the △ B F C are proportional to the three sides of the △ L O I. *Cor.*
16. From whence it follows, that those △ B F C & I O L are ∽. *P.* 5. *B.* 6.
17. It may be demonstrated after the same manner, that all the triangles which form the pyramid B D Q F are ∽ to all the triangles which form the pyramid L H S O, each to each.



And as the baſes of thoſe pyramids are ∽ polygons. (*Prep.* 2.)

18. The pyramid B D Q F is ∽ to the pyramid L H S O. *D.* 9. *B.*11.
But thoſe pyramids being ∽.

19. The pyramid B D Q F : pyramid L H S O = $CB^3 : IL^3$. *P.* 8. *B.*12. *Cor.*
But, C A : B C = I N : I L. (*Arg.* 9.).

20. Therefore invert. B C : C A = I L : I N. *P.* 4. *B.* 5. *Cor.*

21. And alternando, B C : L I = CA : I N. *P.*16. *B.* 5.

22. Conſequently, B C : L I = CD : I H. *P.*15. *B.* 5.

23. Therefore, three times the ratio of B C to L I is = to three times the ratio of C D to I H, *that is*, $BC^3 : LI^3 = CD^3 : IH^3$. *P.*11. *B.* 5.
But $CB^3 : IL^3$ = pyramid B D Q F : pyramid LHSO. (*Arg.*19).

24. Conſequently, pyramid BDQF : pyramid LHSO = $CD^3 : IH^3$. *P.*11. *B.* 5.
But the cone B F E : Z = $CD^3 : IH^3$. (*Sup.*).

25. Therefore, the pyram. BDQF : pyram. LHSO = cone BFE : Z. *P.*11. *B.* 5.
But the pyramid B D Q F being < cone B E F. *Ax.*8. *B.* 1.

26. The pyramid L H S O will be alſo < Z. *P.*14. *B.* 1.
But the pyramid L H S O is > Z. (*Arg.* 2.).

27. Conſequently, the pyram. LHSO will be < & > Z. (*Arg.*2. & 26).

28. Which is impoſſible.

29. Therefore, the ſuppoſition of Z < the cone L O M or L T G H M S R I O is falſe.

30. From whence it follows, that the cone B F E is not to a magnitude less than the cone L O M, in the triplicate ratio of the diameter C D to the diameter I H.

II. Supposition.

Let Z be > the cone L O M.

II. Preparation.

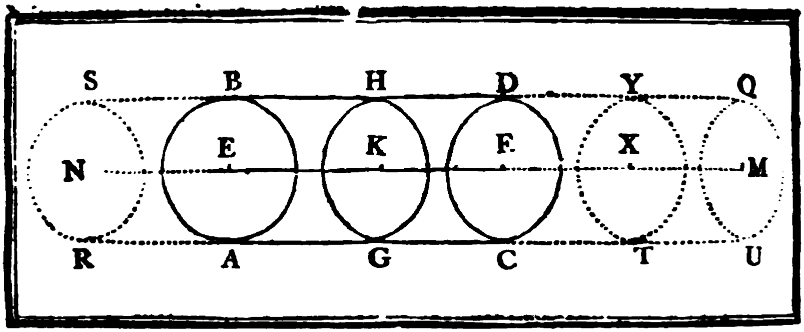
Take a magnitude X, such that Z : cone B F E = cone L O M : X.

BECAUSE Z is > than the cone L O M. *(II. Sup.)*.
31. The cone B F E will be > X. P.14. B. 5.
But C D³ : I H³ = cone B F E : Z. *(Sup.)*.
32. Therefore, invert. I H³ : C D³ = Z : cone B F E. P. 4. B. 5. Cor.
But Z : cone B F E = cone L O M : X. *(II. Prep.)*.
33. Consequently, I H³ : C D³ = cone L O M : X. P.11. B. 5.
And it has been demonstrated *(Arg. 30.)*, that a cone is not to a magnitude less than another cone in the triplicate ratio of the diameters of their bases.
34. Therefore, X is not < the cone B F E.
But X is < the same cone. *(Arg. 31.)*.
35. From whence it follows, that X will be < the cone, & will not be < at the same time.
36. Which is impossible.
37. Therefore, the supposition of Z being > the cone L O M, is false.
Therefore, the magnitude Z being neither < nor > the cone L O M. *(Arg. 29. & 37.)*.
38. It will be equal to it.
39. Consequently, the cone B F E : cone L O M = C D³ : I H³. P. 7. B. 5.

Which was to be demonstrated. I.

The cylinder B *a* *b* E, being triple of the cone B F E.
And the cylinder L *c* *d* M, the triple of the cone L O M. P.10. B.12.
40. The cylinder B *a* *b* E : cylinder L *c* *d* M = C D³ : I H³. P.15. B. 5.

Which was to be demonstrated. II.



PROPOSITION XIII. *THEOREM XIII.*

IF a cylinder (A B D C) be cut by a plane (H G) parallel to its opposite planes (B A & D C): It divides the cylinder into two cylinders (A B H G & G H D C), which are to one another as their axes, (E K & K F) (*that is*, the cylinder A B H G : cylinder G H D C = axis E K : axis K F).

Hypothesis.	Thesis.
The cylin. A D *is cut by a plane* H G, *plle. to the opposite planes* A B & D C.	*Cylin.* A H : *cylin.* H C = *axis* E K : *axis* F K.

Preparation.

1. Produce the axis E F of the cylinder A B D C both ways towards N & M. *Pos.* 2. *B.* 1.
2. In the axis N M produced, take several parts = to E K & F K; as E N = E K, & F X, &c. each = F K. *P.* 3. *B.* 1.
3. Thro' those points N, X & M pass the planes S R, T Y & V Q, plle. to the opposite planes B A & D C.
4. From the points N, X & M, describe on those planes the ⊙ S R, T Y & V Q each = to the opposite ⊙ B A & D C. *Pos.* 3. *B.* 1.
5. Complete the cylinders S A, C Y & T Q.

Demonstration.

BECAUSE the axes F X & X M of the cylinders D T & T Q are equal to the axis F K, of the cylinder G D. (*Prep.* 2).

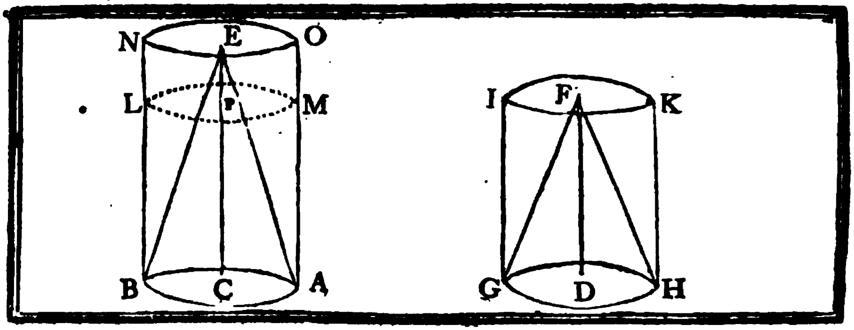
1. Those cylinders D T, T Q & G D will be to one another as their bases. *P.* 11. *B.* 12.

But those bases are equal. (*Prep.* 4).

2. Therefore, those cylinders T D, T Q & G D are also equal. *P.* 14. *B.* 5.

But there are as many equal cylinders C Y, T Q &c. which together are equal to the cylinder G Q, as there are parts F X, X M, &c. each equal to the axis K F, which together are equal to M K.

3. Conſequently, the cylinder G Q or G H Q V is the ſame multiple of the cylinder G H D C, that the axis K M is of the axis K F.
4. It may be demonſtrated after the ſame manner, that the cylinder R S H G is the ſame multiple of the cylinder A B H G, that the axis N K is of the axis E K.
5. Therefore, according as the cylinder G H Q V is >, =, or < the cylinder G H D C, the axis K M will be >, =, or < the axis F K. And according as the cylinder R S H G is >, =, or < the cylinder A B H G, the axis N K will be >, =, or < than the axis E K.
6. Conſequently, cylinder A B H G : cylinder G H D C = axis E K : axis F K. D. 5. B. 5.



PROPOSITION XIV. *THEOREM XIV.*

CYLINDERS (NOAB & IKHG), and cones (BEA & GFH) upon equal bases (B A & G H) : are to one another as their altitudes (C E & D F).

Hypothesis.

The cylinders N O A B *&* G I K H, *as also the cones* B E A *&* G F H, *have equal bases.*

Thesis.

I. Cylinder N O A B : *cylinder* I K H G = *alt.* C E : *alt.* D F.

II. Cone B E A : *cone* G F H = *alt.* C E : *alt.* D F.

Preparation.

1. In the axis of the greater cylinder A O N B, take a part P C = to the altitude of the cylinder G I K H.
2. Thro' the point P, pass a plane L M, plle. to the base BA, which will divide the cylinder A O N B into two cylinders, viz. B A M L & L M O N.

Demonstration.

BECAUSE the cylinder B N O A is cut by a plane plle. to its base, (*Prep.* 2.).

1. The cylinder N O M L : cylinder L M A B = P E : P C. *P.*13. *B.*12.
2. Consequently, cylinder N O M L + L M A B : cylinder L M A B = P E + P C : P C. *P.*18. *B.* 5.

But the cylinder N O M L + L M A B is = to the cylin. B N O A, P E + P C = E C. *Ax.*1. *B.* 1.

Moreover, the cylinder L M A B is = to cylinder I G H K, & P C = D F. (*Prep.* 1.).

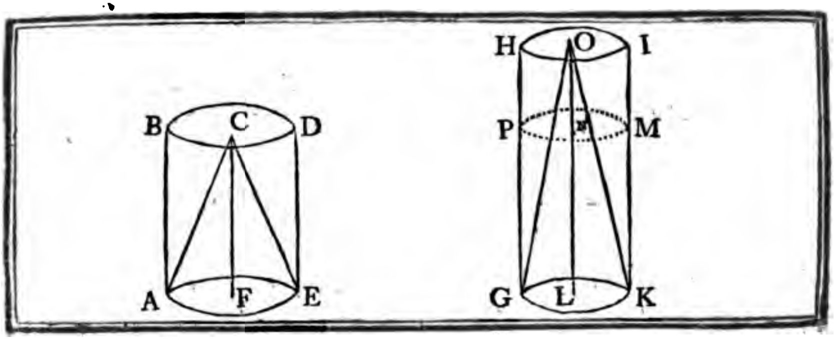
3. Therefore, the cylinder B N O A : cylinder I G H K = alt. E C : alt. D F. *P.* 7. *B.* 1.

Which was to be demonstrated. I.

The cone B E A is the third part of the cylinder B N O A.
And the cone G F H the third part of the cylinder G I K H. } *P.*10. *B.*12.

4. Consequently, the cone B E A : cone G F H = alt. E C : alt. D F. *P.*15. *B.* 5.

Which was to be demonstrated. II.



PROPOSITION XV. THEOREM XV.

THE bases (A E & G K), and altitudes (C F & O L), of the equal cylinders (A B D E & G H I K), and cones (A C E & G O K) : are reciprocally proportional, (*that is*, the base A E : base G K = alt. L O : alt. C F). And the cylinders and cones whose bases and altitudes are reciprocally proportional : are equal to one another.

Hypothesis.	Thesis.
I. The cylinders A B D E & G H I K *are equal.*	*Base* A E : *base* G K = *alt.* L O :
II. The cones A E C & G O K *are equal.*	*alt.* C E.

Preparation.

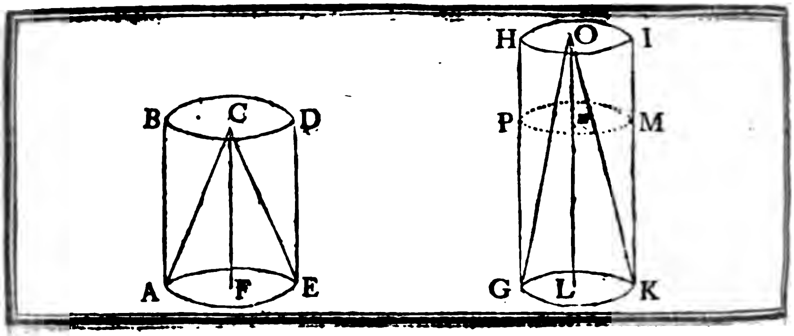
1. From the greater L O, cut off the altitude L N = the altitude C F. *P. 3. B. 1.*
2. Thro' the point N, pass a plane P M plle. to the opposite planes of the cylinder H I K G.

I. DEMONSTRATION.

BECAUSE the cylinder GHIK & PMKG have the same base.

1. The cylinder G H I K : cylinder P M K G = alt. L O : alt L N. *P.14. B.12.*
   But the cylinders A B D E & G H I K are equal. *(Hyp. 1.).*
2. Consequently, the cylinder A B D E : cylinder P M K G = alt. L O : alt. L N. *P. 7. B. 5.*
   Moreover, the cylinders A B D E & P M K G have the same altitude. *(Prep. 1.).*
3. Therefore, the cylinder A B D E : cylinder P M K G = base A E : base G K. *P.11. B.12.*
   But the cylinder A B D E : cylinder P M K G = alt. L O : alt. L N. *(Arg. 2.).*
   And the alt. L N is = to the alt. G F. *(Prep. 1.).*
4. From whence it follows, that base A E : base G K = alt. L O : alt. C F. *P.11. B. 5.* *P. 7. B. 5.*

Which was to be demonstrated.



Hypothesis.
Base G K : *base* A E = *alt.* C F : *alt.* L O.

Thesis.
I. Cyl. A B D E *is* = *to cyl.* G H I K
II. The cone A C E *is* = *to the cone* G O K

II. Demonstration.

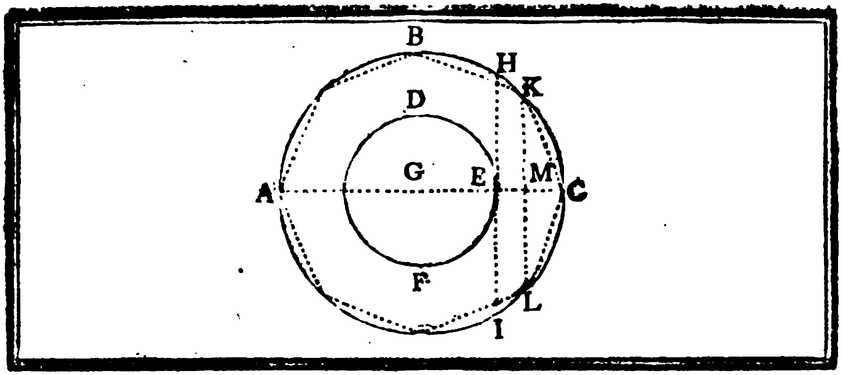
BECAUSE the cylinders G P M K & A B D E, have the same altitude, *(Prep. 2)*.

1. The cylinder G P M K : cylinder A B D E = base G K : base A E. P.11. B.12.
But the base G K : base A E = alt. C F : alt. L O, *(Hyp)*.
2. Consequently, the cyl. G P M K : cyl. ABDE = alt. C F : alt. LO. P.11. B. 5.
Moreover, the cylinders G P M K & H I K G have the same base.
3. Therefore, the cyl. G P M K : cyl. H I K G = alt. LN : alt. LO. P.14. B.12.
But the altitude L N is = to the altitude C F, *(Prep. 1)*.
4. From whence it follows that the cylinder G P M K : cylinder G H I K = altitude C F : altitude L O. P. 7. B. 5.
But the cylinder G P M K : cylinder A B D E = alt. C F : alt. L O. *(Arg. 2)*.
5. Therefore the cylinder G P M K : cylinder A B D E = cylinder G P M K : cylinder G H I K. P.11. B. 5.
6. Consequently, the cylinder A B D E is = to the cylinder G H I K. P.14. B. 5.

Which was to be demonstrated I.

The cones A C E & G O K being each the third part of the cylinders A B D E & G H I K. P.10. B.12.
And those cylinders being equal *(Arg. 6)*.
1. The cone A C E is = to the cone G O K. Ax.7. B. 1.

Which was to be demonstrated II.



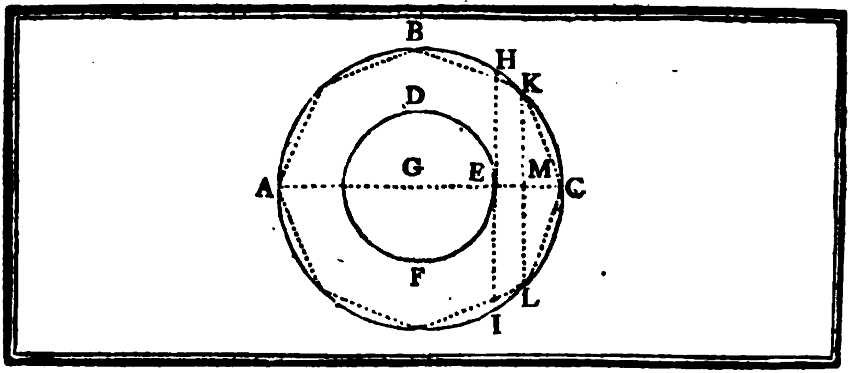
PROPOSITION XVI. *PROBLEM I.*

TWO unequal circles (ABCI & DEF) being given having the ſame center (G): to deſcribe in the greater (A B C I) a polygon of an even number of equal ſides, that ſhall not meet the leſſer circle (D E F).

Given.	Sought.
Two unequal ⊙ A B I *&* D E F *having the ſame center* G.	*To deſcribe in the greater* ⊙ A B I, *a polygon of an even number of equal ſides, that ſhall not be leſſer* ⊙ D E F.

Reſolution.

1. Draw the diameter A C in the greater ⊙ A B I which will cut the ○ of the ⊙ D F in the point E.
2. Thro' the point E, draw the tangent H E I to the ⊙ D E F & produce it until it meets the ○ of the ⊙ A B I in the points H & I. { *P.* 16. *B.* 3. *Poſ.* 2. *B.* 1.
3. Cut the ſemi ⊙ A B C into two equal parts in the point B. *P.* 30. *B.* 3.
4. Divide the ſemi arch B C into two equal parts, & ſo on until the arch K C be < the arch H C. *Lem. B.* 12.
5. Draw the chord K C & apply it around in the ○ of the ⊙ A B C I. { *P.* 1. *B.* 4. *Poſ.* 1. *B.* 1.



Preparation.

From the point K, let fall the ⊥ K M upon the diameter A C, & produce it until it meets the ○ in L. { P. 11. B. 1. / Pos. 1. B. 1.

Demonstration.

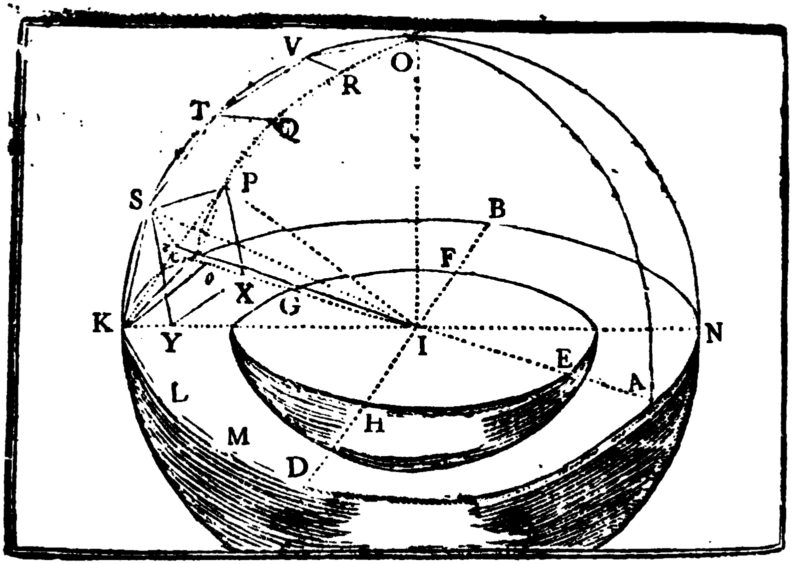
BECAUSE the semi ○ A B C, is divided into two equal parts at the point B. *(Ref. 3.).*
And the divisions have been continued until the arch K C has been attained. *(Ref. 4.).*

1. It follows, that this arch K C will measure the ○, an even number of times without a remainder, (because it measures the semi ○. *Ref. 3. & 4.).*
2. Consequently, the line K C (chord of the arch K C) will be the side of a polygon, having an even number of equal sides inscribed in the ⊙.
   Moreover, the two ∀HEM & KME being two ∟. *(Ref. 2. & Prep).*
3. The line K M or K L is plle. to H E or H I. **P. 28. B. 1.**
   But the line H I is a tangent of the ⊙ D E F in E. *(Ref. 2.).*
4. Consequently, K L does not meet the ⊙ D E F. **D. 35. B. 1.**
   But K C is < K L *(P. 15. B. 3.)* because K C is remoter from the center than K L. *(Prep.).*
5. Much more then K C will not meet the ⊙ D E F. **P. 15. B. 1.**
   And since the other sides of the polygon inscribed in the ⊙ A B C I are each = to K C. *(Ref. 5.).*
6. It may be demonstrated after the same manner, that they do not meet the ⊙ D E F.
7. Consequently, there has been described in the ⊙ A B C I, a polygon having an even number of equal sides, which does not meet the ⊙ D E F.

Which was to be done.

COROLLARY.

THE *line* KL, *which is* ⊥ *to the diameter* AC, & *joins the two sides* KC & LC, *of the polygon which meet at the extremity of this same diameter: does not meet the lesser circle.* (Arg. 4).



PROPOSITION XVII. *PROBLEM II.*

TWO spheres (K O N & G F E H) having the same center (I) being given : to describe in the greater (K O N) a polyhedron (K C S P T Q V R O &c.), the superficies of which shall not meet the lesser sphere.

Given.	Sought.
Two concentric spheres K O N & G F E H.	*I. A polyhedron* KPTRVO *&c. described in the greater sphere* K O N.
	II. The superficies of which polyhedron must not touch the lesser sphere G F E H.

Resolution.

1. Cut the spheres by a plane K B N D passing thro' their center.
2. In the ⊙ A B C D, draw the diameters A C & B D, intersecting each other at right angles. { *Pos.*1. *B.* 1. *P.*12. *B.* 1.
3. In this greater ⊙ ABCD, describe the polygon CKLMD &c. so as not to meet the lesser ⊙ G F E H. *P.*16. *B.*12.
4. Draw the diameter K I N.
5. From the center I, erect on the plane of ⊙ A B C D, the ⊥ I O, & produce it to the surface of the greater sphere in O. { *P.*12. *B.*11. *Pos.*2. *B.* 1.
6. Thro' I O, & the diameters A C, B D, & K N, pass the planes A O C, B O D, & K O N.
7. Divide the arches A O C & K O N into an even number of parts in the points P, Q, R, S, T, & V, &c. so that each of those parts be equal to C K.
8. Draw the straight lines S P, T Q, V R.

I. Preparation.

1. From the points P & S, let fall the ⊥ P X & S Y upon the plane of the ⊙ A B C D. P.12. B.11.
2. Draw Y X.

DEMONSTRATION.

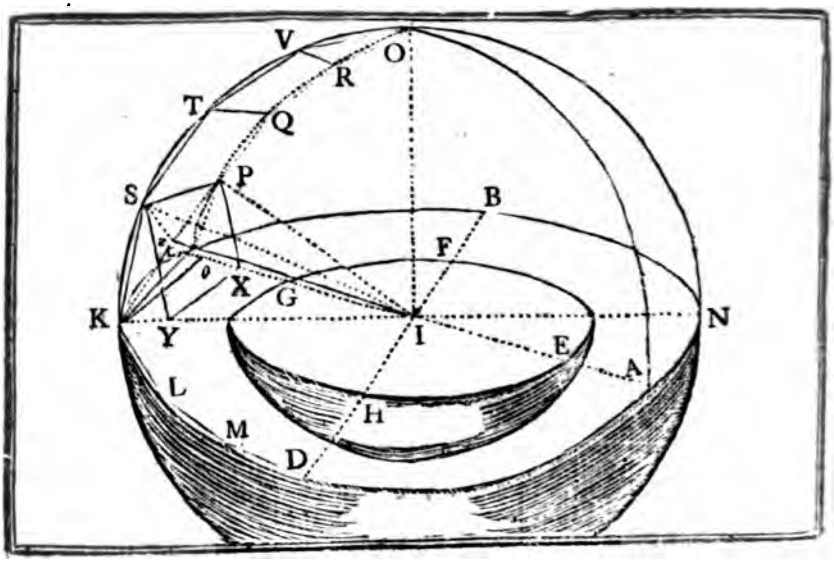
BECAUSE the planes K O N & C O A pass thro' I O. *(Ref.6).*
And that I O is ⊥ to the plane of the ⊙ A B C D. *(Ref. 5.).*
1. Those planes K O N & C O A, are ⊥ to the plane of this ⊙. P.18. B.11.
But the points P & S are in those planes C O A & K O N.
And from those points have been let fall the ⊥ PX & SY. *(I. Prep).*
2. Consequently, the points Y & X are in the lines K N & C A. P.38. B.11.
In the △ C X P & K Y S; ∀ P X C is = ∀ S Y K. *(I. Prep 1).*
Moreover, ∀ PCX = ∀ SKY. *(P.27.B.3),* & CP = KS, *(Ref.7).*
3. Therefore, the sides P X & X C are = to the sides S Y & Y K. P.26. B. 1.
But the rays K I & C I are equal. D.15. B. 1.
Therefore, if the equals X C & Y K be taken from them.
4. The remainders, viz. I X & Y I will be equal. Ax.3. B. 1.
5. Consequently, I X : X C = I Y : Y K. P. 7. B. 5.
6. From whence it follows, that X Y is plle. to K C. P. 2. B. 6.
But P X which is = to S Y *(Arg. 3.)* is also ⊥ on the same plane with S Y. *(I. Prep. 1.).*
7. Therefore, P X is also plle. to Y S. P. 6. B.11.
8. Likewise, S P is = & plle. to X Y. P.33. B. 1.
But X Y is plle. to K C. *(Arg. 6.).*
9. Therefore, S P is also plle. to K C. P. 9. B.11.
10. Consequently, the sides of the quadrilateral figure K S P C are in the same plane. P. 7. B.11.
11. It may be demonstrated after the same manner, that the sides of the quadrilateral figures T Q P S, V R Q F, & of the △ R Q V, are each in the same plane.
12. And as it may be demonstrated in this manner, that the whole sphere is incompassed with such like quadrilateral figures and triangles.
13. Consequently, there has been described in the greater sphere a polyhedron R P C K T V O, &c.

Which was to be demonstrated 1.

II. Preparation.

1. From the center I, let fall on the plane K S P C, the ⊥ I Z. P.11. B.11.
2. Join the points Z P, Z C, Z S, & Z K; S I & P I. Pof.1. B. 1.
3. From the point K, & in the plane A B C D, let fall the ⊥ K *o* on the diameter C A. P.12. B. 1.

BECAUSE in the △ KCI, the line YX is plle. to KC. *(Arg.6).*
14. I C : C K = I X : X Y. P. 2. B. 6.
But I C is > I X. Ax.8. B. 1.
15. Therefore, C K > X Y. P.14. B. 5.
But P S is = to X Y. *(Arg. 8.).*
16. From whence it follows, that C K is also > P S. P. 7. B. 5.
17. It may be demonstrated after the same manner, that S P is > T Q, & T Q > V R.



The ∠ I Z P, I Z C, I Z K, & I Z S are ∟ (*Il. Prep.* 1. *D.* 3. *B.* 11). *D.* 16. *B.* 1
& I C is = I P = I S = I K. *D.* 15. *B.* 1.
Moreover, I Z is common to the △ I Z P, I Z C, I Z K, & IZS. *P.* 47. *B.* 1.
18. Therefore, Z P = Z C = Z K = Z S. *P.* 46. *B.* 1.
19. Conſequently, the ⊙ deſcribed from the center Z, at the diſtance Z P, will paſs thro' the points K, S & C, & the quadrilateral figure R S P C will be deſcribed in a ⊙. *Cor.* 3. *D.* 3. *B.* 4.
But the four ſides of the quadrilateral figure were equal; the arches which ſubtend them will be ſo alſo, & will be each a quadrant of the ○. (*P.* 28. *B.* 3.).
But K S, C K & C P, are equal (*Ref.* 7.) & C K is > S P. (*Arg.* 16.).
20. From whence it is manifeſt, that the three ſides K S, C K, & C P, ſubtend more than the three quadrants of the ○; &, conſequently, C K (which is = to K S & C P) ſubtends more than a quadrant. *P.* 33. *B.* 6.
21. Conſequently, the ∠ C Z K at the center is > ∟.
22. Hence it follows, that the □ of K C is > □ of Z C + □ of Z K. *P.* 12. *B.* 2.
But the □ of Z C is = to the □ of Z K. (*P.* 46. *B.* 1. *Cor.* 3.).
Becauſe, Z C is = to Z K. (*Arg.* 18).
23. Therefore, the □ of K C is > the double of the □ of Z C.
The ∠ A I K is > ∟ (being = ∠ A I D + ∠ D I K, & ∠ D I A being a ∟. *Ref.* 2.).
Moreover, ∠ A I K is = ∠ I C K + ∠ I K C. *P.* 32. *B.* 1.
24. Conſequently, ∠ I C K + ∠ I K C are > ∟.
But ∠ I C K is = to ∠ C K I (*P.* 5 *B.* 1.) becauſe K I is = to C I. *D.* 15. *B.* 1.
25. Therefore, 2 ∠ I C K are > a ∟, & ∠ I C K > half of a ∟. *Ax.* 7. *B.* 1.
26. Wherefore, in △ C *o* K, the ∠ C K *o* is < half a ∟.

But ∠ I C K is > half a L. (*Arg.* 25.).

27. From whence it follows, that in the △ C o K, the side K o, opposite to the ∠ K C o or K C I is > the side C o, opposite to the ∠ C K o. *P.*18. *B.* 1.

28. Consequently, the □ of K C (which is = to the □ of K o + the □ of C o. *P.* 47. *B.* 1.) is < 2 □ of K o.

And it has been demonstrated (*Arg.* 23.) that the □ of K C is > the double of the □ Z C.

29. Wherefore, 2 □ of K o will be > 2 □ of Z C.

30. Hence, the □ of K o is > the □ of Z C.

But the □ of I C is = to the □ of I Z + the □ of Z C.

And the □ of I K (= to the □ of I C. *D.* 15. *B.* 1. & *P.* 46. *B.* 1. *Cor.* 3.) is = to the □ of I o + the □ of K o. } *P.*47. *B.* 1.

31. Therefore, □ of I Z + □ of Z C are = to □ of I o + □ of K o. *Ax.*1. *B.* 1.

Therefore. if from one side be taken the □ of Z C, & from the other the □ K o, (which are unequal, *Arg.* 30).

32. The remainder, viz. the □ of I Z will be > the □ of I o. *Ax.*5. *B.* 1.

33. Consequently, I Z is > I o.

But the line K o, (which is ⊥ to the diameter A C. *II. Prep.* 3. is without the sphere E F G H; & cannot meet it, *P.*16. *B.*12. *Cor.*), *that is,* I o is > I G.

And I o is < I Z. (*Arg.* 33.).

34. Much more then I Z, (which is much > I G) does not meet the surface of the sphere E F G H.

35. Wherefore the plane K S P C, in which Z is the point nearest the center I, does not touch this sphere E F G H.

36. It may be demonstrated after the same manner, that all the other planes which form the polyhedron do not meet the sphere E F G H.

37. Consequently, there has been described in the greater sphere K O N a polyhedron K P T R V O, &c. whose planes do not meet the lesser sphere. Which was to be demonstrated. 11.

COROLLARY.

If in two spheres there be described two similar polyhedrons; those polyhedrons will be to one another in the triplicate ratio of the diameters of the spheres in which they are described: For those polyhedrons being similar, are bounded by the same number of planes similar each to each, (D. 9. B. 11.); *consequently each polyhedron may be divided into pyramids, having all their vertices at the center of the sphere, & for bases the planes of the polyhedron, besides all the pyramids contained in the first polyhedron are similar to all the pyramids contained in the second polyhedron, each to each; consequently, they are to one another, (viz. the pyramids of the first polyhedron to the pyramids of the second) in the triplicate ratio of their homologous sides;* that is, *of the semi diameters of their spheres.* (Cor. P. 8. B. 12.) *From whence it follows,* (P.12. B.5.) *that all the pyramids composing the first polyhedron, are to all the pyramids composing the second polyhedron in the triplicate ratio of the semi diameters of their spheres;* & (P. 11. & 15. B. 5.) *that the first polyhedron is to the second in the triplicate ratio of the diameters of their spheres.*